Randomization, Relaxation, and Complexity

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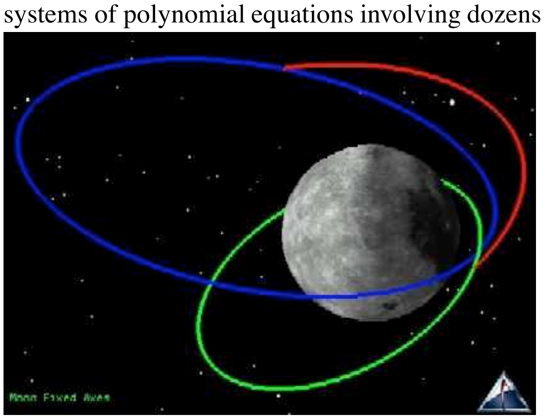
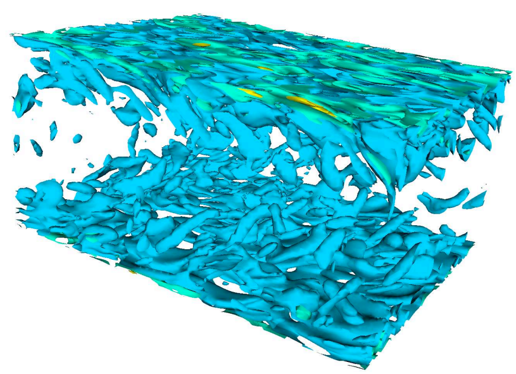
1 Overview of Polynomial System Solving

Systems of polynomial equations arise naturally in applications ranging from the study of chemical reactions to coding theory to geometry and number theory. Furthermore, the complexity of the equations we wish to solve continues to rise: while engineers in ancient Egypt needed to solve quadratic equations in one variable, today we have applications in satellite orbit design and combustive fluid flow hinging on the solution of systems of polynomial equations involving dozens or even thousands of variables.





For example, the left-hand illustration above shows an instance of the orbit transfer problem, while the right-hand illustration above shows a level set for a reactive fluid flow. More precisely, in the first problem, one wants to use N blasts of a rocket to transfer a satellite from an initial orbit to a desired final orbit, using as little fuel as possible. The optimal rocket timings and directions can then be reformulated as the real solutions of a system of $45N$ sparse polynomial equations in $45N$ variables, thanks to recent work of Avendano and Mortari [1]. For reactive fluid flow, a standard technique is to decimate the space into small cubes and obtain an approximation to some parameter function (such as vorticity or temperature) via an expansion into polynomial basis functions. Asking for regions where a certain parameter lies in a certain interval then reduces to solving **millions** of polynomial systems — the precise number depending on the region and size of the cubes.

Far from laying the subject to rest, modern hardware and software has led us to even deeper unsolved problems concerning the hardness of solving. These questions traverse not only algebraic geometry but also number theory, algorithmic complexity, numerical analysis, and probability theory. The need to look beyond computational algebra for new algorithms is thus one of the main motivations behind this workshop.

1.1 Historical Highlights

In the 1980s and 1990s, work in computational algebra culminated in singly exponential complexity bounds for many fundamental problems involving polynomial equations. Highlights include: reducing arbitrary systems to an encoding involving a polynomial in a single variable (a.k.a. rational univariate reduction [31]), bounding Betti numbers of semi-algebraic sets [3], and computing geometric decompositions for complex algebraic sets [17]. 19th century techniques (such as resultants) and more recent techniques (such as Gröbner bases) began to receive increasingly reliable and efficient software implementations, and the limits of computational algebra began to emerge: all of the aforementioned problems, in their decision form, are $\mathbf{NP}$-hard. Furthermore, it also emerged that the classical techniques of computational algebra largely ignore the special structure of **real** solutions. So any new speed-ups must come from new mathematical ideas and/or relaxing the statement of the problem. We now review some more recent advances, in 3 settings: detecting, counting, and approximating solutions.

Detecting Solutions. Thanks to work of Koiran in the 1990s [23], it is now known that the truth of the Generalized Riemann Hypothesis (GRH) implies that deciding the existence of solutions over the complex numbers is doable in polynomial time if and only if $\mathbf{P} = \mathbf{NP}$. This intersection of algebraic complexity with two of the biggest unsolved problems in mathematics attests to the depth of polynomial system solving. One can expand this study of complexity by looking for more special kinds of solutions. For example, deciding the existence of **integral** solutions leads us to even higher complexity classes: The famous negative solution to Hilbert's Tenth Problem in 1970 [28] is a proof of the algorithmic impossibility of deciding the existence of integer solutions to (completely general) systems of polynomial equations.

Caught between $\mathbf{NP}$-hardness and complete intractability, one then clearly hopes that detecting real solutions lies closer to $\mathbf{NP}$, particularly since most applications require just the real solutions of systems of polynomial equations. That detecting real solutions is at least theoretically tractable was proved in the early twentieth century by Tarski [39]. More recently, various results have hinted at the possibility of polynomial-time algorithms in special settings, e.g., real feasibility for quadratic polynomials [2] and certain sparse polynomials [6]. These new algorithms take us farther and farther away from traditional commutative algebra.

Counting Solutions. Work of Bernstein, Khovanski, and Kushnirenko in the 1970s [5, 33] showed that counting the number of complex solutions of a system of sparse polynomials is (with high probability) the same as computing a mixed volume of polytopes. Later, in the 1990s, Dyer and other authors determined the algorithmic complexity of computing volumes and mixed volumes of polytopes [14, 15]. One thus began to see signs that counting complex solutions is close to being a $\#\mathbf{P}$-complete problem. Gurvits then made major advances by finding efficient approximation algorithms for mixed volumes, also unifying earlier quantitative results in convexity via the framework of hyperbolic polynomials [19, 18].

Once viewed from the point of view of toric geometry, the connections between convex geometry and complex algebraic geometry are more natural than surprising. In a more topological vein, there has been much recent progress on understanding the complexity of counting connected (and even irreducible) components of algebraic sets over the complex numbers [10]: one sees new complexity classes, including some from the more recent BSS model of computation [8].

Similar progress was made over the real numbers (see, e.g., [9]), but precise complexity bounds remain more elusive over the real numbers than over the complex numbers. In particular, it was discovered in the 1980s that the number of real solutions for systems of **sparse** polynomials could be dramatically smaller than the number of complex solutions [22]. Taking full advantage of sparsity (or other types of structure) when counting real roots remains a challenging problem in algorithmic complexity.

In a different direction, using toric geometric methods, Huber and Sturmfels presented an algorithm for computing mixed volume, thus counting exactly the number of complex solutions for certain sparse polynomial systems [20]. Even better, their methods also yielded a new numerical method for approximating complex solutions: polyhedral homotopy.

Approximating Solutions. The complexity of numerical solving presents new difficulties not present in the more discrete problems of detecting and counting solutions. In this setting, ideas from numerical linear algebra have entered algebraic geometry via the notion of the **condition number**.

The condition number is an invariant one can now associate to families of semi-algebraic sets [12] to extract important information about the complexity of numerical optimization questions, just as Betti numbers extract important topological information. And while condition numbers are about as difficult to compute as numerical solutions themselves, they admit useful expectation bounds when considered as random variables attached to families of random algebraic sets [37, 27, 11]. This has led to average case complexity bounds for polynomial system solving. Recasting traditional algebraic complexity results to incorporate the condition number is now a lively subarea of algorithmic algebraic geometry. So far, only classical homotopy algorithms have fully benefited from this point of view, so condition number analysis is still an open problem for many other algorithms. For instance, even polyhedral homotopy still lacks rigourous complexity bounds.

Nevertheless, some very recent algorithms show great performance in practice. For instance, Parrilo's seminal work [30] blends 19th century ideas (sums of squares and Hilbert's 17th Problem) with 20th century optimization (semidefinite programming, a.k.a. **SDP**) to yield an efficient algorithm for solving certain relaxations of polynomial systems. Much recent effort in the optimization community has then focussed on quantifying how close these relaxations are to the original systems of equations (see, e.g., [24]).

Extending the idea of numerical conditioning, one can ask what is the most theoretically sound method to solve a numerically ill-posed problem. This leads one to the study of the **nearest** ill-posed problem, and major advances by Zeng and others [21, 40] have already yielded numerically reliable algorithms for problems that would have been impossible to solve with earlier software.

One can also study the geometry of zero sets of random polynomials, independent of numerical conditioning. This has lead to deep connections with several complex variables and mathematical physics [13, 36]. The behavior of real roots of random systems, particularly with respect to sparsity, has proven even more challenging [16, 32, 35].

Goals of the Workshop. The study of systems of polynomial equations has thus led us to a greater understanding of the complexity of detecting, counting, and numerically approximating solutions. However, for many structured systems of equations (e.g., those with few real solutions and many complex solutions), polynomial-time algorithms remain only a tantalizing possibility. Also, on a more fundamental level, many of the advances in polynomial system solving involve so many different techniques that refining them to specially structured systems is daunting. This workshop thus focusses on emerging methods to attain such speed-ups, and the resulting interactions between optimization, theoretical computer science, and algebraic geometry.

2 Emerging Directions

Much how probabilistic methods are just beginning to enter algebraic geometry [35, 29], randomized complexity bounds for polynomial system solving (and precise general estimates on numerical stability) were virtually unknown until recently. In particular, Smale's 17th Problem [38] beautifully captured what was sorely missing from computational algebra:

> "Can a solution of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?"

Smale's statement elegantly highlights 3 issues in polynomial system solving: (1) average case complexity, (2) the notion of approximation for solutions, and (3) the possibility of a polynomial-time solution for a numerical problem known to be $\mathbf{NP}$-hard in its decision form. Indeed, observe how Smale's 17th Problem asks for just **one** complex root, since the number of complex solutions is exponential in the input size (here measured to be the number of monomial terms of the input polynomial system). Note also that Smale's introduction of randomization and approximation (to enable a polynomial-time solution) is very much in parallel to the idea of relaxation in optimization: simplify a seemingly intractable problem by softening the notion of a solution.

While the role of real solutions does not enter in Smale's statement, advances in the study of sparse systems of polynomial equations (a.k.a. **fewnomial systems**) over the real numbers also blossomed in the early 2000s: Li, Rojas, and Wang proved dramatically improved bounds (independent of the degree of the underlying polynomials) for the number of real roots of certain sparse polynomial systems [26]. This was the first significant evidence that the famous earlier bounds of Khovanski [22] could be significantly improved. Furthermore, completely general and explicit bounds over the p-adic rational numbers were initiated in 2004 by Rojas [34], following Lenstra's seminal results in one variable [25].

Smale's 17th Problem was, from a practical point of view, settled positively by Beltran and Pardo in 2008 [4].[1] Based on this advance, and progress in algorithmic fewnomial theory, Rojas began to form new conjectures on the complexity of solving real polynomial systems. (See Section 4 below.)

Other sources for new speed-ups have emerged recently: Pablo Parrilo discovered in his Ph.D. thesis that Semi-definite Programming (SDP) can sometimes be used to maximize multivariate polynomial much faster than the classical methods of computational algebra [30]. Also, perhaps one of the earliest 20th century signs that real solving could go faster than complex solving comes from work of Barvinok: he showed that detecting real roots for homogeneous multivariate quadratic polynomials could be done in polynomial time, contrary to known methods for computational algebra at the time [2].

3 Presentation Highlights

A central activity in our workshop was 22 talks delivered by our diverse group of researchers. Full information (including abstracts, slides for almost all talks, and video for 2 talks) is available from the BIRS website. So we outline the talks below, from the point of view of their major themes. Afterward, we include some information **not** listed at the BIRS website: Details from the talks of Greg Blekherman, Mihai Putinar, Leonid Gurvits, and Victor Vinnikov. (These 4 talks were done on the blackboard without slides.) We then conclude with a condensed list of the talks.

3.1 Algebra of Polynomial System Solving

The talk of **Bernard Mourrain** focussed on moment matrices and border bases as a means of finding a canonical form (for more efficient solving) for certain polynomial systems. **Laura Matusevich** then described deep connections between monomial ideals (which are an important ingredient in Gröbner basis algorithms) and hypergeometric functions. On a related note, **Sue Margulies** spoke on the connection between algorithms for polynomial ideals and the resolution of certain conjectures in graph theory.

Closer to our next theme, **Martin Avendaño** presented an elegant new approach to Descartes' Rule of Signs that connects to an extension of a famous result of Polya: the number of real roots of a univariate polynomial f is **exactly** the number of sign alternations in the ordered coefficient sequence of $(1+x)^N f(x)$ for N sufficiently large.

3.2 Sums of Squares and Real Solving

Chris Hillar spoke on rational solutions to sums of squares certificates of positivity, raising many intriguing open problems. For instance, let $A_0, \ldots, A_n$ be rational $m \times m$ symmetric matrices and define a (rational) spectrahedron to be any set of the form $\{(x_1, ..., x_n) \in \mathbb{R}^n \mid A_0 + x_1 A_1 + \cdots + x_n A_n \geq 0\}$, where the inequality indicates positive semidefiniteness. Determine those real algebraic numbers that can be obtained as the coordinates of a finite spectrahedron. This question is open already for $n=1$!

Martin Harrison, a graduate student at UCSB, spoke on expressing certain non-commutative polynomials as a sum of a minimal number of squares. **Korben Rusek**, a graduate student at Texas A&M, presented a new class of fewnomial bounds which give dramatically sharper upper bounds on the number of real solutions of certain specially structured sparse polynomial systems.

Continuing the topic of fewnomials, **Dan Bates** spoke about a new homotopy algorithm that follows a remarkably small number of solution paths and finds all real solutions of any nondegenerate polynomial system. The number of paths followed is between a certain fewnomial bound recently derived by Bates and Sottile and the true number of real solutions. **Rojas** spoke on an alternative homotopy algorithm, based on a simple modification of polyhedral homotopy, that follows a number of paths that is **exactly** the number of real roots. Rojas' method works for any polynomial system lying outside of a particular discriminant amoeba, thus leading to interesting questions on real random polynomial systems.

[1] Strictly speaking, the problem is still open because Smale asked for a deterministic algorithm, and the solution from [4] is a randomized algorithm with a small, but controllable, failure probability.

3.3 Numerical Methods

Tien-Yien Li spoke about his most recent algorithm for computing the mixed volume of polytopes, and how it leads to one of the fastest current implementations of homotopy solving. **Andrew Sommese** spoke about his methods for homotopy solving. Thanks to his extensive use of parallelization, Sommese's implementation is currently the only software that can beat Li's implementation for certain massive polynomial systems.

Turning to more theoretical issues, **Zhonggang Zeng** lectured on how to reliably solve univariate polynomials that are known to be degenerate, and even how to find the degeneracy structure. **Anton Leykin** then spoke on how homotopy solving can be made completely rigourous (via exact arithmetic and the use of recent quantitative bounds of Shub and Smale), even demonstrating a preliminary implementation in `Macaulay2`.

3.4 Geometry and Complexity

Mounir Nisse, a graduate student from Paris 6, gave us highlights of the connections between complex amoebae and amoebae over non-Archimedean fields. (Amoebae are the images of algebraic sets under a valuation map: over the complex numbers, the valuation is the log-absolute value map.) In particular, Nisse presented very recent work on characterizing **co**-amoebae. Co-amoebae are the image of algebraic sets under the phase map, and are a vital ingredient to a deeper understanding of the geometry of complex algebraic sets. The impact of co-amoebae for algorithmic algebraic geometry will be at least as great as that of amoeba theory.

Pascal Koiran gave an enlightening talk on Valiant's version of the $\mathbf{P}$ versus $\mathbf{NP}$ problem and the derandomization of polynomial identity testing. It turns out that circuit complexity provides a useful link between both problems, and a deeper study leads to the study of shallow circuits with high powered inputs. In particular, one is led to study the number of real roots of polynomials that are sums of products of sparse polynomials. Such polynomials are just beyond the current reach of fewnomial theory, and thus yield fascinating new directions in fewnomial theory.

3.5 Optimization and Beyond

Levant Tuncel gave a timely survey on the state of the art of interior point methods in conic programming. His talk helped clarify some misconceptions behind the complexity of semidefinite programming, and focussed on barrier functions and locally quadratic convergence. **Brendan Ames**, a graduate student at the University of Waterloo, spoke on SDP relaxations for compressive sensing and maximum clique problems via the nuclear norm (sum of singular values) of a matrix.

Jim Renegar gave a stimulating evening talk on the frontiers of optimization. In particular, he spoke about optimizing over hyperbolic cones (a problem which includes SDP as a very special case) and how variations of Smale's α-Theory allow new convergence bounds.

3.6 4 Talks Without Slides

Greg Blekherman: Blekherman considered criteria for determining when a real n-variate homogeneous polynomial of degree $2d$ is convex. (For homogeneous polynomials, convexity clearly implies nonnegativity, and thus convexity is a stronger restriction.) He showed how recent advances on quantifying how often nonnegative forms are sums of squares have analogues in the setting of convexity. In particular, Blekherman proved that there are convex forms that are **not** sums of squares. However, unlike the classical examples of Motzkin and others, not a single convex form is known that is **not** a sum of squares! Blekherman went on to give an elegant sufficient condition for convexity in terms of tight clustering of the values of a form, and then developed some of the quantitative bounds necessary for his existence proof.

Mihai Putinar: Putinar developed a beautiful analytic framework starting from the following basic problem: How does one determine if n given disks are non-overlapping? Putinar related this problem to positive semi-definite matrices and then proceeded to explore connections with orthogonal polynomials and tomography. Via some delicate estimates, he proceeded to prove new growth estimates of complex orthogonal polynomials with respect to certain area measures.

Leonid Gurvits: Gurvits' evening talk was an entertaining tour through hyperbolicity, convex geometry, and physics. First, Gurvits showed how the volume of a scaled Minkowski sum of convex bodies is a hyperbolic polynomial. He then proceeded to an elegant proof of the $\#\mathbf{P}$-hardness of computing the mixed volume of parllelograms. Gurvits then continued by giving a deterministic polynomial-time algorithm for $(1+\sqrt{2})^n$-factor approximation of the mixed volume of any n convex bodies, given access to a weak membership oracle. He then concluded with a fascinating account of the connections between quantum linear optics and the permanents of unitary matrices.

Victor Vinnikov: Vinnikov gave a fascinating talk on constructing determinantal representations of polynomials via noncommutative algebra. These results give deep insights into representations of convex sets as the feasible sets for linear matrix inequalities, i.e., spectrahedra. Such representations have deep implications for optimization as they are behind the question of how much more general hyperbolic programming is than SDP.

3.7 A Condensed List of the Talks (in order of presentation)

March 1, 2010 (monday)
J. Maurice Rojas (Texas A&M): Simple Homotopies for Just Real Roots
Tien-Yien Li (Michigan State): The mixed volume computation: MixedVol-2.0 vs. DEMiCs
Zhonggang Zeng (U Illinois, Carbondale): Solving Ill-posed Algebraic Problems: A Geometric Perspective
Pascal Koiran (ENS Lyons): Shallow circuits with high-powered inputs
Mounir Nisse (Institut de Mathématiques de Jussieu): Complex and Non-Archimedean (Co)Amoebas, and Phase Limit Sets

March 2, 2010 (tuesday)
Chris Hillar (UC Berkeley): Do rational certificates always exists for sum of squares problems?
Greg Blekherman (VBI): Volume of the Cone of Convex Forms and new Faces of the Cone of Sums of Squares
Levant Tuncel (Waterloo): Local Quadratic Convergence of Polynomial-Time Interior-Point Methods for Nonlinear Convex Optimization Problems
Mihai Putinar (UCSB): Discretization of Shapes via Orthogonal Polynomials
Martin Harrison (UCSB): Minimal Sums of Squares in a Free-* Algebra
Susan Margulies (Rice): Vizing's Conjecture and Techniques from Computer Algebra
Brendan Ames (Waterloo): Convex relaxation for the clique, biclique and clustering problems
Leonard Gurvits (Los Alamos National Labs): Mixed Volumes of Parallelograms and Other Cool Things

March 3, 2010 (wednesday)
Bernard Mourrain (INRIA Sophia-Antipolis): Moment matrices and border basis
Laura Matusevich (Texas A&M): Monomial ideals and hypergeometric equations
Jim Renegar (Cornell): Optimizing Over Hyperbolicity Cones By Using Their Derivative Relaxations

March 4, 2010 (thursday)
Dan Bates (Colorado State): Khovanskii-Rolle continuation for finding real solutions of polynomial systems
Andrew Sommese (Notre Dame): Recent work in Numerical Algebraic Geometry
Anton Leykin (Georgia Tech): Certified numerical homotopy continuation
Software Demos (by Dan Bates and Anton Leykin)
Martin Avendaño (Texas A&M): Descartes' Rule of Signs is exact!
Korben Rusek (Texas A&M): On Certain Structured Fewnomials
Victor Vinnikov (Ben-Gurion): Constructing determinantal representations via noncommutative techniques

March 5, 2010 (friday)
Impromptu Problem Session (featuring Leonid Gurvits, Pascal Koiran, and J. Maurice Rojas)

4 Scientific Progress Made

The best part of our workshop was the opportunity for experts who rarely see each other to speak freely about their work in a comfortable environment. An important aspect of these discussions was an impromptu open problem session.

At our problem session, Leonid Gurvits raised intriguing open questions on the approximability of mixed volume: should the best current factor for polynomial-time approximability really be so large? Gurvits also pointed out unusual parallels between polyhedral lifting and recent approaches to Boolean satisfiability.

The questions Pascal Koiran raised revealed that certain advances in fewnomial bounds over the real numbers would enable an attack on a constant-free version of Valiant's Problem, i.e., a variant of the $\mathbf{VP} \stackrel{?}{=} \mathbf{VNP}$ problem. Koiran also pointed out a fascinating recent paper of Aaronson showing that if quantum linear optics is efficiently simulable, then the polynomial hierarchy collapses.

Finally, Rojas pointed out some unusual parallels between real algorithmic fewnomial theory and p-adic algorithmic fewnomial theory. In particular, at a coarse level, the complexity of detecting roots for sparse polynomials has similar complexity in both settings. However, sporadic differences occur already for univariate trinomials: detecting real roots is doable in polynomial-time but detecting p-adic rational roots is only known to be in $\mathbf{NP}$.

To obtain some additional perspective on the advances made during our workshop, it will be useful to return to Smale's 17th Problem (as described in Section 2) and see how the ideas arising from our workshop helped extend this question in a new direction.

DEFINITION 1 *We call an* $f \in \mathbb{R}[x_1, \ldots, x_n]$ *(with* $f(x) = \sum_{i=1}^{n+k} c_i x^{a_i}$, $c_i \neq 0$ *and* $x^{a_i} = x_1^{a_{1,i}} \cdots x_n^{a_{n,i}}$ *for all* i*, and the* a_i *distinct) an* $\boldsymbol{n}$**-variate** $\boldsymbol{(n+k)}$**-nomial**. *We also define* $\mathrm{supp}(f) := \{a_1, \ldots, a_{n+k}\}$ *to be the* **support** *of* f. *The collection of* n*-variate* $(n+k)$*-nomials in* $\mathbb{R}[x_1, \ldots, x_n]$ *is denoted* $\mathcal{F}_{n,n+k}$. *Also, if* $F := (f_1, \ldots, f_n)$ *with* $f_i \in \mathcal{F}_{n,n+k}$ *and* $\mathrm{supp}(f_i) = \{a_1, \ldots, a_{n+k}\}$ *for all* i *then we call* F *an* $\boldsymbol{(n+k)}$**-sparse** $\boldsymbol{n \times n}$ **polynomial system (over** $\mathbb{R}$**)**. $\diamond$

DEFINITION 2 *Let* $\Omega(n,k)$ *denote the maximal number of non-degenerate roots, with all coordinates positive, of any* $(n+k)$*-sparse* $n \times n$ *polynomial system over* $\mathbb{R}$. $\diamond$

CONJECTURE 1. (OPTIMAL REAL FEWNOMIAL BOUNDS) *There are absolute constants* $C_2 \geq C_1 > 0$ *such that, for all* $n, k \geq 2$*, we have* $(n+k)^{C_1 \min\{k-1,n\}} \leq \Omega(n,k) \leq (n+k)^{C_2 \min\{k-1,n\}}$.

CONJECTURE 2. (SPARSE REAL ANALOGUE OF SMALE'S 17TH PROBLEM) *Suppose we fix either* n *or* k*, and we consider random systems* $(n+k)$*-sparse* $n \times n$ *systems* F *over* $\mathbb{R}$*. Then there are uniform algorithms that:*

A: compute a positive integer in polynomial-time that, with high probability, is exactly the number of roots of F *in the positive orthant.*

B: approximate a single solution of F *in* $\mathbb{R}^n$*, on the average, in polynomial time.*

The intuition that the complexity of finding just the real roots of polynomial systems depends only weakly on the number complex roots, for systems of equations with few real roots and many complex roots, is captured in a rigourous way by these last 2 conjectures. Note also how we progress from bounding the number of positive roots, to computing the exact number of positive roots with high probability, to approximating a single positive root efficiently.

Progress toward these conjectures has been made from different points of view. For instance, Rojas' bound over the p-adic numbers, and a more recent bound over the real numbers of Bihan and Sottile [7], provided evidence toward Conjecture 1. Conjecture 2 is heavily based on [6] and recent Chamber Cone methods, the latter covered in the first talk at this workshop.

5 Final Notes

Rojas proposed an AMS Contemporary Mathematics proceedings volume for this workshop which has now been provisionally approved. The editors will be Philippe Pébay, J. Maurice Rojas, and David C. Thompson. As of this writing, we have submissions from the following sets of authors:

Dan Bates & Andrew Sommese
Carlos Beltran & Luis-Miguel Pardo
Anton Leykin
Tien-Yien Li
Zhonggang Zeng

We also have commitments for papers from:

Martin Avedano & Ashraf Ibrahim
Saugata Basu
O. Bastani, C. Hillar, D. Popov, & J. M. Rojas
Bernard Shiffman & Steve Zelditch

All editors and authors are either attendees of our workshop or invitees who were unable to attend.

In closing, we would like to extend our humble thanks for the wonderful facilities and magnificent setting. BIRS is truly a treasure, and it was a privilege to hold our workshop here.

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