Complex and Non-Archimedean (Co)amoebas, and Phase Limit Sets

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Randomization, Relaxation, and Complexity

Join works with P. Johansson, M. Passare.
& F. Sottile

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Summary

Description of the Complex Algebraic Torus

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ the field of Puiseux series.

$$\begin{array}{rcccl} w & : & \mathbb{K}^* & \longrightarrow & \mathbb{C}^* \\ & & a & \longmapsto & w(a) = e^{\mathrm{val}(a) + i \arg(\xi_{-\mathrm{val}(a)})}, \end{array}$$

for any $a \in \mathbb{K}$ with $a = \sum_{j \in A_a} \xi_j t^j$.

Description of the Complex Algebraic Torus

$$
\begin{tikzcd}
 & \mathbb{K}^* \arrow[d, "W"'] \arrow[ddl, "\mathrm{Log}_{\mathbb{K}}"'] \arrow[ddrr, "\mathrm{Arg}_{\mathbb{K}}"] & & \\
 & \mathbb{C}^* \arrow[dl, "\mathrm{Log}"'] \arrow[drr, "\mathrm{Arg}"] & & \\
\mathbb{R} & \mathbb{C} \arrow[l, "\mathrm{Re}"'] \arrow[u, "\exp"'] \arrow[r] & \mathbb{R} \arrow[r, "\pi"] & \mathbb{R}/2\pi\mathbb{Z} = S^1
\end{tikzcd}
$$

We apply the maps coordinatewise.

Description of the Complex Algebraic Torus

Let V_f be the complex algebraic hypersurface defined by the polynomial

$$f(z) = \sum_{\alpha \in \mathrm{supp}(f)} a_\alpha z^\alpha,$$

with $a_\alpha \in \mathbb{C}^*$, and $\mathrm{supp}(f)$ finite subset of $\mathbb{Z}^n$

$$V_f = \{z \in (\mathbb{C}^*)^n \mid f(z) = 0\}$$

Description of the Complex Algebraic Torus

$$
\begin{tikzcd}
 & V_f \subset (\mathbb{C}^*)^n \arrow[dl, "\mathrm{Log}"'] \arrow[drr, "\mathrm{Arg}"] & & \\
\mathscr{A}_{V_f} \subset \mathbb{R}^n & \mathbb{C}^n \arrow[l, "\mathrm{Re}"'] \arrow[u, "\exp"'] \arrow[r] & \mathbb{R}^n \arrow[r, "\pi"] & (S^1)^n \supset co\mathscr{A}_{V_f}
\end{tikzcd}
$$

DEFINITION

The *complex amoeba* of V_f is $\mathscr{A}_{V_f} := \mathrm{Log}(V_f)$

The *complex coamoeba* of V_f is $co\mathscr{A}_{V_f} := \mathrm{Arg}(V_f)$

DEFINITION

The *Non-Archimedean amoeba* of V_f is $\mathscr{A}_{V_f} := \mathrm{Log}_{\mathbb{K}}(V_f)$

The *Non-Archimedean coamoeba* of V_f is $co\mathscr{A}_{V_f} := \mathrm{Arg}_{\mathbb{K}}(V_f)$

Theorem (Nisse, 2009)

Let V be a complex algebraic hypersurface defined by a polynomial f with Newton polytope Δ. Let us denote by τ_f the subdivision of Δ dual to the spine of the amoeba of V. Then there exists a complex tropical hypersurface $V_{\infty,f}$ satisfying the following :

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(i) The closure of the coamoebas of $V_{\infty,f}$ and V in the real torus $(S^1)^n$ have the same homotopy type;

(ii) The lifting of the coamoeba of $V_{\infty,f}$ in the universal covering of the torus $(S^1)^n$ contains an arrangement $\mathscr{H}$ of codual hyperplanes to the set of edges of τ_f which determine completely the topology of the complex coamoeba of V.

Theorem (Johansson-Nisse-Passare, 2009)

Let k, and m be two positives natural integers , and $\mathscr{P}(k) \subset (\mathbb{C}^*)^{k+m}$ be an affine linear space of dimension k. Then, the dimension of the (co)amoeba $(co)\mathscr{A}_k$ of $\mathscr{P}(k)$ satifies the following :

$$k+1 \leq \dim((co)\mathscr{A}_k) \leq \min\{2k, k+m\}.$$

In particular, if $\mathscr{P}(k)$ is in general position, then the dimension of its (co)amoeba is maximal.

For example, there are two types of amoebas of lines in $(\mathbb{C}^*)^{1+m}$ for $m > 1$, amoebas with boundary and without boundary. All real line in $(\mathbb{C}^*)^{1+m}$ for $m \geq 1$ are with boundary.

Coamoebas of some complex algebraic plane curves

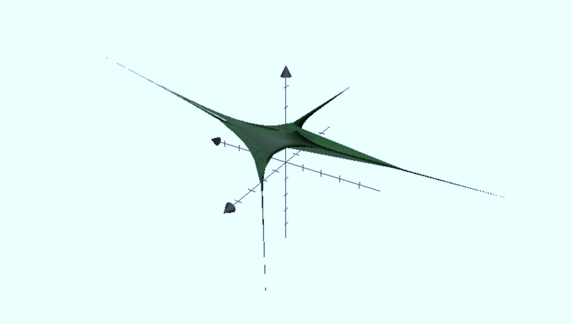

Figure: Amoeba of real line in $(\mathbb{C}^*)^3$

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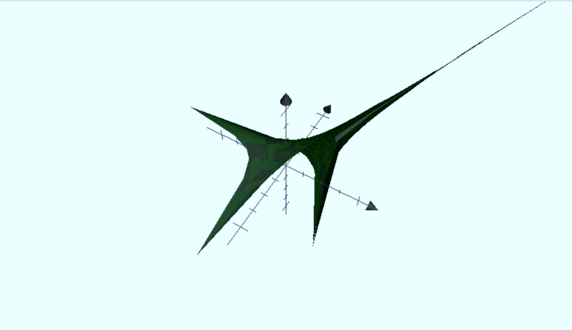

Figure: Amoeba of real line in $(\mathbb{C}^*)^3$

Theorem (Johansson-Nisse-Passare, 2009)

Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety with defining ideal $\mathcal{I}(V)$. Then the (co)amoeba of V is given as follows :

$$(co)\mathscr{A}(V) = \bigcap_{f \in \mathcal{I}(V)} (co)\mathscr{A}(V_f)$$

Let $\mathcal{S}(V)$ be the set of sequences $\{z_n\} \subset V$ such that z_n converge to the infinity. Let $q = \{z_n\}$ be an element of $\mathcal{S}(V)$, and $acc(q)$ be the set of accumulation points, in the real torus $(S^1)^n$, of the sequence $\{\mathrm{Arg}(z_n)\}$.

Definition (Nisse-Sottile, 2009)

Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety. The *phase limit set* of V is the subset of the real torus $(S^1)^n$ denoted by $\mathscr{P}^\infty(V)$ and defined by :
$\mathscr{P}^\infty(V) := \cup_{q \in \mathcal{S}(V)} acc(q)$.

Theorem (Nisse-Sottile, 2009)

Let V be an algebraic variety of dimension k in $(\mathbb{C}^*)^n$. Let $co\mathscr{A}$ be its coamoeba and $\mathscr{P}^\infty(V)$ its phase limit set. Then $\overline{co\mathscr{A}} = co\mathscr{A} \cup \mathscr{P}^\infty(V)$, where $\overline{co\mathscr{A}}$ denotes the closure of $co\mathscr{A}$ in the universal covering of the real torus. Moreover, $\mathscr{P}^\infty(V)$ is the union of some arrangement $\mathscr{H}(V)$ of k-torus and the coamoebas of some complex algebraic varieties of dimension l with $l \leq k-1$.

Non-Archimedean Coamoebas

Theorem (Nisse-Sottile, 2009)

Let V be an algebraic variety over $\mathbb{K}$ with defining ideal $\mathcal{I}(V)$, and with non-Archimedean amoeba $\mathscr{A}_{\mathbb{K}}(V)$. Then, its non-Archimedean coamoeba is the union of the non-Archimedean coamoebas of the varieties with defining ideals $in_w(\mathcal{I}(V))$ for $w \in \mathrm{Vert}(\mathscr{A}_{\mathbb{K}}(V))$:

$$co\mathscr{A}_{\mathbb{K}}(V) = \cup_{w\in \mathrm{Vert}(\mathscr{A}_{\mathbb{K}}(V))} co\mathscr{A}_{\mathbb{K}}(V(in_w(\mathcal{I}(V)))).$$

Moreover, each $co\mathscr{A}_{\mathbb{K}}(V(in_w(\mathcal{I}(V))))$ is a complex coamoeba of varieties with maximally sparse defining polynomials, and such that the spine of their amoebas contains only one vertex.

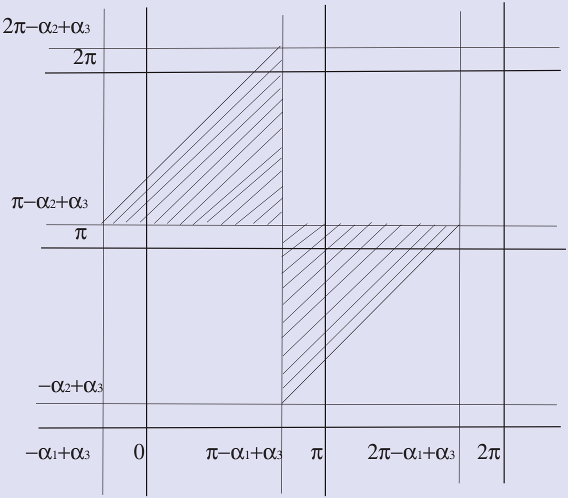

Figure: The coamoeba of the line in $(\mathbb{C}^*)^2$ defined by the polynomial $f(z,w) = r_1 e^{i\alpha_1} z + r_2 e^{i\alpha_2} w + r_3 e^{i\alpha_3}$ where r_i are real positive numbers and $\alpha_1 > \alpha_2 > \alpha_3 > 0$.

Coamoebas of some complex algebraic plane curves

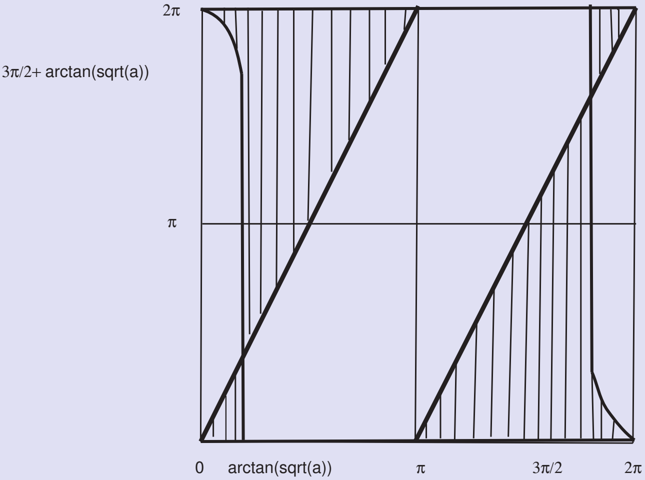

Figure: Coamoeba of parabola with solid amoeba (not Harnack).

Coamoebas of some complex algebraic plane curves

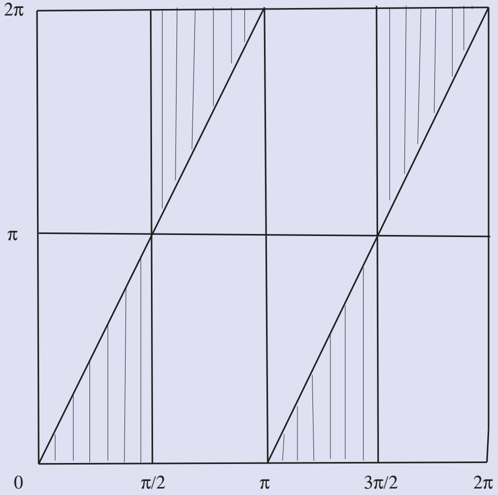

Figure: Coamoeba of a complex tropical parabola with solid amoeba

Coamoebas of some complex algebraic plane curves

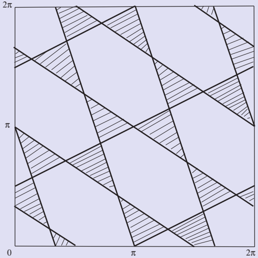

Figure: The coamoeba of the curve defined by the polynomial $f(z, w) = w^3z^2 + wz^3 + 1$

Coamoebas of some complex algebraic plane curves

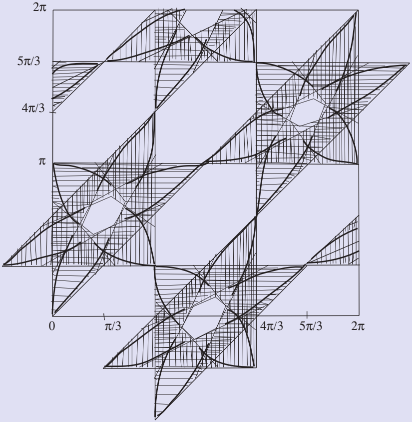

Figure: Coamoeba of a cubic with solid amoeba

Coamoebas of some complex algebraic plane curves

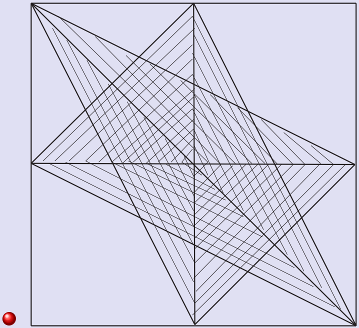

Figure: Coamoeba of a Harnack curve