Monomial Ideals and Hypergeometric Equations

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The Punchlines

- Good (or bad) examples are hard to find.
- There are open about monomial ideals, and even about squarefree monomial ideals.
- Homological algebra can be easier than combinatorics. Combinatorics is very hard.
- Having an explicit formula for a quantity does not imply fully understanding it.
A-hypergeometric (GKZ) systems

Definition
The Weyl Algebra $D$ (over $\mathbb{C}$) is given by generators $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_n$, and relations

$$x_ix_j = x_jx_i; \quad \partial_i\partial_j = \partial_j\partial_i; \quad \partial_ix_j = x_j\partial_i + \delta_{ij}.$$
Let \( A = (a_{ij}) \in \mathbb{Z}^{d \times n} \); \( \text{rank}(A) = d > n \), \([1 \ldots 1] \in \text{Rowspan}(A) \).
The ideal

\[
I_A = \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]
\]

is called a toric ideal.
Define the Euler operators \( E_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j \) for \( i = 1, \ldots d \).

**Definition**

The \( A \)-hypergeometric system with parameter \( \beta \in \mathbb{C}^d \) (GKZ system) is:

\[
H_A(\beta) = I_A + \langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle \subseteq D_n
\]

\( A \)-hypergeometric systems have interesting solutions.
A-hypergeometric (GKZ) Systems

**Definition**

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$$H_A(\beta) = \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle + \langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle \subseteq D_n$$

**Example (Roots of sparse polynomials)**

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & k_1 & \cdots & k_m \end{bmatrix}, \quad 0 < k_1 < \cdots < k_m \in \mathbb{N}; \quad \beta = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

The solutions of $H_A(\beta)$ are spanned by the roots of:

$$x_0 + x_1 t^{k_1} + \cdots + x_m t^{k_m} = 0,$$

considered as functions of $x_0, \ldots, x_m$. 

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**A-hypergeometric (GKZ) Systems**

**Definition**

The *A*-hypergeometric system with parameter $\beta \in \mathbb{C}^d$ is

$$H_A(\beta) = \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle + \langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle \subseteq D_n$$

**Example (Sparse systems)**

System of $m$ equations in $m$ unknowns $t_1, \ldots, t_m$; $A_i$ the support of $i$th equation $f_i = 0$; $J(t_1, \ldots, t_m) = \det(\partial f_i / \partial t_j)$ the Jacobian.

$(T_1, \ldots, T_m)$ a root; $T_j$ is a function of the coefficients of the $f_i$.

$$A = \{e_1\} \times A_1 \cup \cdots \cup \{e_m\} \times A_m$$ (Cayley trick).

For any $u \in \mathbb{N}^m$, $T^u / J(T)$ is $A$-hypergeometric.

Principal $A$-determinant: singular locus of $H_A(\beta)$. 

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To summarize:

- Roots of polynomials are hypergeometric.
- Roots of sparse systems are (almost) hypergeometric.
- Mostly, get transcendental solutions: e.g. Gauss, Appell, Lauricella, etc.

Long range goal: understand hypergeometric functions.

Start by studying the differential equations.
The holonomic rank

Define \( \text{rank}(H_A(\beta)) = \dim_{\mathbb{C}}(\{ \text{Solutions of } H_A(\beta) \}) \).

- (GKZ): \( \text{rank}(H_A(\beta)) \geq \text{vol}(A) \) for all \( \beta \), with equality for \( \beta \) generic.
- (GKZ/MMW): Necessary and sufficient condition on \( A \) for constant rank.
- (SST): \( \text{rank}(H_A(\beta)) \leq 2^{2d} \text{vol}(A) \), for all \( \beta \). Most likely far from optimal. Highest example: \( \text{rank}(H_A(\beta)) = \text{vol}(A) + 2 \).
- (MW): Examples with \( \text{rank}(H_A(\beta)) = \text{vol}(A) + d - 1 \).
- (Okuyama): Generalization and rank formulas for \( d = 3 \).
- (Berkesch): Complete rank formulas.

Can we improve the upper bound using the formulas?
Towards better bounds and worse examples

- The formulas are very complicated: multiple sums with signs and binomial coefficients.
- Try to get improvements in the proof by SST. First (hard) step:
  \[ \text{rank}(H_A(\beta)) \leq \text{rank}(\text{in}_w(I_A) + \langle E - \beta \rangle). \]
- Second step: \( \Delta_w = \text{triangulation of } A, \)
  \[ \text{rank}(\text{in}_w(I_A) + \langle E - \beta \rangle) \leq \text{arithmetic volume of } \Delta_w. \]
- Third step: \( 2^{2d} \text{vol}(A) \) bounds the arithmetic volume.
- Goal: Improve Step 2.

Theorem (Berkesch-M.)

There is a combinatorial formula for \( \text{rank}(\text{in}_w(I_A) + \langle E - \beta \rangle). \)
Definition

Given $\Delta$ a simplicial complex on $\{1, \ldots, n\}$, let

$$I_\Delta = \langle \prod_{i \in \tau} t_i \mid \tau \not\in \Delta \rangle = \bigcap_{\text{facets } \sigma \in \Delta} \langle t_i \mid t_i \not\in \sigma \rangle \subseteq \mathbb{C}[t_1, \ldots, t_n].$$

We call $\mathbb{C}[\Delta] = \mathbb{C}[t_1, \ldots, t_n]/I_\Delta$ the Stanley–Reisner ring of $\Delta$.

- Every squarefree monomial ideal is of the form $I_\Delta$ for some $\Delta$.
- $\dim(\mathbb{C}[\Delta]) = \dim(\Delta) + 1 =: d$.
- $\deg(I_\Delta) =$ number of top-dimensional facets of $\Delta$.
- For $\sigma \in \Delta$, define

$$\text{link}(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}.$$
Theorem (Reisner)
\( \mathbb{C}[\Delta] \) is a Cohen–Macaulay ring if and only if for all \( \sigma \in \Delta \),
\[ \tilde{H}_i(\text{link}(\sigma), \mathbb{C}) = 0, \quad \text{for } i < \dim(\text{link}(\sigma)). \]

A sequence \( E = E_1, \ldots, E_d \) of linear forms is a linear system of parameters (or Isop) for \( \mathbb{C}[\Delta] \) if \( \mathbb{C}[\Delta]/\langle E \rangle \) is zero dimensional.

Proposition
If \( E \) is an Isop for \( \mathbb{C}[\Delta] \),
\[ k_0(\Delta) := \dim_{\mathbb{C}} (\mathbb{C}[\Delta]/\langle E \rangle) \geq \deg(I_{\Delta}), \]
and equality holds if and only if \( \mathbb{C}[\Delta] \) is Cohen–Macaulay.
Some comments on $k_0(\Delta)$

- $k_0(\Delta)$ does not depend on the choice of lsop.
- $k_0(\Delta) \leq 2^d |\{\text{facets of } \Delta\}|$, because $\{t^\sigma \mid \sigma \in \Delta\}$ spans the vector space $\mathbb{C}[\Delta]/\langle E \rangle$. This is used in the bound for a. vol.
- If $\Delta \subseteq \Delta'$ are simplicial complexes of the same dimension, then an lsop for $\mathbb{C}[\Delta']$ is also an lsop for $\mathbb{C}[\Delta]$, and
  $$k_0(\Delta) \leq k_0(\Delta').$$
- If $\Delta \subseteq \Delta'$ as above, and $\Delta'$ is Cohen–Macaulay,
  $$k_0(\Delta) \leq k_0(\Delta') = \deg(I_{\Delta'}) = |\{\text{facets of } \Delta'\}|.$$
- The notation $k_0$ is for Koszul.
Example: Disconnected simplices

Let $\Delta$ consist of $r$ disconnected 1-dimensional simplices.

$\Delta$

Not Cohen–Macaulay

$\Delta'$

Cohen–Macaulay

$$k_0(\Delta) \leq k_0(\Delta') = r + r - 1.$$
Example: Disconnected simplices

Let $\Delta$ consist of $r$ disconnected 2-dimensional simplices.

\[
k_0(\Delta) \leq k_0(\Delta') = r + 2(r - 1).
\]

This way, can see that if $\Delta$ consists of $r$ disconnected simplices of dimension $d - 1$, then

\[
k_0(\Delta) \leq r + (d - 1)(r - 1).
\]
Computing $k_0(\Delta)$

**Theorem (Berkesch, M)**

There is a formula of the form

$$k_0(\Delta) = \deg(I_\Delta) + \text{sums of binomial coefficients with signs,}$$

and terms appearing here depend only on the combinatorics of $\Delta$.

**Corollary**

*Let $\Delta$ consist of $r$ disconnected $(d - 1)$-dimensional simplices, Then*

$$k_0(\Delta) = r + (d - 1)(r - 1).$$

**Proof.**

Seven lines of regrouping and using combinatorial identities. □
The proof of the Theorem is homological (there is a nice spectral sequence).

A formula without alternating signs would be desirable.

Is there an upper bound for $k_0(\Delta)$ better than $2^d |\{\text{ facets of } \Delta\}|$? Or is this tight?

Given $\Delta$ is there always $\Delta' \supseteq \Delta$ Cohen–Macaulay of the same dimension, such that and $k_0(\Delta) = k_0(\Delta')$?
Buchsbaum complexes

Definition
A simplicial complex $\Delta$ is **Buchsbaum** if it is pure and, for all $\sigma \in \Delta \setminus \emptyset$, $\tilde{H}_i(\text{link}(\sigma), \mathbb{C}) = 0$, for $i < \dim(\text{link}(\sigma))$.

If $\Delta$ is Buchsbaum, but not Cohen–Macaulay, this means that $\text{link}(\sigma) = \Delta$ has homology.

**Theorem (Schenzel)**
If $\Delta \neq \emptyset$ is Buchsbaum, 
$k_0(\Delta) = |\{\text{facets of } \Delta\}| + \sum_{j=2}^{d} \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-i-1} \tilde{h}_{i-1}(\Delta)$. 

- Gives a one line computation of $k_0(\text{Disconnected Simplices})$.
- Can produce a Buchsbaum $\Delta$ with: 
  
  $k_0(\Delta) = |\{\text{facets}\}| + 2^d - 1 + (d + 1)d(d - 1)/2$. 

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Can produce a Buchsbaum $\Delta$ with:

$$k_0(\Delta) = |\{\text{facets}\}| + 2^d - 1 + (d + 1)d(d - 1)/2.$$ 

This example is honestly exponential (the number of facets is polynomial in $d$).

Does this $\Delta$ come up in the hypergeometric situation? Probably not: initial ideals of toric ideals are very special.

Idea: use Schenzel’s method in the hypergeometric context: impose conditions on $A$ to get better formulas and maybe worse examples.
Towards the non squarefree case: Distractions

Proposition

Let $\Delta$ be a simplicial complex, $E$ an Isop, and $\beta \in \mathbb{C}^d$. Then

$$\dim_{\mathbb{C}}(\mathbb{C}[\Delta]/\langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle) \geq \deg(I_\Delta), \quad \forall \beta \in \mathbb{C}^d,$$

with equality if $\beta$ is generic. Equality holds for all $\beta \in \mathbb{C}^d$ if and only if $\mathbb{C}[\Delta]$ is Cohen–Macaulay.

Definition

If $I \subseteq \mathbb{C}[t_1, \ldots, t_n]$ is a monomial ideal, its distraction $\tilde{I}$ is obtained by replacing, in each minimal generator, powers of the variables by descending factorials. For instance,

$$t_1^4t_2^2t_3 \quad \text{is replaced by} \quad t_1(t_1-1)(t_1-2)(t_1-3)t_2(t_2-1)t_3.$$
$k_0$ in the non squarefree case

- $I$ is squarefree if and only if $I = \tilde{I}$.
- The zero set of $\tilde{I}$ is the Zariski closure of the exponents of the monomials not in $I$.

Let $I \subseteq \mathbb{C}[t]$ a monomial ideal, and $d = \dim(\mathbb{C}[t]/I)$. Choose $E_1, \ldots, E_d$ linear forms such that $\dim_{\mathbb{C}}(\mathbb{C}[t]/(\tilde{I} + \langle E - \beta \rangle)) < \infty$ for all $\beta \in \mathbb{C}^d$.

Theorem

$k_0(I, E - \beta) := \dim_{\mathbb{C}}(\mathbb{C}[t]/(\tilde{I} + \langle E - \beta \rangle)) \geq \deg(I), \quad \forall \beta \in \mathbb{C}^d$,

with equality for generic $\beta$. Equality holds for all $\beta$ if and only if $\mathbb{C}[t]/I$ is Cohen–Macaulay.
More Comments and Questions

- One can use this Theorem to reduce the Cohen–Macaulayness of $\mathbb{C}[t]/I$ to the Cohen–Macaulayness of a finite collection of simplicial complexes.

- One can also write a formula for $k_0(I; E - \beta)$ using the formula for the squarefree case.

- The set $\{\beta \in \mathbb{C}^d \mid k_0(I; E - \beta) > \deg(I)\}$ can be written explicitly in terms of $E$ and the local cohomology of $\mathbb{C}[t]/I$.

- The $\max\{k_0(I; E - \beta) \mid \beta \in \mathbb{C}^d\}$ depends on the choice of linear forms $E$.

- What is this maximum when $I = \text{in}_w(I_A)$ and $E$ comes from the rows of $A$? (upper bound $2^{2d} \text{vol}(A)$, [SST].)

- There is an analogous criterion to decide whether a binomial ideal is Cohen–Macaulay, but one needs hypergeometric differential equations.