Optimizing Over Hyperbolicity Cones By Using Their Derivative Relaxations

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- $p : \mathbb{R}^d \to \mathbb{R}$ homogeneous polynomial of degree n
- $p(e) > 0$

Defn: The polynomial p is

"hyperbolic in direction e"

if for all $x \in \mathbb{R}^d$, the univariate polynomial

$\lambda \mapsto p(\lambda e - x)$ has only real roots.

Roots: $\lambda_{1,e}(x) \leq \lambda_{2,e}(x) \leq \cdots \leq \lambda_{n,e}(x)$

"eigenvalues of x (in direction e)"

LP:

- $p(x) = x_1, \ldots, x_n$
- $e > 0$

$$\lambda \mapsto p(\lambda e - x) = (\lambda e_1 - x_1) \cdots (\lambda e_n - x_n)$$

Eigenvalues of x in direction e: $\quad \frac{x_1}{e_1}, \ldots, \frac{x_n}{e_n}$

SDP:

- $p(x) = \det(x)$
- $e \succ 0$

$$\lambda \mapsto \det(\lambda e - x) = \det(e)\,\det(\lambda I - e^{-1/2} x e^{-1/2})$$

Eigenvalues of x in direction e

= traditional eigenvalues of $e^{-1/2} x e^{-1/2}$

$$\lambda_{1,e}(x) \le \lambda_{2,e}(x) \le \cdots \le \lambda_{n,e}(x) \qquad \text{roots of } \lambda \mapsto p(x - \lambda e)$$

Hyperbolicity Cone:

$$\begin{aligned} \Lambda_{++} \quad &:= \{x : 0 < \lambda_{1,e}(x)\} \\ &= \text{connected component of} \\ &\qquad \{x : p(x) > 0\} \text{ containing } e \end{aligned}$$

Gårding (1959): p is hyperbolic in direction e for all $e \in \Lambda_{++}$

Corollary: Λ_{++} is a convex cone

Corollary: $x \mapsto \lambda_{n,e}(x)$ is a convex function

Bauschke, Güler, Lewis & Sendov:

If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex and permutation-invariant then $x \mapsto f(\vec{\lambda}_e(x))$ is convex

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Lax, Vinnikov and Helton Theorem:

Every 3-dimensional hyperbolicity cone is a slice of a PSD cone.

Cor: Faces of hyperbolicity cones are exposed.

Chua: Every homogeneous cone is a slice of a PSD cone.

ϕ a univariate polynomial

If ϕ has only real roots then:

- ϕ' has only real roots.
- Roots are interlaced: $\lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \cdots \leq \lambda'_{n-1} \leq \lambda_n$

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p a multivariate polynomial

$p'_e(x) := \langle \nabla p(x), e \rangle$ (directional derivative)

If p is hyperbolic in direction e then:

- p'_e is hyperbolic in direction e.
- $\Lambda_+ \subseteq \Lambda'_{e,+}$

Inductively:

$$p_e^{(i+1)}(x) = \langle \nabla p_e^{(i)}(x), e \rangle$$

$$\Lambda_+ = \Lambda_{e,+}^{(0)} \subseteq \Lambda_{e,+}^{(1)} \subseteq \cdots \subseteq \Lambda_{e,+}^{(n-1)} = \text{a halfspace}$$

$$p_e^{(i)}(x) = i!\, p(e)\, E_{n-i}(\vec{\lambda}_e(x))$$

where E_k = elementary symmetric polynomial of degree k

$$\Lambda_{e,+}^{(i)} = \{x : E_k(\vec{\lambda}_e(x)) \geq 0, \quad k = 1, \ldots, n-i\}$$

Hyperbolic Program (HP):

$$\begin{array}{rl} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}$$

Introduced by Güler (mid-90's) in context of ipm's:

"Central Path" $= \{x(\eta) : \eta > 0\}$

where $x(\eta)$ solves

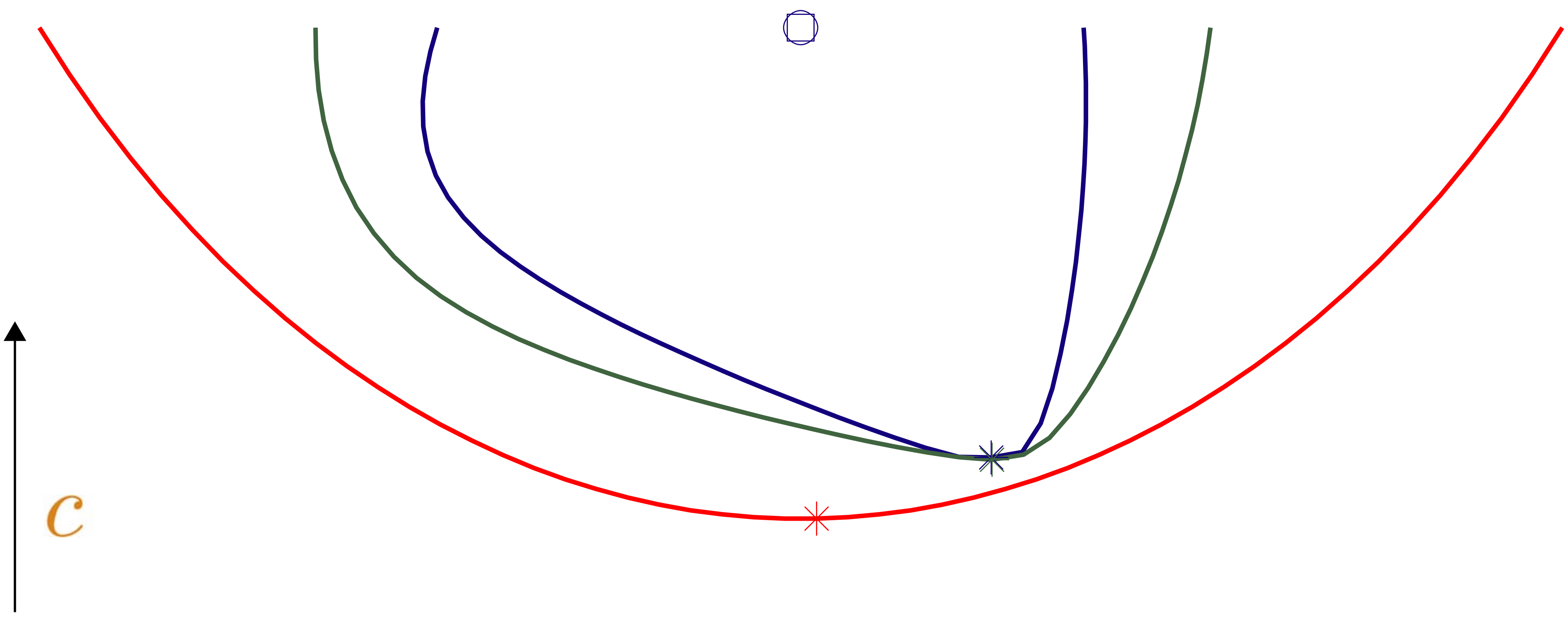
$$\begin{array}{rl} \min & \eta\,\langle c, x\rangle - \ln p(x) \\ \text{s.t.} & Ax = b \end{array}$$

$O(\sqrt{n})\log(1/\epsilon)$ iterations suffice

to reduce $\alpha := \langle c, x\rangle - \langle b, y\rangle$ to $\epsilon\,\alpha$

$$\begin{array}{ll} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_+ \end{array}$$

$$\begin{array}{ll} \min & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \in \Lambda_{+,e}^{(i)} \end{array}$$



Thm: Fix $\alpha, \beta > 0$.

If q_1, q_2 are hyperbolic in direction e

and $k < \deg(q_1) + \deg(q_2)$

then

$$\sum_{j=0}^{k} \binom{k}{j} \alpha^j \beta^{k-j} {q_1}^{(j)} {q_2}^{(k-j)}$$

is hyperbolic in direction e.

Pf:

- $Q(x,t) := q_1(x + t\alpha e) q_2(x + t\beta e)$
- Hyperbolic in direction $(0,1)$
- $(e,0)$ in hyperbolicity cone of Q, hence of $Q^{(k)}$
- Thus, $x \mapsto Q^{(k)}(x,0)$ is hyperbolic in direction e $\square$

Consequence: Can morph directly from $\Lambda_+^{(k)}$ to Λ_+

Downside: Don't gain facial structure along the way

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax = b \\ & x \in \Lambda_+ \end{aligned}$$

$z =$ optimal solution

If $z \notin \partial \Lambda'_{e,+}$ then z solves

$$\begin{aligned} \min_x \quad & -\ln\langle c, e - x \rangle - \frac{p(x)}{p'_e(x)} \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

How good is Newton's method at solving the latter problem?

Thm: If p is hyperbolic in direction e

then p/p'_e is a concave function on $\Lambda'_{e,++}$

Pf:

- $q(x,t) := tp(x)$ is hyperbolic in direction $(e,1)$
- Hence, $q'_{(e,1)}$ is hyperbolic in direction $(e,1)$
- Hyperbolicity cone of $q'_{(e,1)}$ is epigraph of $x \mapsto -p(x)/p'_e(x)$

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax = b \\ & x \in \Lambda_+ \end{aligned}$$

$$z = \text{optimal solution}$$

If $z \notin \partial \Lambda'_{e,+}$ then z solves

$$\begin{aligned} \min_x \quad & -\ln\langle c, e - x \rangle - \frac{p(x)}{p'_e(x)} \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

How good is Newton's method at solving the latter problem?

A general theorem on Newton's method (Smale, Guler, ...)

$$\begin{array}{rl} \min & f(x) \\ \text{s.t.} & Ax = b \end{array}$$

Let z denote optimal solution

For u satisfying $Au = 0$, let $\phi_u(t) := f(z + tu)$, and define

$$\gamma := \sup_{u,\, k>2} \left| \frac{\phi_u^{(k)}(0)}{(k-2)!\, \phi_u^{(2)}(0)^{\frac{k}{2}}} \right|^{\frac{1}{k-2}}$$

Thm: If x satisfies $Ax = b$ and

$$\langle x - z, \nabla^2 f(z)(x - z)\rangle < \frac{1}{36\,\gamma^2}$$

then Newton's method initiated at x converges quadratically.

For interior-point methods:

$$f(x) = \eta \langle c, x \rangle - \ln p(x)$$

$$\gamma \leq 1$$

So $\|x - x(\eta)\|_{\nabla^2 f(x(\eta))} < \frac{1}{6} \;\Rightarrow\;$ quadratic convergence

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For present context:

$$f(x) = -\ln\langle c, e - x\rangle - \frac{p(x)}{p'_e(x)}$$

γ can be arbitrarily large

("Inversely proportional to curvature of $\partial\Lambda_+$ at z")

$$f(x) = -\ln\langle c, e - x\rangle - \frac{p(x)}{p'_e(x)}$$

Nonetheless, something meaningful can be said ...

Thm:

$$\gamma \leq \frac{4}{\min\{\|x - z\|_{\nabla^2 f(z)} : Ax = b \text{ and } x \in \partial\Lambda'_{e,+}\}}$$

In other words, quadratic convergence occurs on nearly the largest "ball" within reason.

Limitation of theorem:

$\| \ \|_{\nabla^2 f(z)}$ reflects curvature of $\partial\Lambda_+$ at z,

not shape of $\Lambda'_{e,+}$ around z

That shape is reflected by Hessian of $h(x) := -\ln p'_e(x)$

If $\| \ \|_{\nabla^2 f(z)}$ is (nearly) a scalar multiple of $\| \ \|_{\nabla^2 h(z)}$

then Newton's domain of convergence

is *truly* the largest within reason