Monomial Ideals and Hypergeometric Equations

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The Punchlines

- Good (or bad) examples are hard to find.
- There are open about monomial ideals, and even about squarefree monomial ideals.
- Homological algebra can be easier than combinatorics. Combinatorics is very hard.
- Having an explicit formula for a quantity does not imply fully understanding it.

A-hypergeometric (GKZ) systems

Definition

The Weyl Algebra D (over $\mathbb{C}$) is given by generators $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_n$, and relations

$$x_i x_j = x_j x_i \; ; \quad \partial_i \partial_j = \partial_j \partial_i \; ; \quad \partial_i x_j = x_j \partial_i + \delta_{ij}.$$

A-hypergeometric (GKZ) systems

Let $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$; $\mathrm{rank}(A) = d > n$, $[1 \dots 1] \in \mathrm{Rowspan}(A)$.
The ideal

$$I_A = \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle \subseteq \mathbb{C}[\partial_1, \dots, \partial_n]$$

is called a toric ideal.
Define the Euler operators $E_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j$ for $i = 1, \dots d$.

Definition

The A-hypergeometric system with parameter $\beta \in \mathbb{C}^d$ (GKZ system) is:

$$H_A(\beta) = I_A + \langle E_1 - \beta_1, \dots, E_d - \beta_d \rangle \subseteq D_n$$

A-hypergeometric systems have interesting solutions.

A-hypergeometric (GKZ) Systems

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Example (Roots of sparse polynomials)

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & k_1 & \dots & k_m \end{bmatrix}, \quad 0 < k_1 < \cdots < k_m \in \mathbb{N}; \quad \beta = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The solutions of $H_A(\beta)$ are spanned by the roots of:

$$x_0 + x_1 t^{k_1} + \cdots + x_m t^{k_m} = 0,$$

considered as functions of $x_0, \dots, x_m$.

A-hypergeometric (GKZ) Systems

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Example (Sparse systems)

System of m equations in m unknowns $t_1, \ldots, t_m$; A_i the support of ith equation $f_i = 0$; $J(t_1, \ldots, t_m) = \det(\partial f_i / \partial t_j)$ the Jacobian.

$(T_1, \ldots, T_m)$ a root; T_j is a function of the coefficients of the f_i.

$$A = \{e_1\} \times A_1 \cup \cdots \cup \{e_m\} \times A_m \text{ (Cayley trick).}$$

For any $u \in \mathbb{N}^m$, $T^u / J(T)$ is A-hypergeometric.

Principal A-determinant: singular locus of $H_A(\beta)$.

A-hypergeometric (GKZ) Systems

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$$H_A(\beta) = \langle \partial^u - \partial^v \mid A \cdot u = A \cdot v \rangle + \langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle \subseteq D_n$$

To summarize:

- Roots of polynomials are hypergeometric.
- Roots of sparse systems are (almost) hypergeometric.
- Mostly, get transcendental solutions: e.g. Gauss, Appell, Lauricella, etc.

Long range goal: understand hypergeometric functions.

Start by studying the differential equations.

The holonomic rank

Define $\mathrm{rank}(H_A(\beta)) = \dim_{\mathbb{C}}(\{\text{Solutions of } H_A(\beta)\})$.

- (GKZ): $\mathrm{rank}(H_A(\beta)) \geq \mathrm{vol}(A)$ for all β, with equality for β generic.
- (GKZ/MMW): Necessary and sufficient condition on A for constant rank.
- (SST): $\mathrm{rank}(H_A(\beta)) \leq 2^{2d}\mathrm{vol}(A)$, for all β. Most likely far from optimal. Highest example: $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(A) + 2$.
- (MW): Examples with $\mathrm{rank}(H_A(\beta)) = \mathrm{vol}(A) + d - 1$.
- (Okuyama): Generalization and rank formulas for $d = 3$.
- (Berkesch): Complete rank formulas.

Can we improve the upper bound using the formulas?

Towards better bounds and worse examples

- The formulas are very complicated: multiple sums with signs and binomial coefficients.
- Try to get improvements in the proof by SST. First (hard) step:

$$\mathrm{rank}(H_A(\beta)) \leq \mathrm{rank}(\mathrm{in}_w(I_A) + \langle E - \beta \rangle).$$

- Second step: $\Delta_w =$ triangulation of A,

$$\mathrm{rank}(\mathrm{in}_w(I_A) + \langle E - \beta \rangle) \leq \text{ arithmetic volume of } \Delta_w.$$

- Third step: $2^{2d}\mathrm{vol}(A)$ bounds the arithmetic volume.
- Goal: Improve Step 2.

Theorem (Berkesch-M.)

There is a combinatorial formula for $\mathrm{rank}(\mathrm{in}_w(I_A) + \langle E - \beta \rangle)$.

Shift gears: Stanley–Reisner rings

Definition
Given Δ a simplicial complex on $\{1, \ldots, n\}$, let

$$I_\Delta = \left\langle \prod_{i \in \tau} t_i \;\middle|\; \tau \notin \Delta \right\rangle = \bigcap_{\text{facets } \sigma \in \Delta} \langle t_i \mid t_i \notin \sigma \rangle \subseteq \mathbb{C}[t_1, \ldots, t_n].$$

We call $\mathbb{C}[\Delta] = \mathbb{C}[t_1, \ldots, t_n]/I_\Delta$ the Stanley–Reisner ring of Δ.

- Every squarefree monomial ideal is of the form I_Δ for some Δ.
- $\dim(\mathbb{C}[\Delta]) = \dim(\Delta) + 1 =: d$.
- $\deg(I_\Delta) =$ number of top-dimensional facets of Δ.
- For $\sigma \in \Delta$, define

$$\operatorname{link}(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \varnothing, \tau \cup \sigma \in \Delta\}.$$

Cohen–Macaulay Stanley–Reisner rings

Theorem (Reisner)

$\mathbb{C}[\Delta]$ *is a Cohen–Macaulay ring if and only if for all* $\sigma \in \Delta$*,*

$$\widetilde{H}_i(\mathrm{link}(\sigma), \mathbb{C}) = 0, \quad \text{for i} < \dim(\mathrm{link}(\sigma)).$$

A sequence $E = E_1, \ldots, E_d$ of linear forms is a linear system of parameters (or lsop) for $\mathbb{C}[\Delta]$ if $\mathbb{C}[\Delta]/\langle E \rangle$ is zero dimensional.

Proposition

If E *is an lsop for* $\mathbb{C}[\Delta]$*,*

$$k_0(\Delta) := \dim_{\mathbb{C}} \big(\mathbb{C}[\Delta]/\langle E \rangle\big) \geq \deg(I_\Delta),$$

and equality holds if and only if $\mathbb{C}[\Delta]$ *is Cohen–Macaulay.*

Some comments on $k_0(\Delta)$

- $k_0(\Delta)$ does not depend on the choice of lsop.
- $k_0(\Delta) \leq 2^d \, |\{\text{facets of } \Delta\}|$, because $\{t^\sigma \mid \sigma \in \Delta\}$ spans the vector space $\mathbb{C}[\Delta]/\langle E \rangle$. This is used in the bound for a. vol.
- If $\Delta \subseteq \Delta'$ are simplicial complexes *of the same dimension*, then an lsop for $\mathbb{C}[\Delta']$ is also an lsop for $\mathbb{C}[\Delta]$, and

$$k_0(\Delta) \leq k_0(\Delta').$$

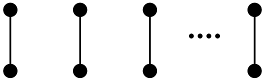
- If $\Delta \subseteq \Delta'$ as above, and Δ' is Cohen–Macaulay,

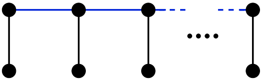
$$k_0(\Delta) \leq k_0(\Delta') = \deg(I_{\Delta'}) = |\{\text{facets of } \Delta'\}|.$$

- The notation k_0 is for Koszul.

Example: Disconnected simplices

Let Δ consist of r disconnected 1-dimensional simplices.



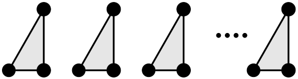


$$k_0(\Delta) \leq k_0(\Delta') = r + r - 1.$$

Example: Disconnected simplices

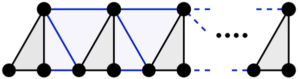
Let Δ consist of r disconnected 2-dimensional simplices.

Δ



Not Cohen–Macaulay

Δ'



Cohen–Macaulay

$$k_0(\Delta) \leq k_0(\Delta') = r + 2(r-1).$$

This way, can see that if Δ consists of r disconnected simplices of dimension $d-1$, then

$$k_0(\Delta) \leq r + (d-1)(r-1).$$

Computing $k_0(\Delta)$

Theorem (Berkesch, M)

There is a formula of the form

$$k_0(\Delta) = \deg(I_\Delta) + \text{sums of binomial coefficients with signs},$$

and terms appearing here depend only on the combinatorics of Δ.

Corollary

Let Δ consist of r disconnected $(d-1)$-dimensional simplices, Then

$$k_0(\Delta) = r + (d-1)(r-1).$$

Proof.

Seven lines of regrouping and using combinatorial identities. $\square$

Comments and Questions

- The proof of the Theorem is homological (there is a nice spectral sequence).
- A formula without alternating signs would be desirable.
- Is there an upper bound for $k_0(\Delta)$ better than $2^d\,|\{\text{facets of }\Delta\}|$? Or is this tight?
- Given Δ is there always $\Delta' \supseteq \Delta$ Cohen–Macaulay of the same dimension, such that and $k_0(\Delta) = k_0(\Delta')$?

Buchsbaum complexes

Definition

A simplicial complex Δ is Buchsbaum if it is pure and, for all $\sigma \in \Delta \smallsetminus \varnothing$, $\widetilde{H}_i(\mathrm{link}(\sigma), \mathbb{C}) = 0, \quad \text{for i} < \dim(\mathrm{link}(\sigma))$.

If Δ is Buchsbaum, but not Cohen–Macaulay, this means that $\mathrm{link}(\sigma) = \Delta$ has homology.

Theorem (Schenzel)

If $\Delta \neq \varnothing$ is Buchsbaum,
$k_0(\Delta) = |\{\text{facets of } \Delta\}| + \sum_{j=2}^{d} \binom{d}{j} \sum_{i=1}^{j-1} (-1)^{j-i-1} \widetilde{h}_{i-1}(\Delta)$.

- Gives a one line computation of k_0(Disconnected Simplices).
- Can produce a Buchsbaum Δ with:

$$k_0(\Delta) = |\{\text{facets}\}| + 2^d - 1 + (d+1)d(d-1)/2.$$

Buchsbaum complexes and beyond

- Can produce a Buchsbaum Δ with:

$$k_0(\Delta) = |\{\text{facets}\}| + 2^d - 1 + (d+1)d(d-1)/2.$$

- This example is honestly exponential (the number of facets is polynomial in d).
- Does this Δ come up in the hypergeometric situation? Probably not: initial ideals of toric ideals are very special.
- Idea: use Schenzel's method in the hypergeometric context: impose conditions on A to get better formulas and maybe worse examples.

Towards the non squarefree case: Distractions

Proposition

Let Δ be a simplicial complex, E an lsop, and $\beta \in \mathbb{C}^d$. Then

$$\dim_{\mathbb{C}}(\mathbb{C}[\Delta]/\langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle) \geq \deg(I_\Delta), \quad \forall \beta \in \mathbb{C}^d,$$

with equality if β is generic. Equality holds for all $\beta \in \mathbb{C}^d$ if and only if $\mathbb{C}[\Delta]$ is Cohen–Macaulay.

Definition

If $I \subseteq \mathbb{C}[t_1, \ldots, t_n]$ is a monomial ideal, its distraction $\tilde{I}$ is obtained by replacing, in each minimal generator, powers of the variables by descending factorials. For instance,

$$t_1^4 t_2^2 t_3 \quad \text{is replaced by} \quad t_1(t_1-1)(t_1-2)(t_1-3)t_2(t_2-1)t_3.$$

k_0 in the non squarefree case

- I is squarefree if and only if $I = \tilde{I}$.
- The zero set of $\tilde{I}$ is the Zariski closure of the exponents of the monomials not in I.

Let $I \subseteq \mathbb{C}[t]$ a monomial ideal, and $d = \dim(\mathbb{C}[t]/I)$. Choose $E_1, \ldots, E_d$ linear forms such that $\dim_{\mathbb{C}}(\mathbb{C}[t]/(\tilde{I} + \langle E - \beta \rangle)) < \infty$ for all $\beta \in \mathbb{C}^d$.

Theorem

$$k_0(I, E - \beta) := \dim_{\mathbb{C}}(\mathbb{C}[t]/(\tilde{I} + \langle E - \beta \rangle)) \geq \deg(I), \quad \forall \beta \in \mathbb{C}^d,$$

with equality for generic β. Equality holds for all β if and only if $\mathbb{C}[t]/I$ is Cohen–Macaulay.

More Comments and Questions

- One can use this Theorem to reduce the Cohen–Macaulayness of $\mathbb{C}[t]/I$ to the Cohen–Macaulayness of a finite collection of simplicial complexes.
- One can also write a formula for $k_0(I; E-\beta)$ using the formula for the squarefree case.
- The set $\{\beta \in \mathbb{C}^d \mid k_0(I; E-\beta) > \deg(I)\}$ can be written explicitly in terms of E and the local cohomology of $\mathbb{C}[t]/I$.
- The $\max\{k_0(I; E-\beta) \mid \beta \in \mathbb{C}^d\}$ depends on the choice of linear forms E.
- What is this maximum when $I = \mathrm{in}_w(I_A)$ and E comes from the rows of A? (upper bound $2^{2d}\mathrm{vol}(A)$, [SST].)
- There is an analogous criterion to decide whether a binomial ideal is Cohen–Macaulay, but one needs *hypergeometric differential equations*.