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Atmospheric Dynamics: Equations and Wave Phenomena

John Methven,

Department of Meteorology,

University of Reading

Thanks for slides from Hylke De Vries & Sarah Dance

Lecture Outline

1. Fundamental properties of the atmosphere
2. Dynamical similarity and scaling
3. Considerations for rotating planet
4. Scale analysis and approximate balances
5. Wave phenomena
   - Gravity waves
   - Rossby waves
   - Equatorial waves
6. Large-amplitude waves
7. Shear instability
8. Active research areas

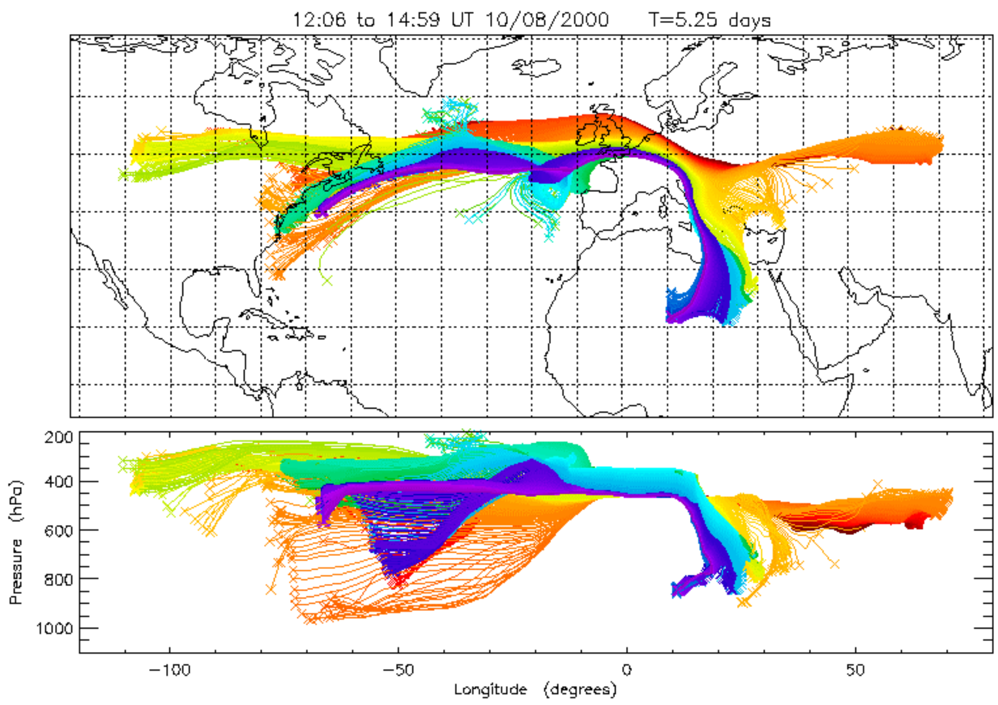
Challenges of Atmospheric Dynamics

Governing equations known for more than a century, but…

Too complex to be solved analytically

Support an amazing variety of phenomena: **waves**, **turbulence**…

Atmosphere *chaotic* ⇒ trajectories from two similar initial conditions separate at exponential rate on average. *(Edward Ott, Tuesday)*



Trajectory can refer to path of an air-mass (as shown here) or sequence of states evolving from particular initial condition. If *x* is state vector and *A* is dynamical operator:

$$\frac{D\underline{x}}{Dt} = A(\underline{x}, t)$$

Cyclone tracks in ensemble forecasts

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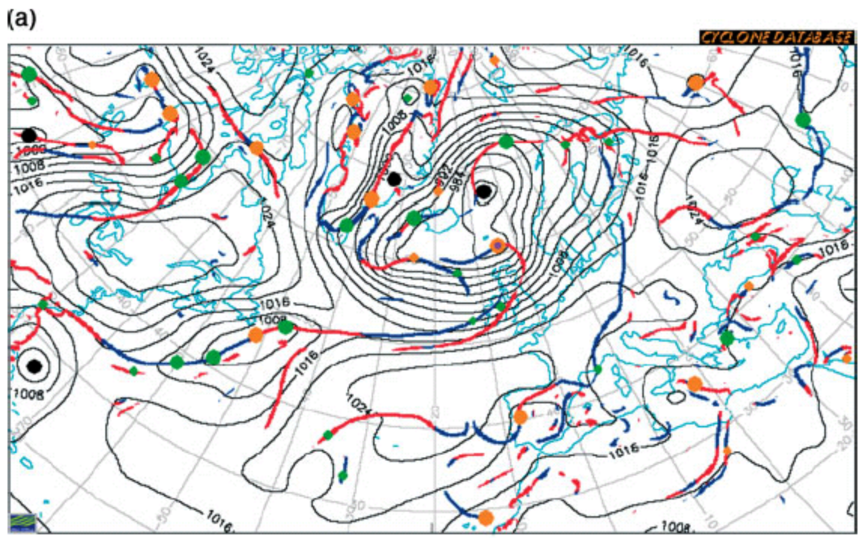
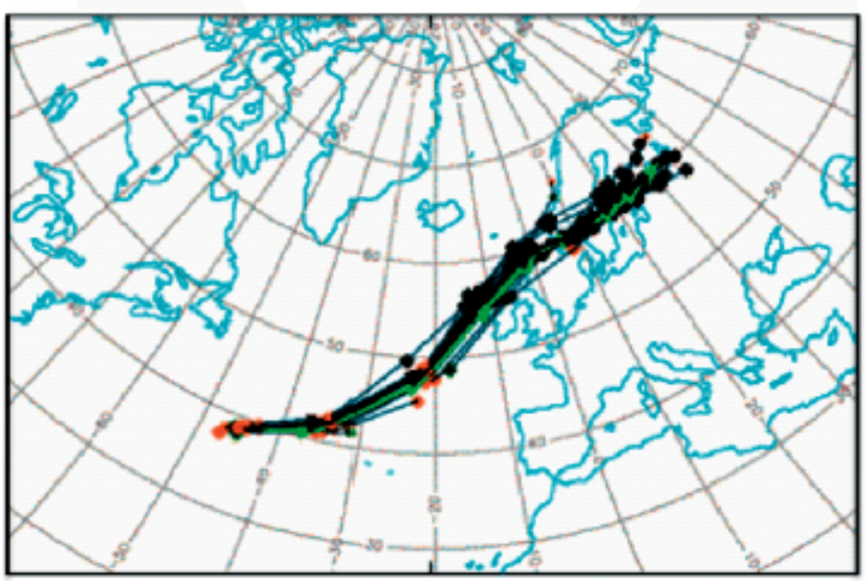


Cyclones at various stages of development and their associated fronts identified from model

Hewson and Titley, *Met Apps,* 2010



Tracks of one cyclonic centre in the MOGREPS ensemble

It is not only the airmass trajectories that are chaotic

Cyclone tracks in ensemble forecasts

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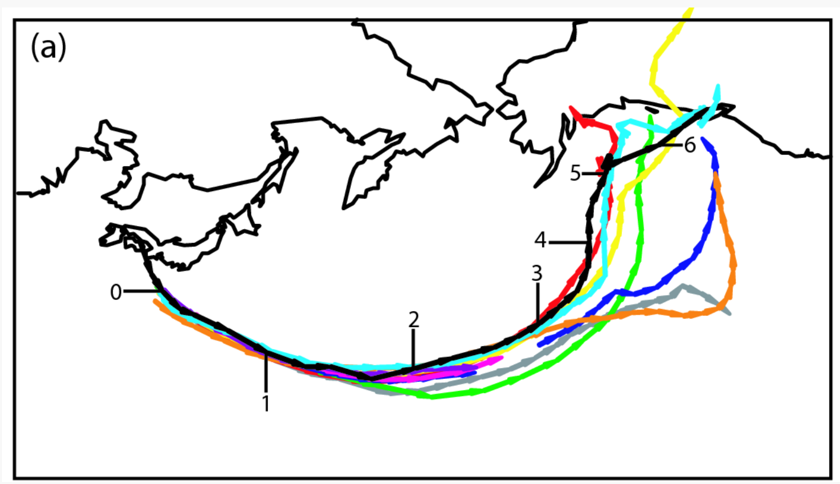
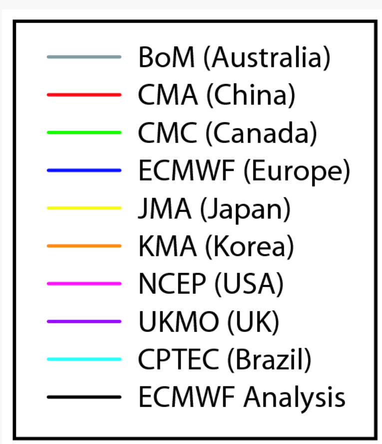
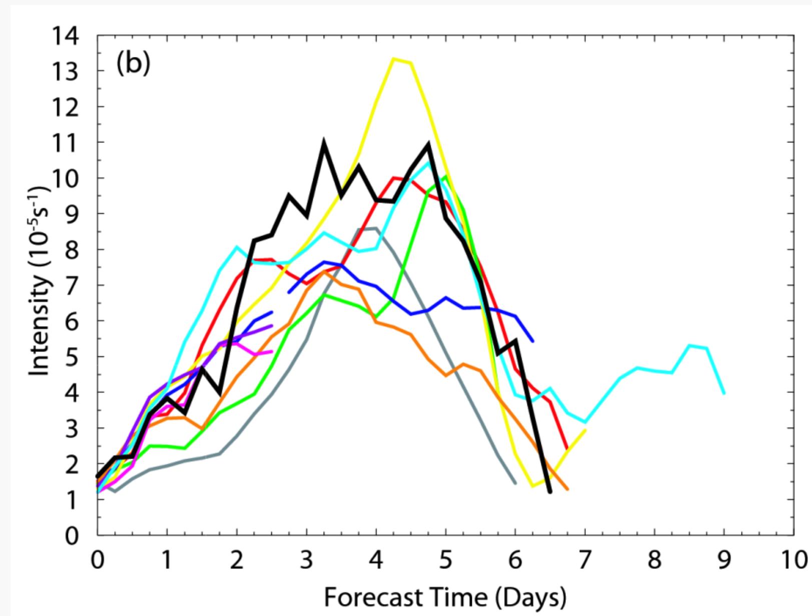
Lizzie Froude, ESSC, *Weather and Forecasting,* 2010







Using Kevin Hodges' cyclone tracking algorithm on ensemble forecasts from different centres archived on TIGGE database @ ECMWF.

Example of the control forecasts for one cyclone collated from 9 operational centres

…atmosphere is *chaotic*, but that is not all!

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Non-local nature of the atmosphere

1. **Material conservation** of properties

⇒ long-range transport of “air masses”

2. **Balance** between variables, mediated by fast wave propagation

⇒ action-at-a-distance.

3. Eddy-mean flow **interaction**

Depends on form of average used to define background.

“Eddy” could mean wave or coherent structure such as a vortex.

1. Macoscropic properties of fluids

- A fluid does not have a preferred shape.
- When a fluid is subject to a stress, it deforms continuously.
- For many, so-called Newtonian fluids, the rate of change of strain is proportional to the stress (Newton's Law of Viscosity).
- A lump of fluid is irreversibly deformed when it is subject to a stress, and never recovers its original shape.

Underlying molecular reality

- Experience suggests that, at a macroscopic (laboratory) scale, liquids and gases look like continua with smoothly varying fields of density, velocity and pressure.
- This is an approximate description, since fluids are made of discrete molecules.

Consider scales L >> mean free path between collisions of molecules

$\Rightarrow$ Continuum hypothesis

1. General conservation laws

Describe the transport of property, *q*, by fluxes, *F*, with sources, *S*:

$$\frac{\partial(\rho q)}{\partial t} + \nabla.\underline{F} = \rho S$$

where ρ= air density. Integrating over volume, *V*, gives:

$$\int_V \frac{\partial(\rho q)}{\partial t} dV + \int_{\partial V} \underline{F}.\underline{n} dS = \int_V \rho S dV$$

If there are no sources and boundary of *V* does not move:

$$\frac{dM}{dt} + \int_{\partial V} \underline{F}.\underline{n} dS = 0$$

If there is no net flux across the boundary, the total amount of *q* is conserved (*M*=constant).

⇒ *global conservation*

Conservation of momentum

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Rate of change of momentum in V:

$$\frac{d}{dt}\int_V \rho\mathbf{u}dV$$

Flux of momentum across the surface A (per unit time):

$$\int_A \rho\mathbf{u}\left(\mathbf{u}\cdot\mathbf{n}\right)dS$$

By Newton's second law:

$$\frac{d}{dt}\int_V \rho\mathbf{u}dV+\oint_A \rho\mathbf{u}\left(\mathbf{u}\cdot\mathbf{n}\right)dS = \underbrace{\int_V \mathbf{F}dV}_{\text{body forces}} + \underbrace{\oint_S \tau dS}_{\text{surface stresses}}$$

Molecular stresses

It can be shown (see Batchelor 1967, ch 1) that

$$\tau_i = \sigma_{ij} n_j$$

where

- σ is a second rank tensor (matrix) known as the Cauchy stress tensor
- **n** is the unit normal to the surface

Thus, we can use Gauss' divergence theorem to give

$$\oint_S \tau_i dS = \oint_S \sigma_{ij} n_j dS = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV.$$

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Normal stress - pressure

In a fluid at rest, no tangential stresses act and the normal stress is isotropic, (i.e., acts equally in all directions) so we define

$$\sigma_{ij} = -p\delta_{ij} + \sigma_{ij}^{(d)}$$

where $\sigma_{ij}^{(d)}$ is the *deviatoric* stress and is equal to zero when the fluid is a rest.

p = pressure (force per unit area from molecular motion and collision)

Action at a distance is mediated through pressure

Tangential stress

For Newtonian fluids, tangential stresses, due to friction between two layers of fluid as they slide across one another, are proportional to the gradient of velocity (Newton's law of viscosity):

$$\sigma_{ij}^{(d)} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where μ is called the dynamic viscosity (or just viscosity).

Since conservation law holds for arbitrary fixed volumes, it must also hold for the integrand:

$$\rho \frac{Du_i}{Dt} = F_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Navier-Stokes equation

Equations of fluid dynamics

- Only assuming the *continuum hypothesis*

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0$$

Conservation of mass

$$\frac{\partial (\rho \underline{u})}{\partial t} + \nabla.(\rho \underline{u}\,\underline{u}) = -\rho \nabla \Phi - \nabla p + \rho \underline{B}$$

Conservation of momentum

ρ=density

$\underline{u}$=velocity

Φ=geopotential

p=pressure

$\underline{B}$=friction + body forces

Thermodynamics 1

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Need another equation relating pressure to fluid state:

e.g., Atmosphere treated as an ideal gas (*R*=gas constant for air)

$$p = \rho R T$$

But, system not closed without a prediction for temperature.

Bring on thermodynamics!

Definition (The first law of thermodynamics)

Heat (δQ) entering a fluid element is used either to change the internal energy (δU, *temperature*) of the parcel or to do work (δW) against the surrounding fluid, i.e.,

$$\delta Q = \delta U + \delta W.$$

Thermodynamics 2

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Use second law of dynamics to define entropy, *s,* for a reversible process:

$$\delta Q = Tds$$

with first law $\Rightarrow$ $Tds = du + pd\left(\frac{1}{\rho}\right)$

Specific heat capacity at fixed volume

u is a functions of state (p, T) $\Rightarrow$

Heat input at fixed volume increases internal energy & T $\delta Q_V = du = c_V dT$

$$\Rightarrow \quad ds = c_V \frac{dT}{T} + R\frac{dT}{T} - R\frac{dp}{p}$$

adiabatic process: *ds=0*

Defines potential temperature, $\theta = T\left(\frac{p}{p_o}\right)^{-\frac{R}{c_p}}$ $\Rightarrow s = c_p \ln\left(\frac{\theta}{\theta_o}\right)$

*Specific heat capacity at constant pressure is defined by $\delta Q_p = c_p dT = (c_V + R)dT$

Equations of fluid dynamics

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- Only assuming the *continuum hypothesis*

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0$$ Conservation of mass

$$\frac{\partial (\rho \underline{u})}{\partial t} + \nabla.(\rho \underline{u}\,\underline{u}) = -\rho\nabla\Phi - \nabla p + \rho \underline{B}$$ Conservation of momentum

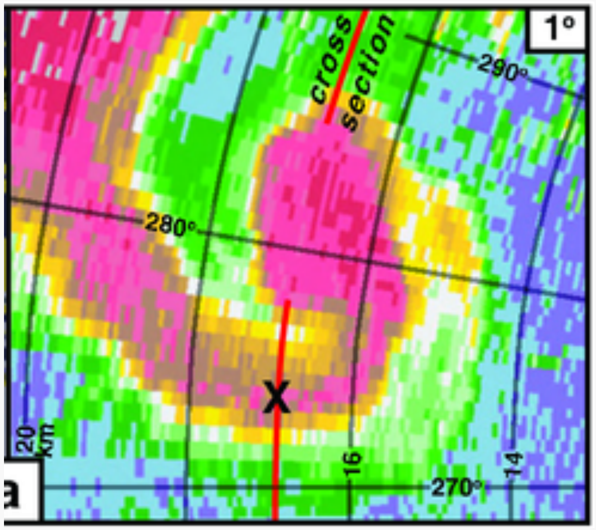
$$\frac{\partial (\rho s)}{\partial t} + \nabla.(\rho s \underline{u}) = \rho Q$$ Conservation of entropy

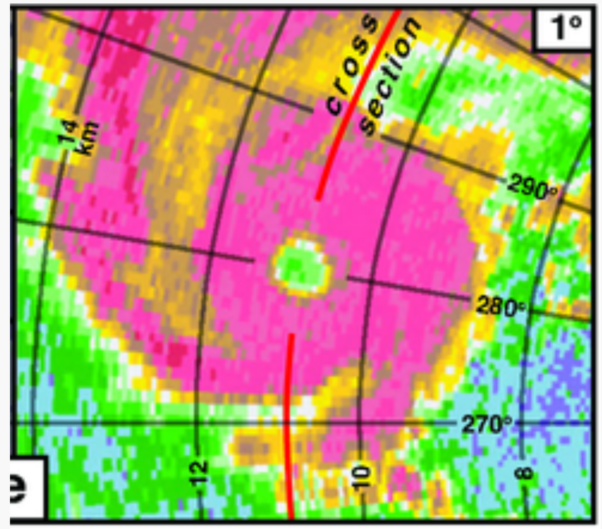
ρ=density $\underline{u}$=velocity

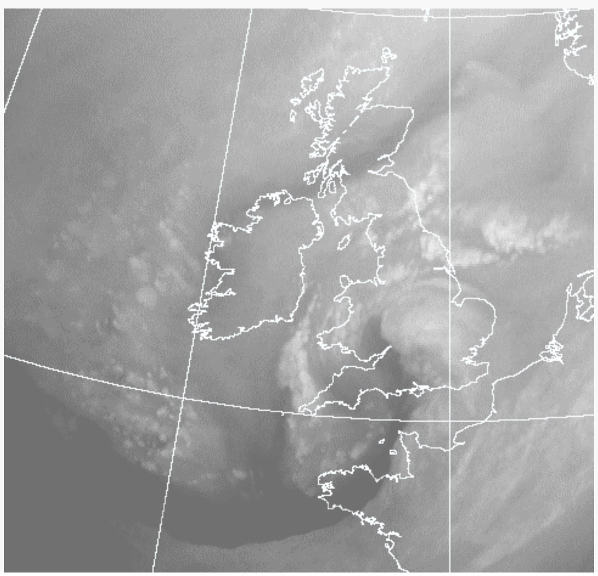
Φ=geopotential p=pressure $\underline{B}$=friction + body forces

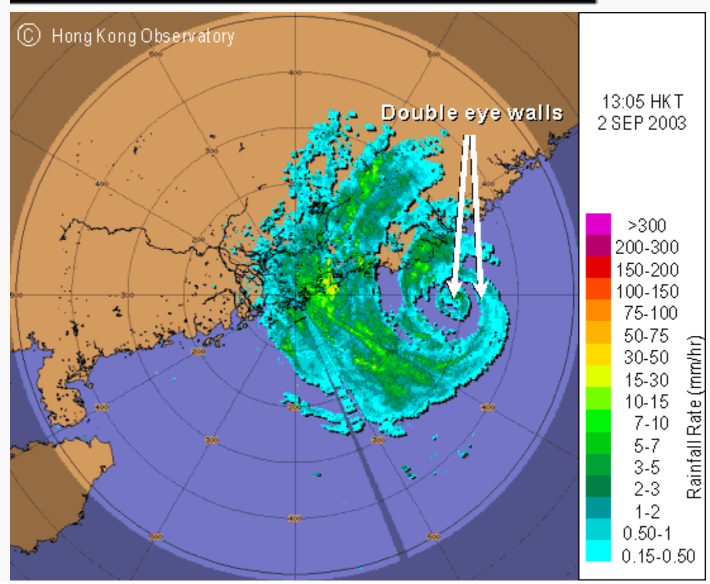
$s(p.\rho)$=specific entropy Q=entropy source (from diabatic processes)

2. Dynamical similarity
What phenomenon is this?

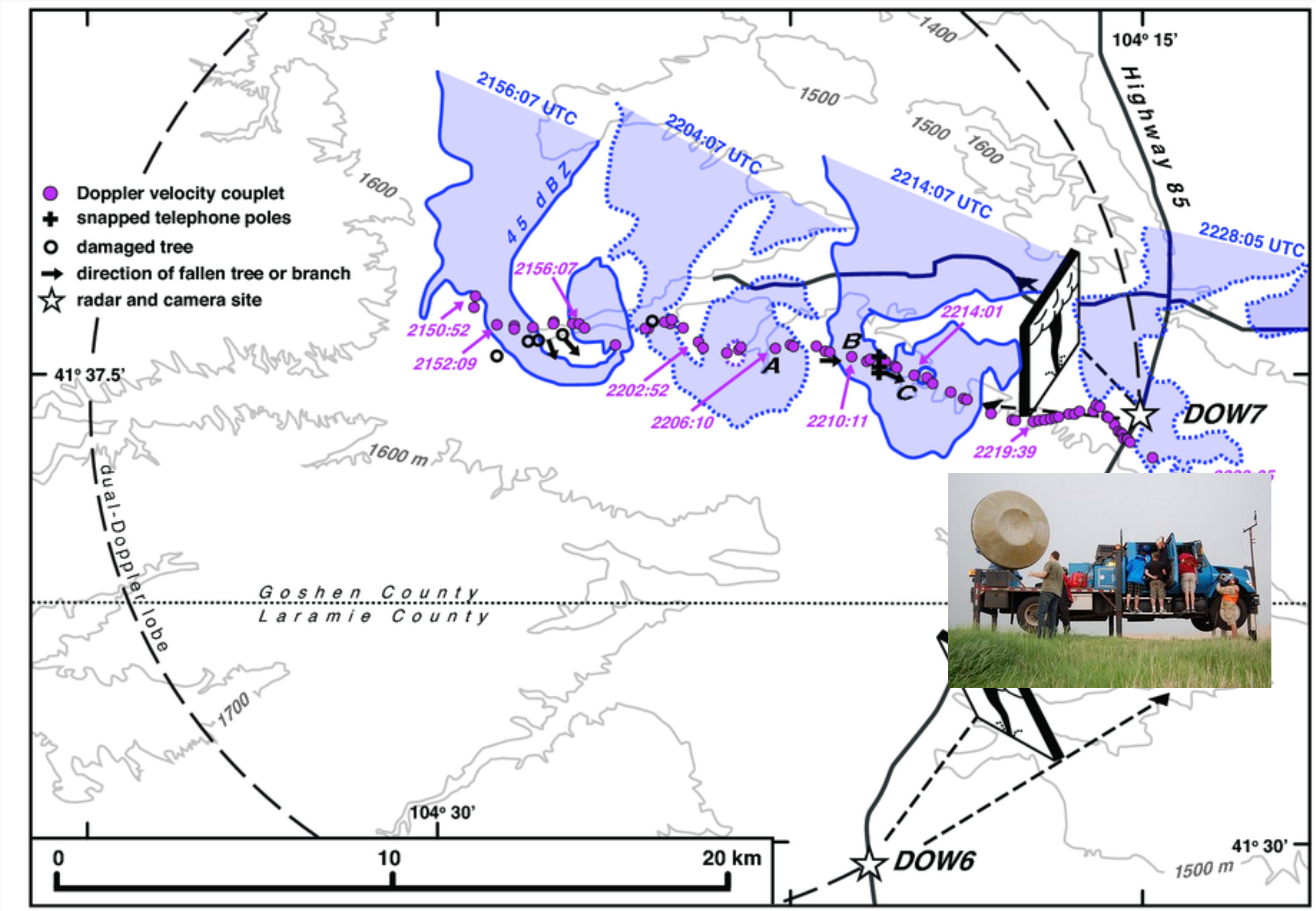








Hook echo with embedded tornado

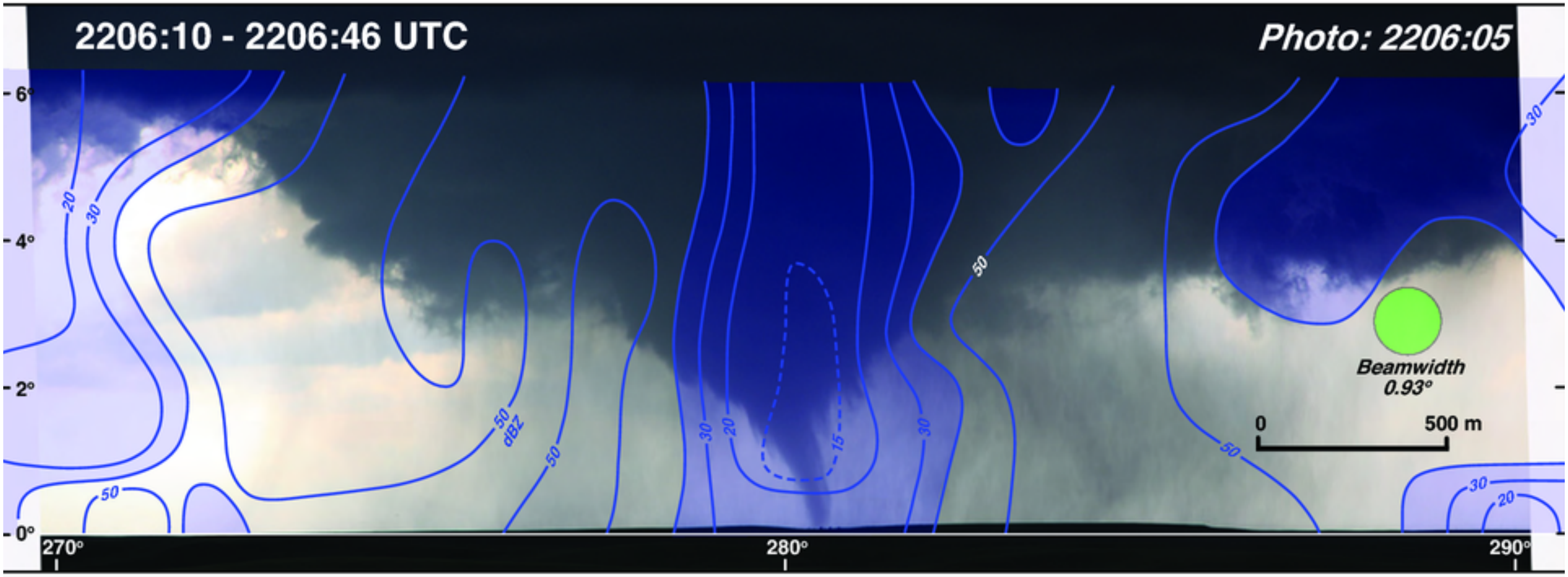


Wakimoto *et al, MWR,* 2011

Tornado during VORTEX2 expt

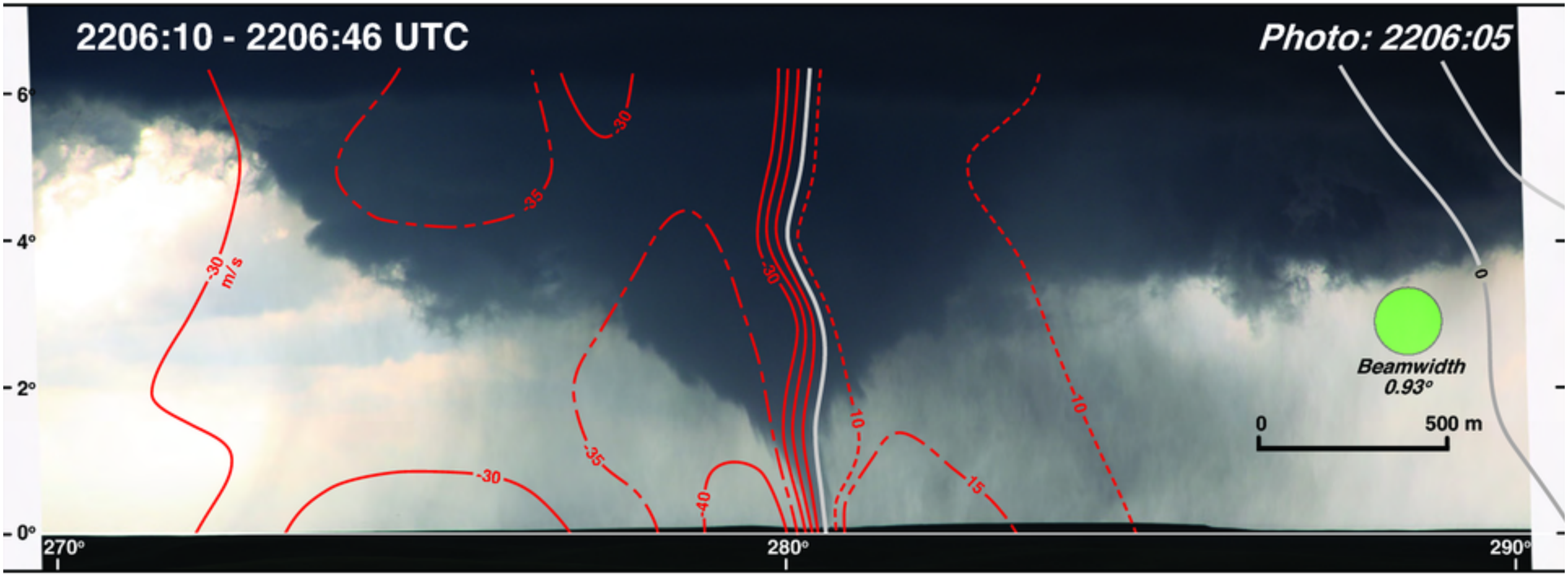
Wakimoto *et al, MWR,* 2011

Tornado photo + radar reflectivity



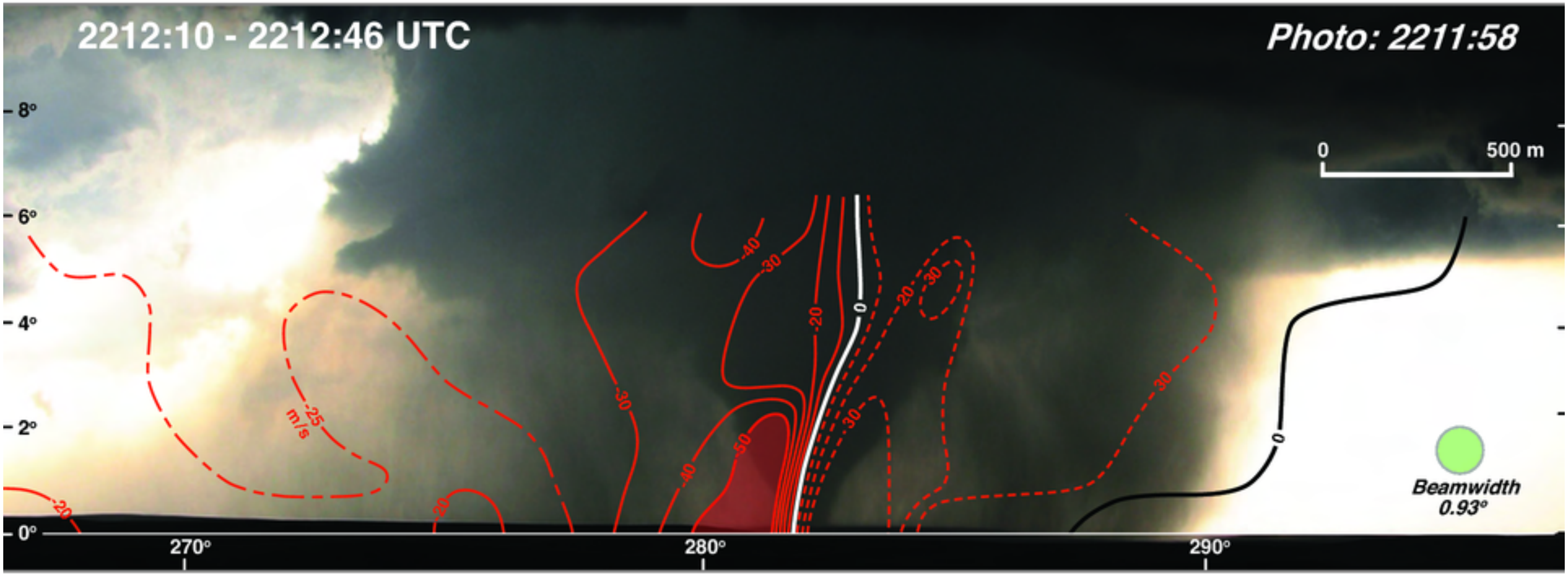
Wakimoto *et al*, *MWR*, 2011

Tornado photo + Doppler radar winds



Wakimoto *et al*, *MWR*, 2011

Tornado photo + Doppler radar winds



Wakimoto *et al*, *MWR*, 2011

2. Dynamical similarity

- Similar structure and evolution even though scales are vastly different - why?
- Take much simpler case:
  - incompressible flow (ρ = constant)
  - Non-rotating frame

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \underline{u}$$

- Non-dimensionalise variables by characteristic values *U and P*

$$\underline{u} = U\hat{\underline{u}} \qquad p = P\hat{p}$$

Non-dimensional equation

$$\left(\frac{L}{UT}\right)\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}}.\hat{\nabla}\hat{\underline{u}} = -\frac{PL}{\rho U^2}\hat{\nabla}\hat{p} + \left(\frac{1}{\mathrm{Re}}\right)\hat{\nabla}^2\hat{\underline{u}}$$

where *T and L are time and space scales associated with flow*

$\mathrm{Re} = \frac{UL\rho}{\mu} = \frac{UL}{\nu}$ is the Reynolds number

ρ, ν known for fluid - how are *T, L, U, P* related?

If *Re >> 1, fluid experiences almost no viscous drag*

Fluid parcels only accelerate where there is a pressure gradient

$P \sim \frac{\rho U^2}{L}$ $T \sim \frac{L}{U}$

Re is the only free parameter

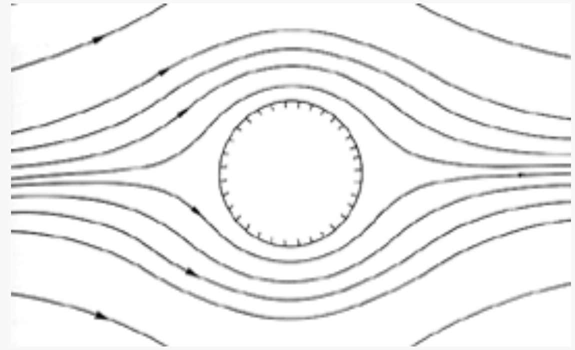
$\Rightarrow$ solution depends only on *Re,* ICs and BCs.

Flow dependence on *Re*

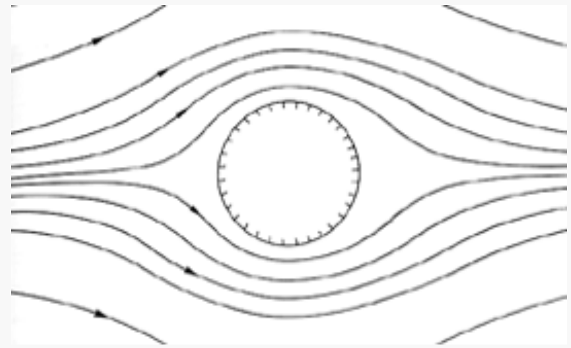
In the lab, nature of barotropic flow around cylinder (uniform inflow) depends only on Reynolds number

ν = molecular viscosity

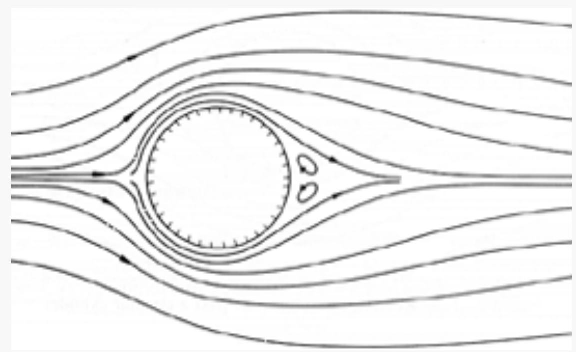
$$\mathrm{Re} = \frac{UL}{\upsilon}$$



Re«1



Re~4

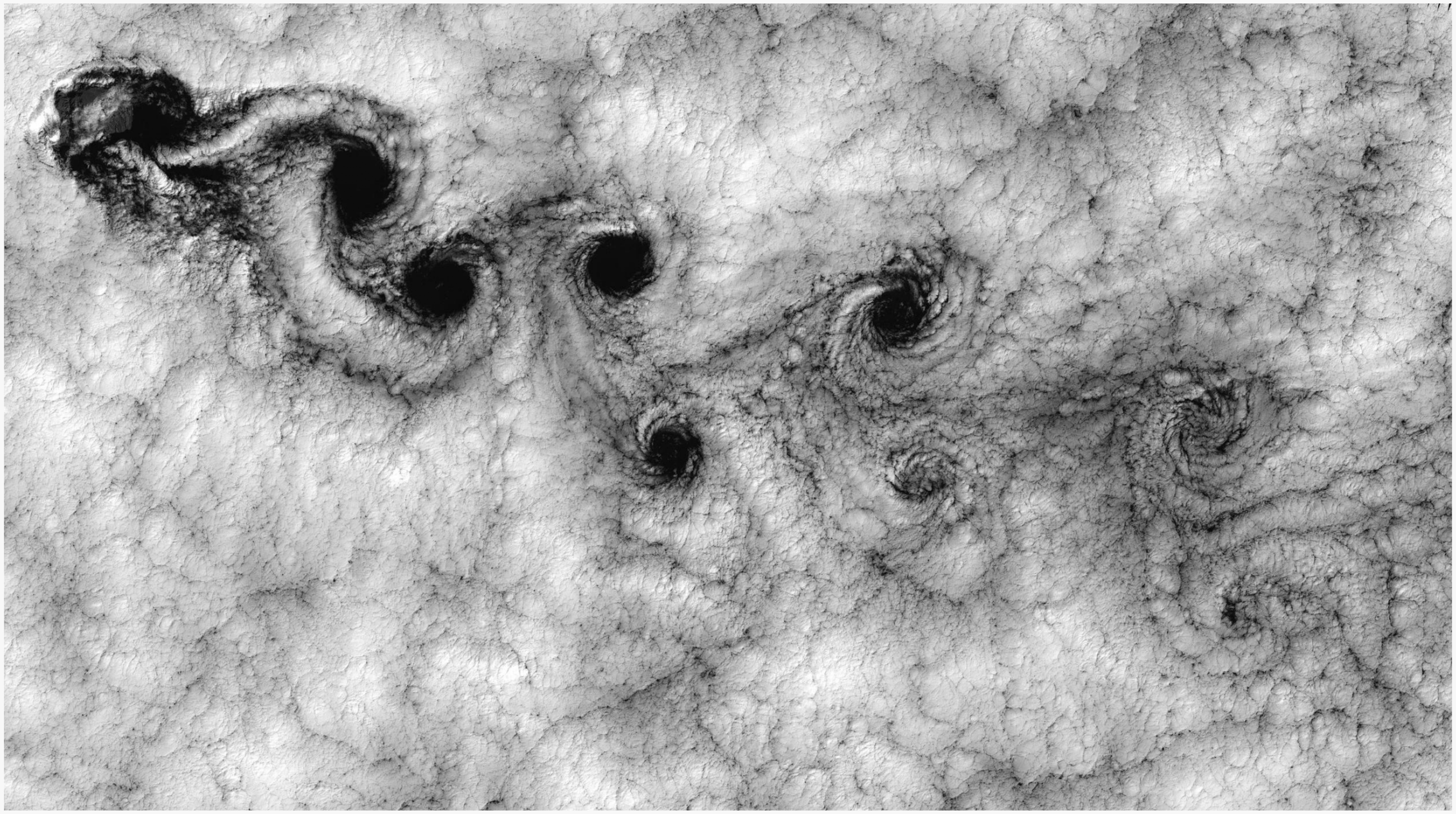


Re~40

Von Karman vortex street

40<Re<200

Von Karman vortex streets seen in atmosphere



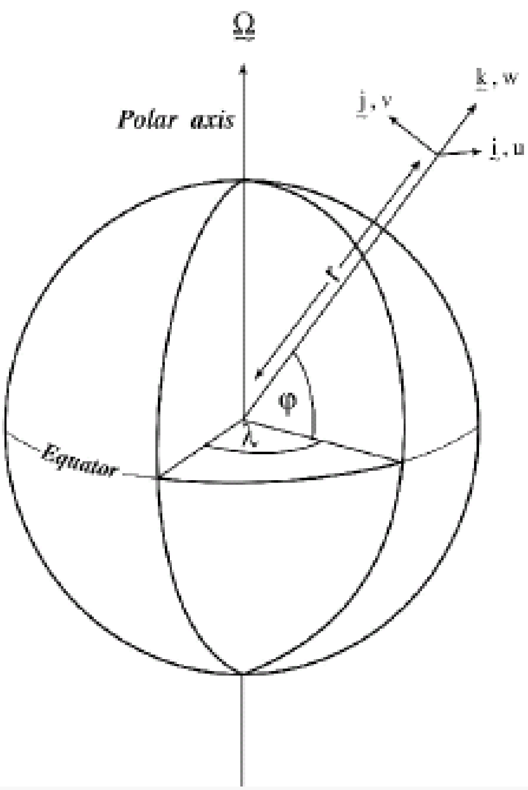
$Re_{eff}=UL/\kappa \approx 100$

K = eddy diffusivity associated with turbulence in boundary layer

Re based on molecular viscosity ~ 10^7

3. Effects of Planetary Rotation

- The Earth is (nearly) a sphere
- Spherical polar coordinates are the natural choice of coordinate system



radius r

longitude λ

latitude ϕ

Orthogonal unit vectors **i**, **j**, **k** parallel to each axis.

Ω = rotation rate of Earth

$= 7.292 \times 10^{-5}\ s^{-1}$

Rotating vectors (and observers)

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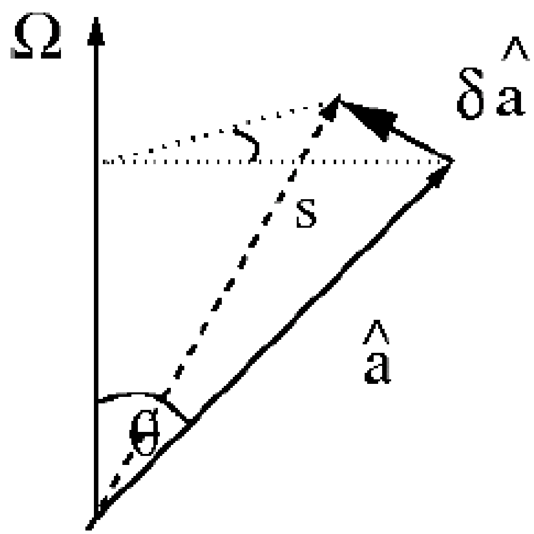
Now suppose that $\hat{\mathbf{a}}$ is a unit vector fixed in the rotating frame of reference.

In an inertial frame, $\hat{\mathbf{a}}$ will remain at a constant magnitude but will change its direction.



$$|\delta\hat{\mathbf{a}}| = s\Omega\delta t$$

$$s = \sin\theta = \frac{|\underline{\Omega} \wedge \hat{\mathbf{a}}|}{\Omega},$$

$$|\delta\hat{\mathbf{a}}| = |\underline{\Omega} \wedge \hat{\mathbf{a}}|\delta t.$$

$$\frac{d_A\hat{\mathbf{a}}}{dt} = \underline{\Omega} \wedge \hat{\mathbf{a}}$$

We conclude that for any vector field V,

$$\frac{d_A\mathbf{V}}{dt} = \frac{d_R\mathbf{V}}{dt} + \underline{\Omega} \wedge \mathbf{V}.$$

Acceleration in a rotating frame

Velocity is related to position vector **r** by

$$\mathbf{u} = \frac{D\mathbf{r}}{Dt}$$

So comparing an inertial and a rotating frame of reference we have

$$\mathbf{u}_A = \frac{D_A\mathbf{r}}{Dt} = \frac{D_R\mathbf{r}}{Dt} + \underline{\Omega} \wedge \mathbf{r},$$

Velocity in a rotating frame

$$\mathbf{u}_A = \mathbf{u}_R + \underline{\Omega} \wedge \mathbf{r}.$$

$$\frac{D_A\mathbf{u}_A}{Dt} = \frac{D_A}{Dt}\left(\mathbf{u}_R + \underline{\Omega} \wedge \mathbf{r}\right)$$

Acceleration in a rotating frame

$$\frac{D_A\mathbf{u}_A}{Dt} = \frac{D_R\mathbf{u}_R}{Dt} + 2\underline{\Omega} \wedge \mathbf{u}_R + \underline{\Omega} \wedge (\underline{\Omega} \wedge \mathbf{r}).$$

Momentum eqn in rotating frame

$$\frac{D_A u_A}{Dt} = g - \frac{1}{\rho}\nabla p$$

We can now write this in a rotating frame of reference

$$\frac{D_R \mathbf{u}_R}{Dt} = -2\underline{\Omega} \wedge \mathbf{u}_R - \underline{\Omega} \wedge (\underline{\Omega} \wedge \mathbf{r}) + g - \frac{1}{\rho}\nabla p.$$

"Coriolis force"

(-Coriolis acceleration)

Centrifugal force

(-centripetal acceleration)

"Apparent forces" to observer since non-inertial

i.e., do no work – just part of acceleration in absolute frame

Spherical geopotential approximation

Point on Earth's surface viewed from rotating frame experiences an apparent "centrifugal force":

$$\mathbf{F}_{CF} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \Omega^2 s\, \mathbf{l} = \nabla \left(\frac{\Omega^2 s^2}{2} \right) \tag{6}$$

Earth bulges such that its surface is almost a *geoid*: a surface of constant *geopotential*, Φ. Viewed from the rotating frame, masses experience *apparent gravity*:

$$\mathbf{g} = -\nabla\Phi = -\nabla \left(\Phi^* - \frac{\Omega^2 s^2}{2} \right) \tag{7}$$

where $-\nabla\Phi^*$ is the Newtonian gravitational acceleration. However, the equatorial bulge is small ($a_{eq} = 6378\text{km}$, $a_{pole} = 6357\text{km}$) so that:

$$\frac{a_{eq} - a_{pole}}{a} \approx \frac{1}{300} \quad \textit{and} \quad \frac{\Omega^2 a}{g} \approx \frac{1}{290}$$

Spherical geopotential approximation

- use orthogonal spherical coordinates
- use constant $g \approx 9.81\,\text{m}\,\text{s}^{-2}$ directed to Earth's centre

3-D Euler equations on a sphere

$$\frac{Du}{Dt} = \frac{uv}{r}\tan\phi - \frac{uw}{r} + 2\Omega\sin\phi v - 2\Omega\cos\phi w - \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda},$$

$$\frac{Dv}{Dt} = -\frac{u^2}{r}\tan\phi - \frac{vw}{r} - 2\Omega\sin\phi u - \frac{1}{\rho r}\frac{\partial p}{\partial\phi}$$

$$\frac{Dw}{Dt} = \frac{(u^2+v^2)}{r} + 2\Omega\cos\phi u - g_e - \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

where the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r}.$$

Euler eqns =
Navier Stokes
without viscosity

Usually simplified by making *traditional (shallow atmosphere) approximation*

z/a << 1 where r = a+z and a = Earth's radius

Primitive equations

In fact, it is necessary to neglect certain small terms in order to ensure the approximated equations conserve angular momentum and kinetic energy:

$$\begin{aligned}
\frac{Du}{Dt} &= \frac{uv}{a}\tan\phi + 2\Omega\sin\phi v - \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} \\
\frac{Dv}{Dt} &= -\frac{u^2}{a}\tan\phi - 2\Omega\sin\phi u - \frac{1}{\rho a}\frac{\partial p}{\partial\phi} \\
\frac{Dw}{Dt} &= g_e - \frac{1}{\rho}\frac{\partial p}{\partial z}.
\end{aligned}$$

The resulting *primitive equations* (PEs) are the basis of almost all atmosphere and ocean *general circulation models* (GCMs) except the Met Office Unified Model which does not make shallow atmosphere approximation.

- Biggest errors are in Tropics associated with neglect of horizontal component of Coriolis acceleration.[1]
- Effects of spherical geopotential approximation are unknown.[2]

[1] White, Hoskins, Roulstone and Staniforth (2005) Consistent approximate models of the global atmosphere: shallow, deep, hydrostatic, quasi-hydrostatic and non-hydrostatic. *Quart. J. Roy. Met. Soc.*, **131**, 2081-2107.

[2] White, Staniforth and Wood (2008) Spheroidal coordinate systems for modelling global atmospheres. *Quart. J. Roy. Met. Soc.*, **134**, 261-270.

4. Scale analysis of equations

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Further approximations are usually based on *scaling* where the typical spatial and length scales associated with motions of interest are assumed. For example:

Planar approximation ($L/a \ll 1$)
Can use local Cartesian coordinates where $(dx, dy) = (a \cos \phi_0 \, d\lambda, a \, d\phi)$.
Can drop spherical metric terms from PEs.

Anelastic approximation ($\Delta\rho/\rho_0 \ll 1$)
Mass conservation equation becomes $\nabla.(\rho_r \mathbf{u}) = 0$ where $\rho_r(z)$ is a reference density.

Incompressible ($\Delta\rho/\rho_0 \ll 1$ and $H \ll H_\rho$)
Huge density height scale, H_ρ, implies $\rho_r \approx$ constant and therefore $\nabla.\mathbf{u} = 0$.

Hydrostatic balance ($H/L \ll 1$)
Vertical momentum equation reduces to:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \qquad (8)$$

Balance in extratropical cyclones

"Large-scale" weather associated with cyclones and therefore rotational flow.

Consider terms that dominate momentum and thermodynamic equations using *scaling analysis* as rough guide to approximation.

$L=1/k$= horizontal length scale of motions of interest

H= vertical length scale

V= horizontal velocity scale

Geostrophic balance

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Horizontal components of momentum equation

(use planar approx. Valid if L/a«1 where a=Earth's radius)

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Rate of change of y-velocity component

D/Dt refers to **material derivative** (change following trajectories)

Coriolis acceleration due to rotation of coord frame

$$f = 2\Omega \sin\phi$$

Pressure-gradient force.

If small Rossby number, Ro=V/(fL), last two terms dominate

⇒ *Geostrophic flow* $u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$ *e.g., zonally symmetric flow*

Geostrophic flow

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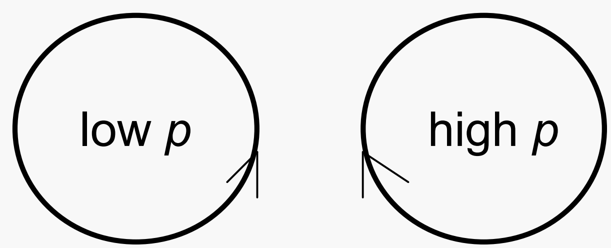
Define *geostrophic streamfunction* $$\psi_g = \frac{p'}{f_0 \rho_r(z)}$$

Geostrophic flow components are then

$$u_g = -\frac{\partial \psi_g}{\partial y} \qquad v_g = \frac{\partial \psi_g}{\partial x}$$

Geostrophic relative vorticity:

$$\xi_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \nabla^2 \psi_g$$



For a pressure wave:

$$p' = p_0 \sin kx \qquad \xi_g = -k^2 \psi_g = -k^2 \frac{p'}{f_0 \rho_r}$$

$p' < 0 \Rightarrow f\xi > 0$, definition of *cyclone*

Anticlockwise in NH where *$f>0$.*

Hydrostatic balance revisited

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Consider vertical momentum equation:

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Define static reference state:

$$\rho = \rho_r(z) + \rho'(x, y, z, t)$$

Anelastic approximation. $\rho'/\rho \ll 1$ *and* $H_\theta >> H_\rho$

$$\frac{Dw}{Dt} \approx \frac{g\theta'}{\theta_0} - \frac{\partial}{\partial z}\left(\frac{p'}{\rho_r}\right)$$

θ = potential temperature

Hydrostatic approximation. $H/L \ll 1$

Buoyancy

$$b' = \frac{g\theta'}{\theta_0} \approx f_0 \frac{\partial \psi_g}{\partial z}$$

Thermal wind balance

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Geostrophic wind and buoyancy all expressed as gradients of streamfunction – *therefore related:*

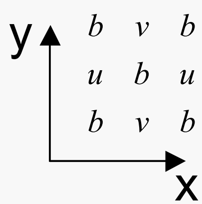
$$\frac{\partial u_g}{\partial z} = -\frac{\partial^2 \psi_g}{\partial z \partial y} = -\frac{1}{f_0}\frac{\partial b'}{\partial y} \qquad \frac{\partial v_g}{\partial z} = \frac{\partial^2 \psi_g}{\partial z \partial x} = \frac{1}{f_0}\frac{\partial b'}{\partial x}$$

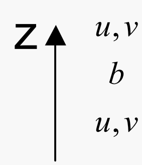
Thermal wind balance is fundamental in atmosphere and ocean.

Met Office Unified Model has made representation of TWB as compact as possible by choosing staggering of variables on grid.

Example of building dynamical knowledge into model design.

Arakawa-C grid





Charney-Phillips grid

Predicting geostrophic flow evolution

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Geostrophic and hydrostatic balance are *diagnostic*

- the time derivatives have been neglected.

Flow evolution depends on *ageostrophic flow:*

$$\begin{pmatrix} u_{ag} \\ v_{ag} \\ w_{ag} \end{pmatrix} = \begin{pmatrix} u - u_g \\ v - v_g \\ w \end{pmatrix}$$

e.g.,
$$\frac{Dv}{Dt} + (f - f_0)u + f_0 u_{ag} = 0$$

Quasi-geostrophic theory is obtained at next order and predicts vorticity evolution:

$$D_g(f + \xi_g) = f_0 \frac{1}{\rho_r} \frac{\partial(\rho_r w)}{\partial z}$$

→ Vortex stretching increases absolute vorticity following geostrophic flow

and evolution of buoyancy:

$$D_g b' + N^2 w = 0$$

→ Advection of reference buoyancy downwards increases buoyancy following geostrophic flow

Define geostrophic material derivative

$$D_g = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

QG potential vorticity

Vorticity and buoyancy evolution both depend on vertical motion, *w.*

Eliminating *w* gives: $D_g q = 0$

Meaning that **QG potential vorticity**, *q,* is *conserved* following the geostrophic flow, where

$$q = f + \frac{\partial^2 \psi_g}{\partial x^2} + \frac{\partial^2 \psi_g}{\partial y^2} + \frac{1}{\rho_r}\frac{\partial}{\partial z}\left(\rho_r \frac{f_0^2}{N^2}\frac{\partial \psi_g}{\partial z}\right)$$

Given distribution of *q* and boundary conditions, can *invert PV*

$q' = q - f = L(\psi_g)$ to find $\psi_g = L^{-1}(q')$

Solution of QG system is to advect QGPV contours with the geostrophic flow (like a tracer) over one time-step. Then invert the new PV distribution to infer new flow and buoyancy:

$$u_g = -\frac{\partial \psi_g}{\partial y} \qquad v_g = \frac{\partial \psi_g}{\partial x} \qquad b' = f_0 \frac{\partial \psi_g}{\partial z}$$

Action-at-a-distance

Assuming density and *N* are constants and re-scaling the height coordinate so that $\hat{z} = (N / f_0) z$

$$q' \approx \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \hat{z}^2} \right) \psi_g = \nabla^2 \psi_g$$

Poisson equation. Other examples from physics:

q'=point charge; ψ=electric field potential

q'=force at a point on drum; ψ=drum skin displacement

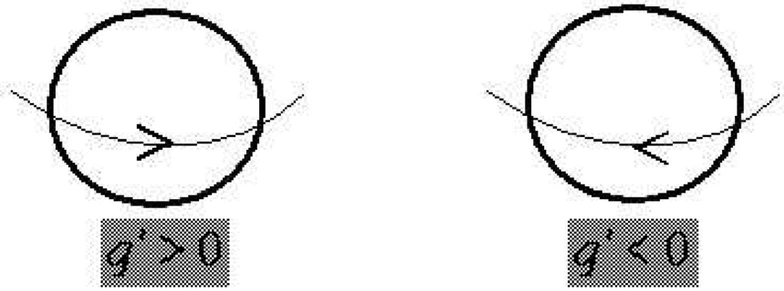
Means that a point **PV anomaly** *induces* flow far away.

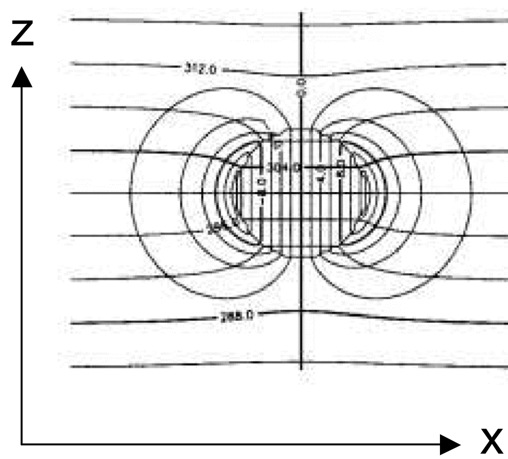
The induced streamfunction is symmetrical about point in re-scaled coordinates.

$\Rightarrow$ **natural aspect ratio** $\frac{L}{H} \approx \frac{N}{f_0} \approx 100$

Inversion of a ball of uniform PV

Anomalies of q have both circulation and temperature anomalies, i.e. $\psi \propto - q'$



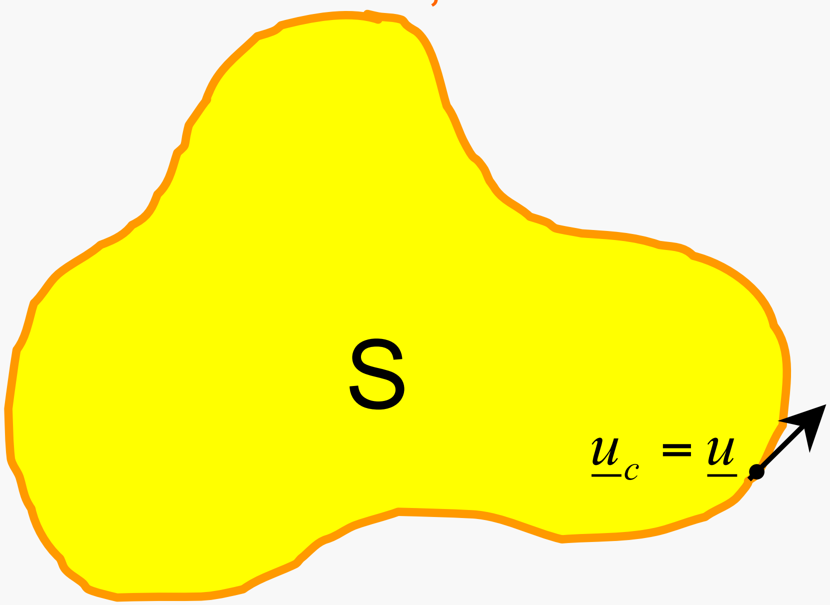


θ, ν for positive PV anomaly

More from Heini in next lecture!

Integral of dynamics – circulation

Material contour, ∂S



$$C = \int_{\partial S} \underline{u}.d\underline{l} = \iint_S (\nabla \times \underline{u}).d\underline{S}$$

$$\frac{dC}{dt} = \int \left\{ \frac{D\underline{u}}{Dt}.d\underline{l} + \underline{u}.\frac{D(d\underline{l})}{Dt} \right\}$$

$$= \int \left\{ \left(-\frac{\nabla p}{\rho} - \nabla \Phi \right).d\underline{l} + \underline{u}.d\underline{u} \right\}$$

$$= -\int_{\partial S} \frac{dp}{\rho}$$

Kelvin's circulation theorem

If $\rho=\rho(p)$ then C is invariant.

True for ideal gas $\rho=p/(RT)$, if S stays on an *isentropic surface*

(potential temperature, $\theta = \theta(T, p)$ = constant).

Ertel potential vorticity

University of Reading

Consider a fluid parcel of mass δM enclosed by the material contour within an isentropic layer of depth $\delta\theta$

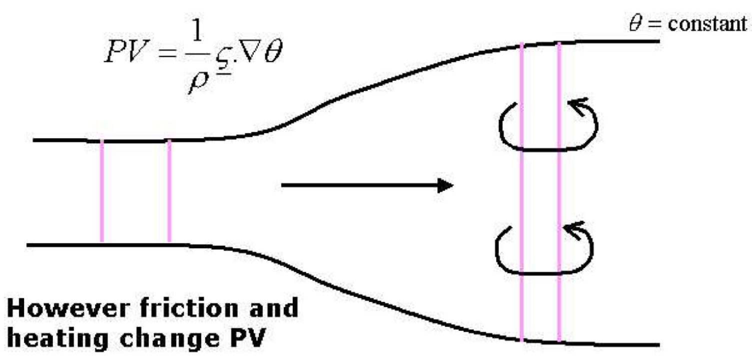
Ertel PV

$$P \underset{\partial S \to 0}{=} \frac{C\delta\theta}{\delta M} = \frac{\underline{\zeta}.\underline{n}\delta S\delta\theta}{\sigma\delta S\delta\theta} = \frac{\zeta_\theta}{\sigma}$$

Component of absolute vorticity normal to isentropic surfaces

Isentropic density

Hans Ertel and Carl-Gustav Rossby showed in early 1940s that for any adiabatic frictionless flow PV is conserved:



Material conservation of PV follows from conservation of mass and circulation

$$\frac{DP}{Dt} = 0$$

PV diagnostics

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Ertel PV is conserved by the unapproximated dynamical equations following adiabatic, frictionless flow.

- Pragmatic approach is to calculate Ertel PV as diagnostic from model variables

  ⇒ approximately follows motion of air along surfaces of constant potential temperature (isentropic surfaces)

  ⇒ *imagine* the flow and stratification anomalies associated with PV anomalies and they way in which they would influence evolution

- More quantitatively obtain the *balanced flow* by inverting the PV distribution using a balance approximation.

5. Wave Phenomena

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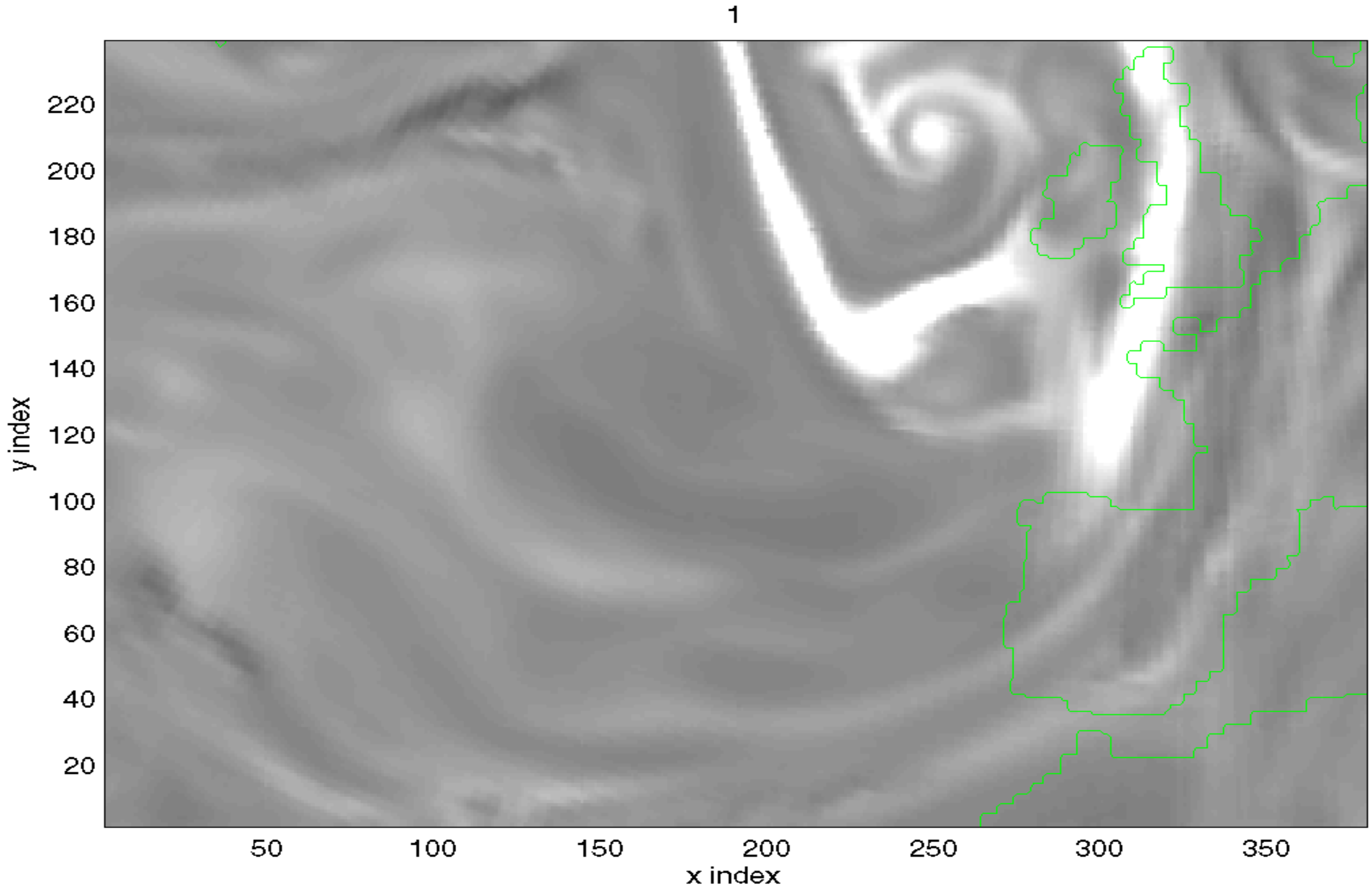
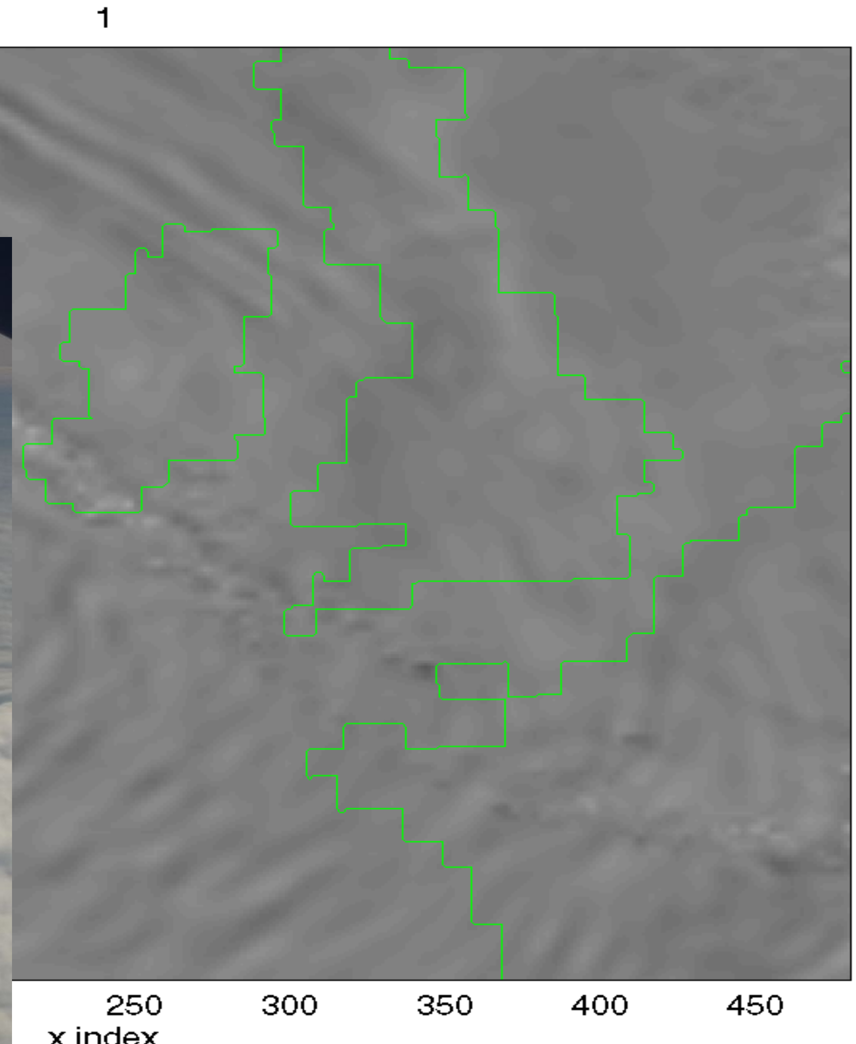
- Wave propagation
  - Signal moves relative to flow velocity (without transporting mass)
  - Wave crests move at phase velocity
  - Wave packets move at group velocity
- Balanced flow in atmospheres and oceans
  - Potential vorticity advected as tracer
    - Balanced component of flow found by *inverting* PV distribution (with BCs)
    - "slow (smooth) evolution"
  - Unbalanced flow dominated by fast wave motions
    - E.g., gravity waves and sound waves



Potential vorticity at 6.5km (white = high values; stratospheric air)

Met Office Unified Model – simulation (*Jeffrey Chagnon)*

$\Delta t = 300s$ $\Delta x \approx 12km$ 38 levels 3 hour frames



itude (white = high values > 0.5ms^{-1})

imited area (*Jeffrey Chagnon)*

38 levels 1 hour frames

Gravity waves

- Diverse forms and generation mechanisms
- Rely on stable stratification $\dfrac{\partial \theta}{\partial z} > 0$
  so that parcel displaced downwards (adiabatically) has high buoyancy relative to its surroundings.
- Interplay between gravity and pressure gradients
- On timescales ~ *1/f* Coriolis effect influences parcel motions ⇒ *inertia-gravity waves*
- GW activity generated by:
  - flow over orography (e.g., lee waves),
  - convective updrafts (and heating),
  - Spontaneous imbalance associated with *balanced motion* (e.g., at fronts, curved jets)

Rossby waves

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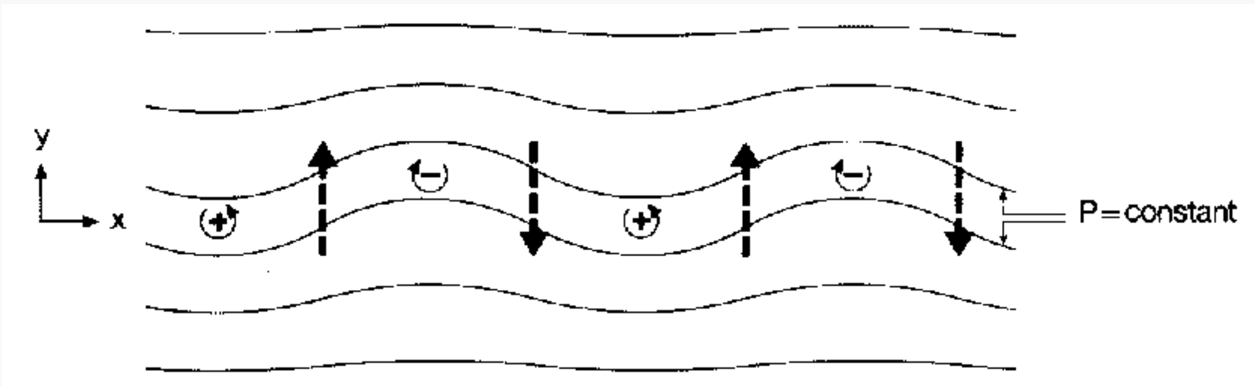
Rossby waves propagate on horizontal PV gradients.



high PV

$$\bar{q}_y > 0$$

low PV

Air displaced to south carries high PV and forms +ve *q'*

$\Rightarrow$ *q'>0* **induces** cyclonic circulation

$\Rightarrow$ advects air southwards on western flank

$\Rightarrow$ wave pattern propagates **westwards**

Phase speed: $$c_p = \bar{u} - \frac{v_k}{kq_k}\frac{\partial \bar{q}}{\partial y}$$

Fig: Hoskins, McIntyre and Robertson (1985), QJ

Equatorial Waves

In tropics, although Coriolis parameter is small its gradient is important to large-scale waves

$\Rightarrow$ Equatorial beta-plane $f = \beta y = 2\Omega\phi$

Solutions obtained (Gill, Matsuno) for atmosphere described by single-layer *shallow water equations* perturbed from rest

Deriving Shallow Water Eqns

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The shallow water equations are obtained by integrating the PEs over a fluid layer of depth h. From hydrostatic balance:

$$\begin{aligned} p_{top} - p_{bot} &= -\rho_o g h \\ \Rightarrow p' &= \rho_o g \eta \end{aligned}$$

assuming that there are no pressure perturbations on the free surface, the pressure gradient terms become $-g\frac{\partial \eta}{\partial x}$ and $-g\frac{\partial \eta}{\partial y}$.
The mass conservation equation integrated vertically is:

$$\begin{aligned} \left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} h + [w]_{bot}^{top} &= 0 \\ \left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} h + \frac{Dh}{Dt} &= 0 \\ \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} &= 0 \end{aligned}$$

where (u, v) is now the depth-average velocity.

Equatorial Waves

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

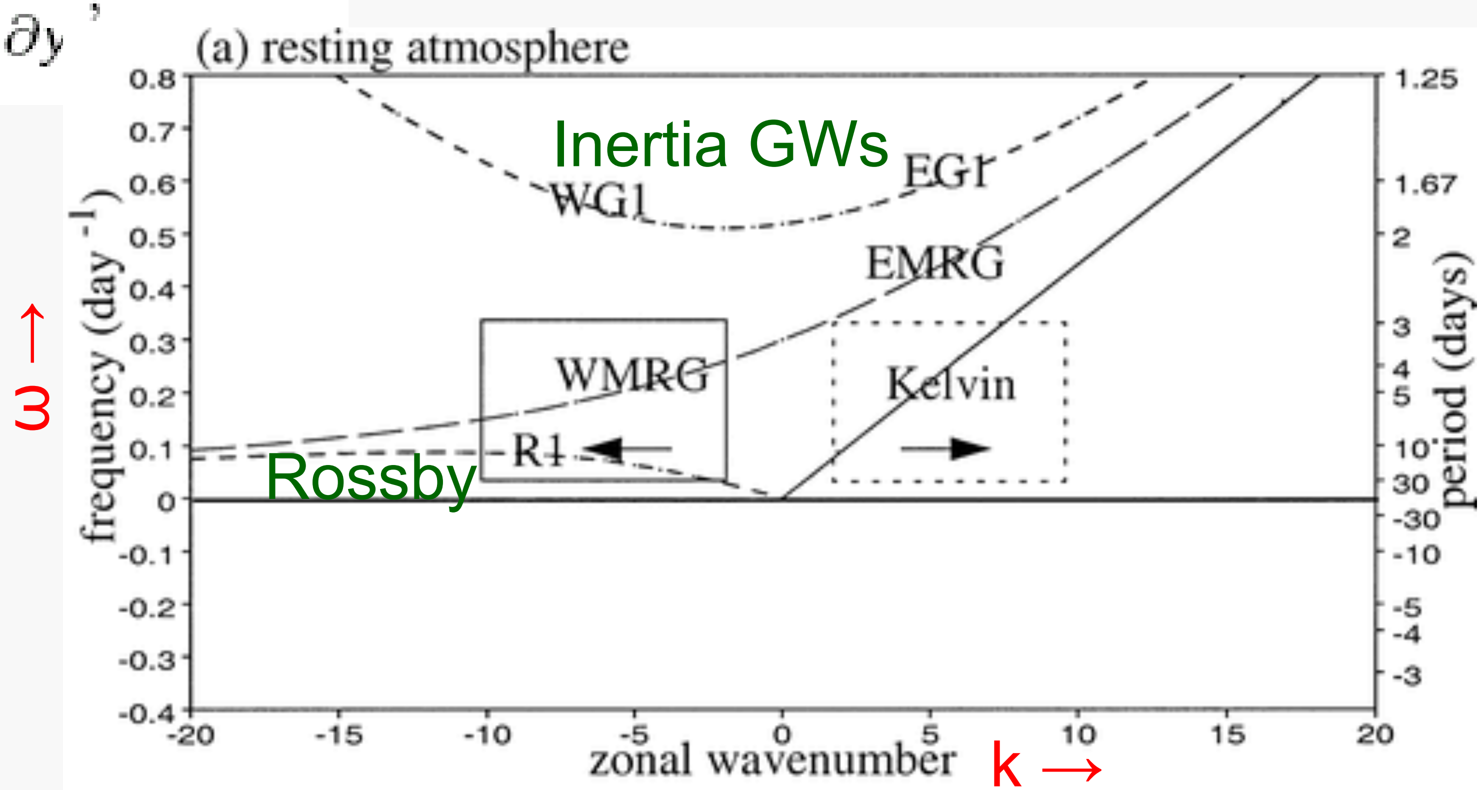
$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y},$$

Eqns linearised about a state at rest.

Solutions below for dry atmosphere.

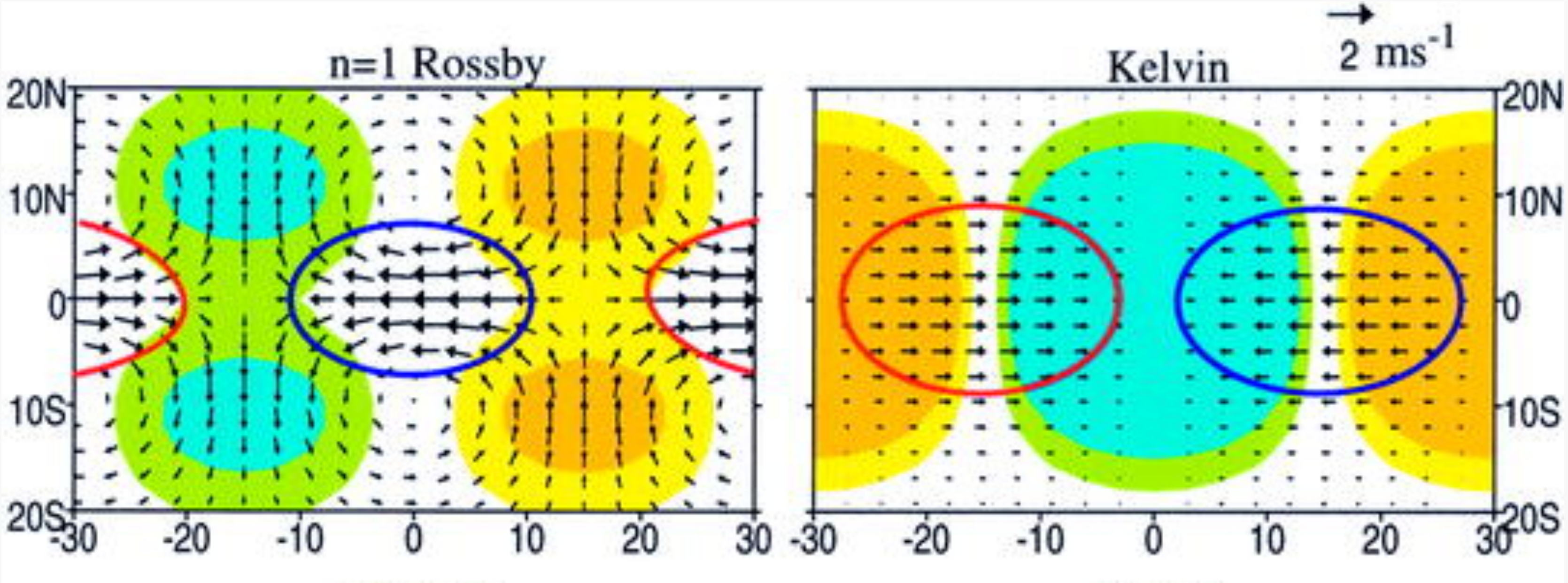
Surprisingly similar in moist model with
3 vertical structure modes
(Khouider and Majda, JAS, 2006)



MRG = mixed
Rossby-gravity

Aka "Yanai wave"

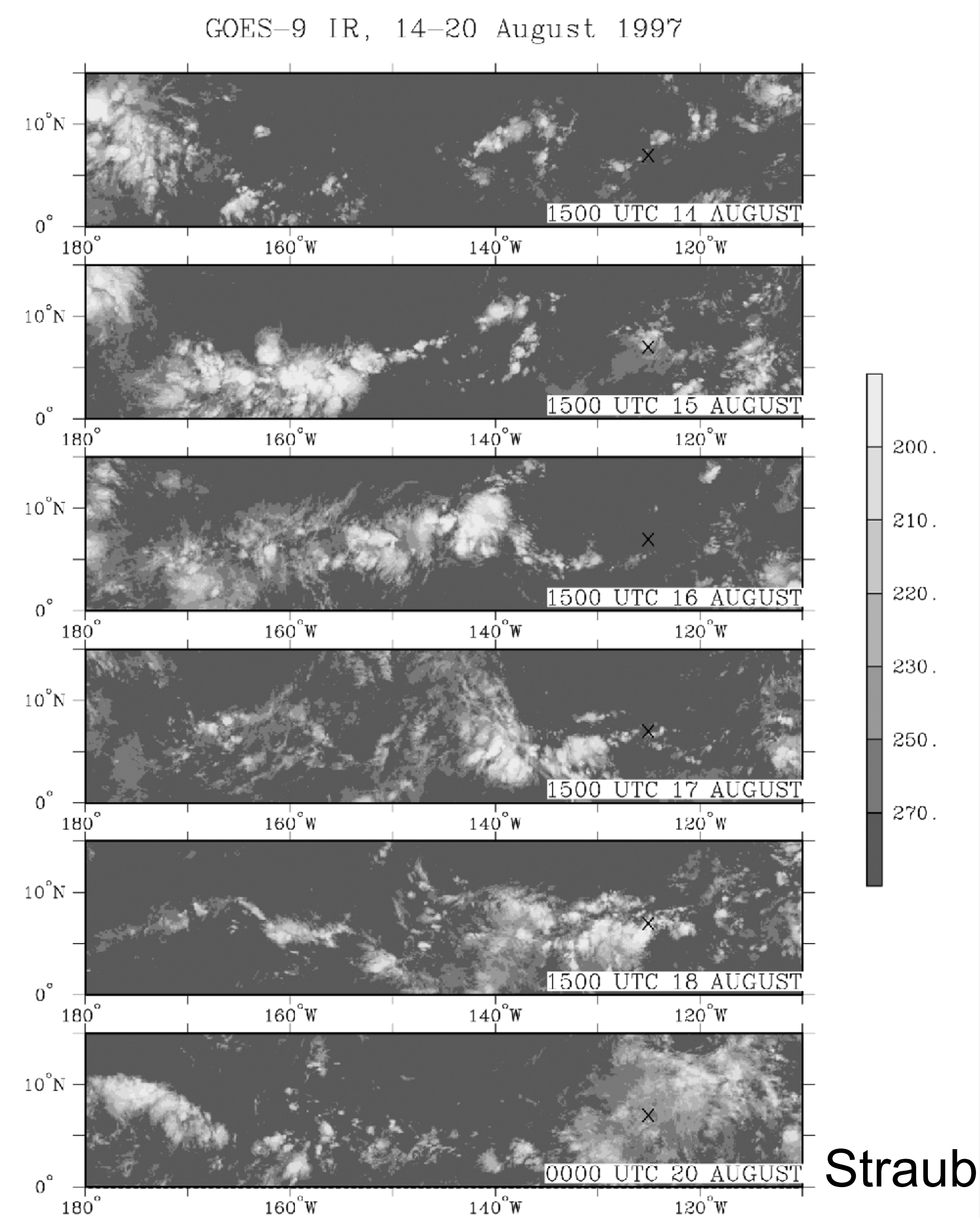
Equatorial Wave Structures

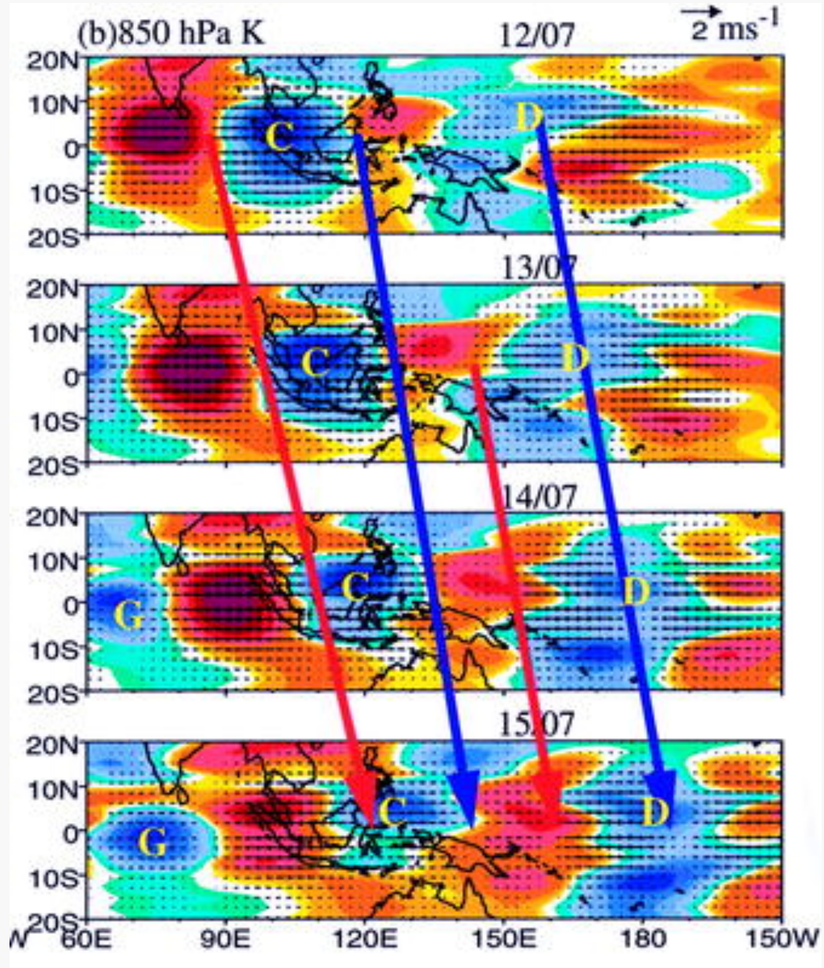


Colour shading: convergence/divergence

Yang, Hoskins and Slingo, *JAS, 2003*

Equatorial Waves: Observations





Yang et al (2003)

Straub and Kiladis (2003)

6. Large Amplitude Waves

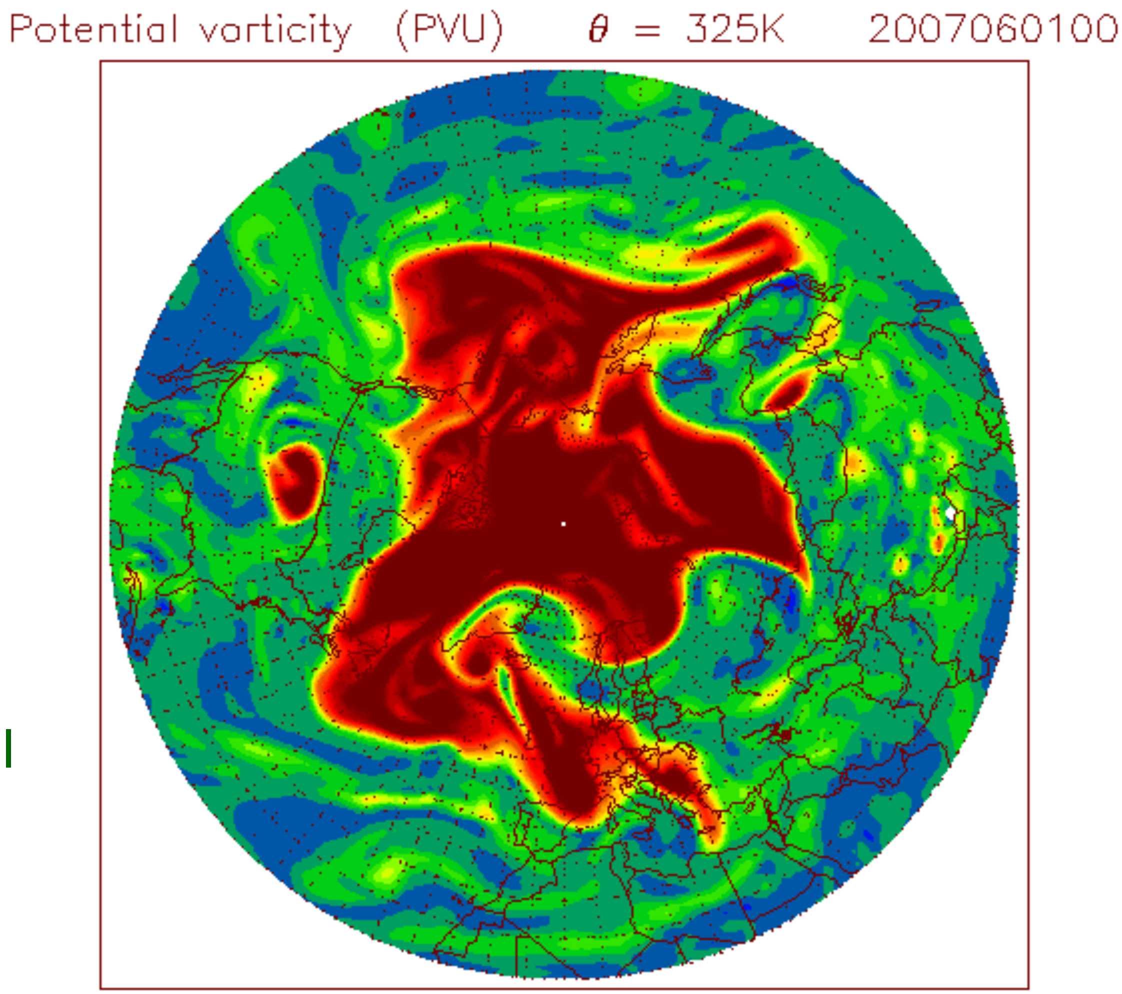
Large amplitude waves are always present in the atmosphere.

Implicit assumption is that PV anomalies can be defined by air parcel displacement from some background state.

But, what is this state?

Assume background has the same **circulation** as full flow but is zonally symmetric *(modified Lagrangian mean)*

⇒ if flow is adiabatic and frictionless, the background is steady



ERA-Interim re-analyses (T255, L60)

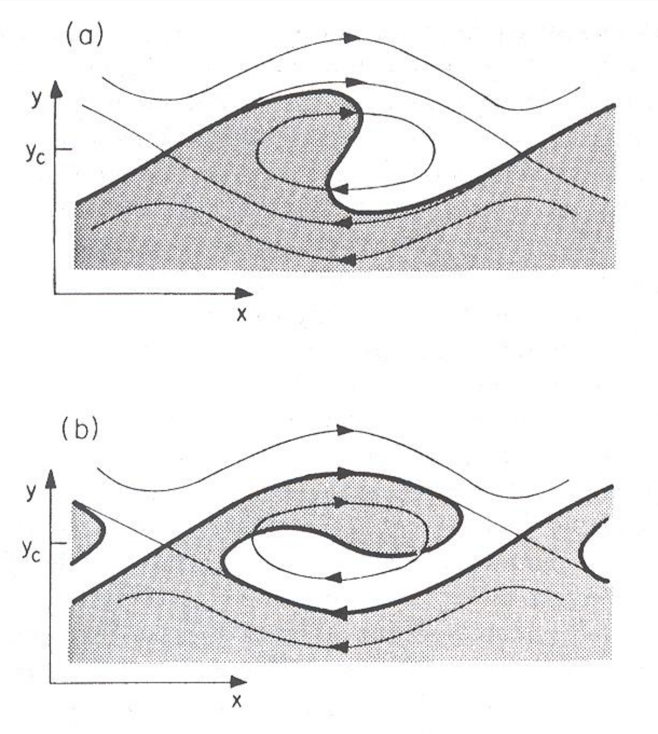
Rossby wave breaking

Theory so far for small amplitude waves (small wave slope).

At large amplitude PV waves are deformed irreversibly by shear.

A Rossby wave critical layer exhibits a wave breaking paradigm.

Viewed from frame moving with phase speed of wave:



Streamfunction has *cat's eye* pattern.

Outside cat's eye PV ≈ parallel to streamlines.

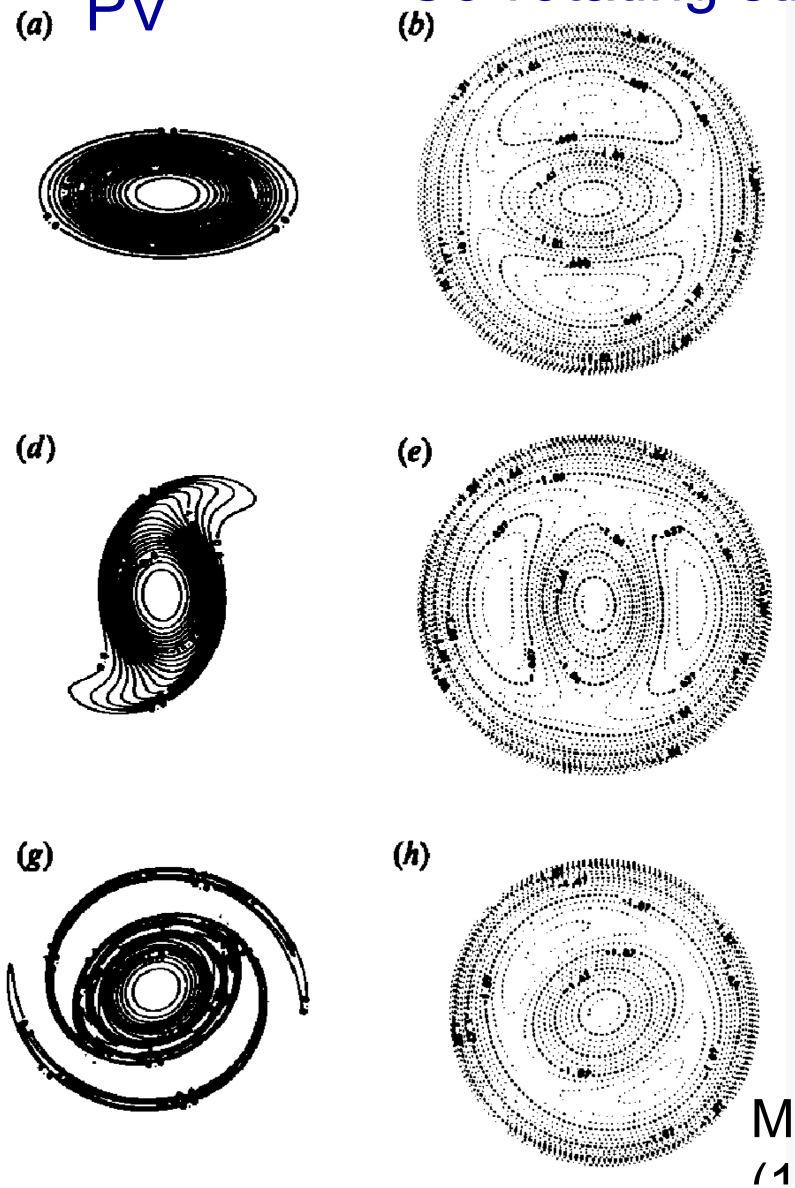
PV contours crossing hyperbolic points (by extra perturbation) are wrapped anticyclonically within cat's eye.

From Andrews *et al* [1987] after Haynes [1985].

Vortex erosion by filamentation

PV

Co-rotating streamfunction



If vortex becomes too elliptical, PV contours can break

⇒ formation of PV filaments

⇒ vortex area and circulation ends up smaller

⇒ PV gradients on vortex edge are sharper

Melander, McWilliams and Zabusky (1987)

Vortex Erosion Process

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Rossby waves exist where there are PV gradients.

Regions of tight PV gradients (“PV steps”) act as *Rossby wave guides.*

+ve PV steps are associated with westerly jets by PV inversion.

Examples: stratospheric polar vortex edge, tropopause (and jetstream), Gulf Stream, zonal jets on Jupiter.

But, why do PV steps emerge?

Rossby wave breaking ⇒ PV filaments drawn off polar vortex

Chaotic stirring within *surf zone* ⇒ stretching and folding of PV contours

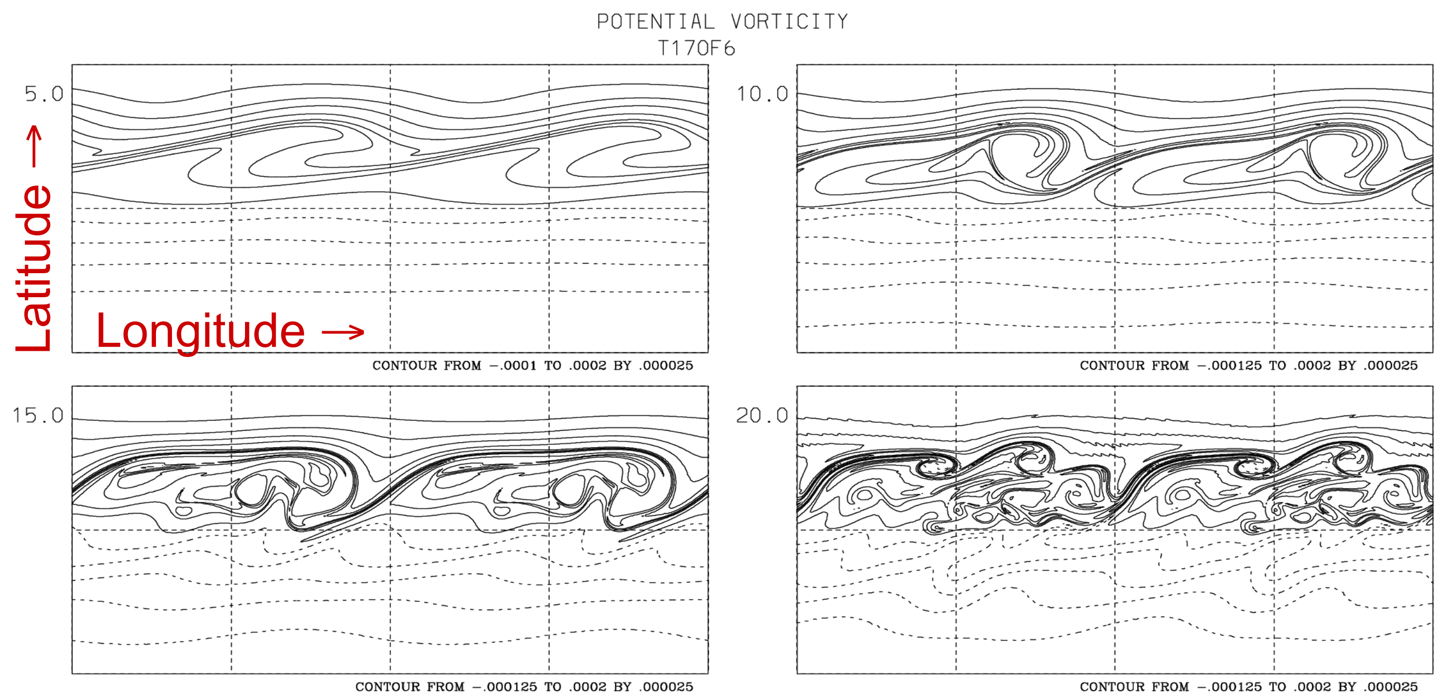
⇒ dissipation at small scales

Net effect: mass between PV contours in vortex edge region transferred irreversibly into surf zone ⇒ PV gradient sharper on edge.

E.g., a single layer model like the atmosphere

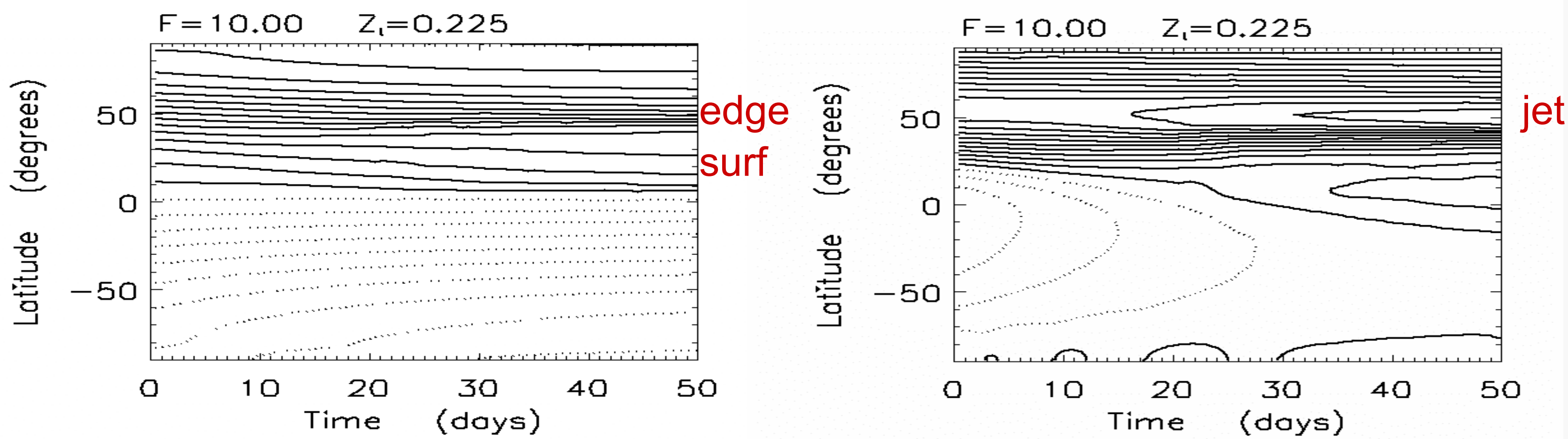
from Methven, QJ, 2003

Rossby waves break, forming filaments which roll-up into eddies

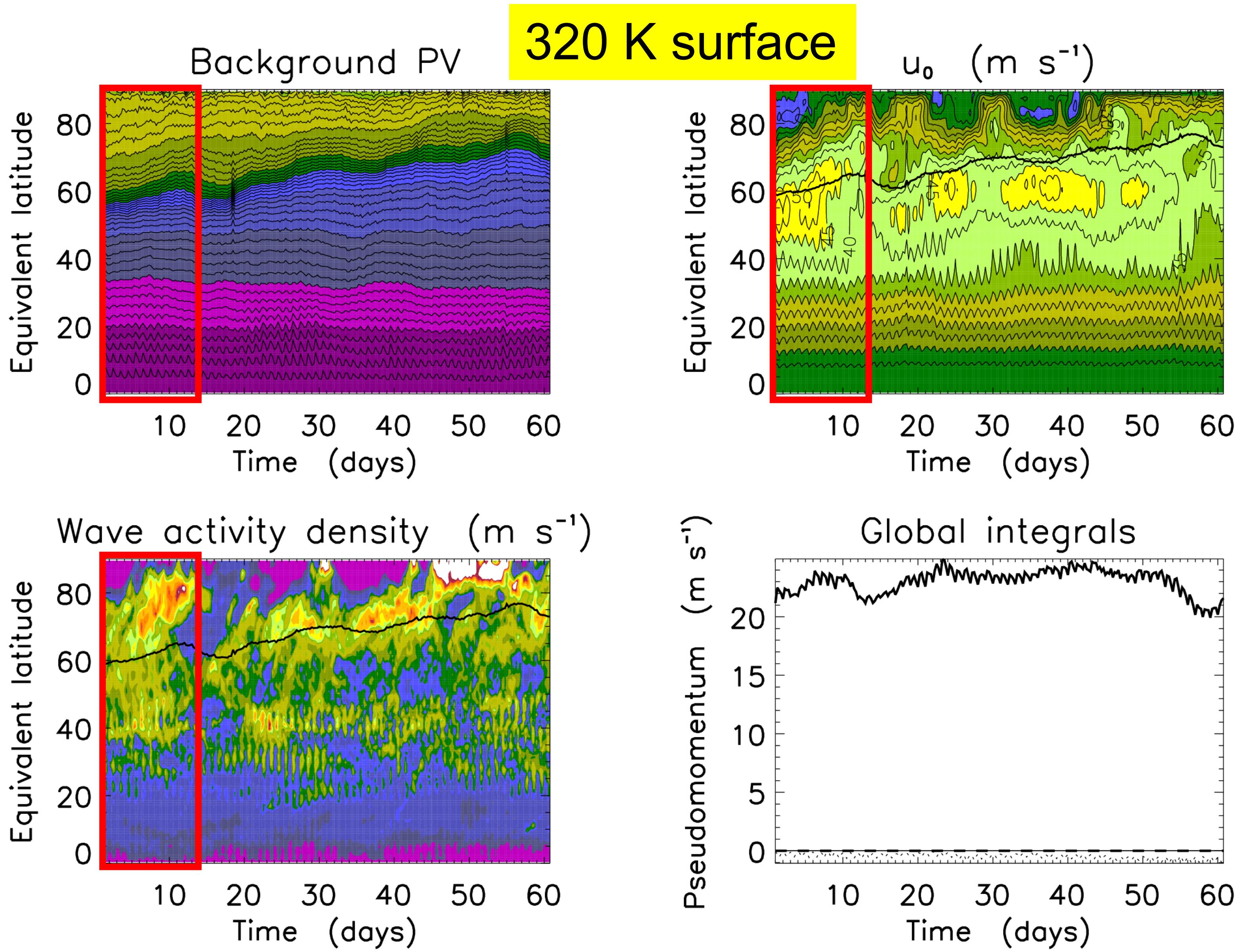


Evolution of background state PV

…and flow from PV inversion



MLM state for June-July 2007



Rossby wave breaking and vortex erosion

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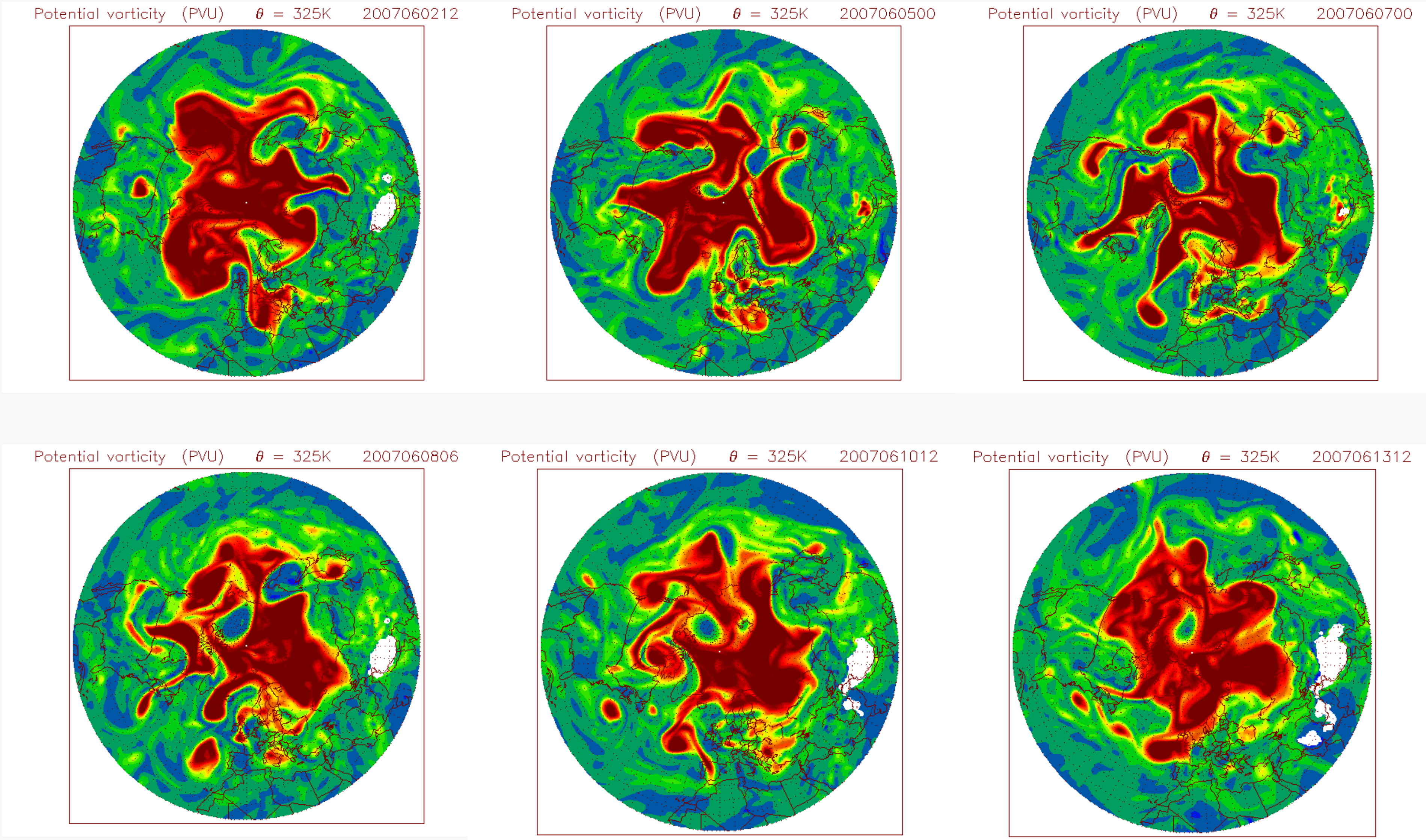
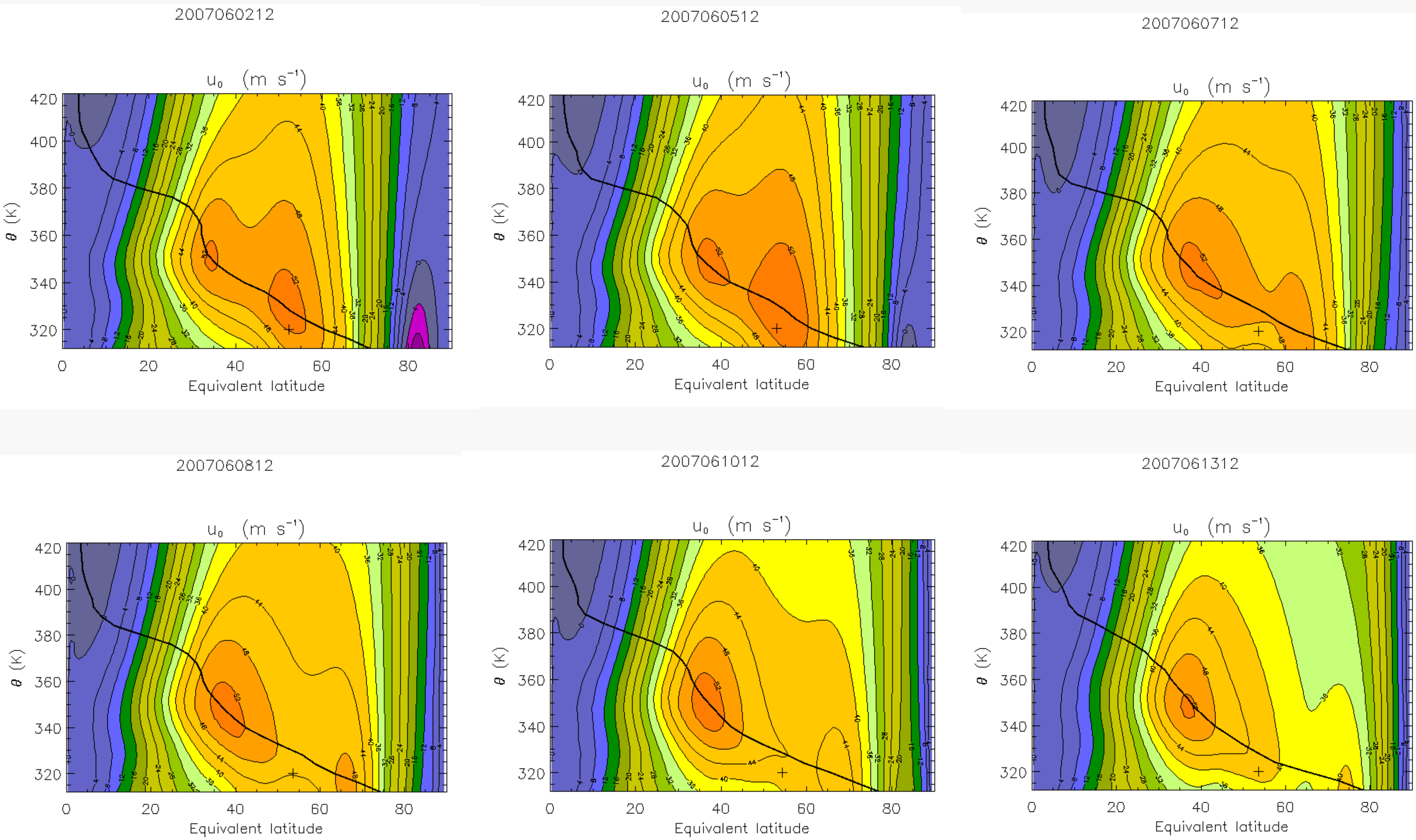


Poleward migration of the polar jet in background state



7. Shear instability

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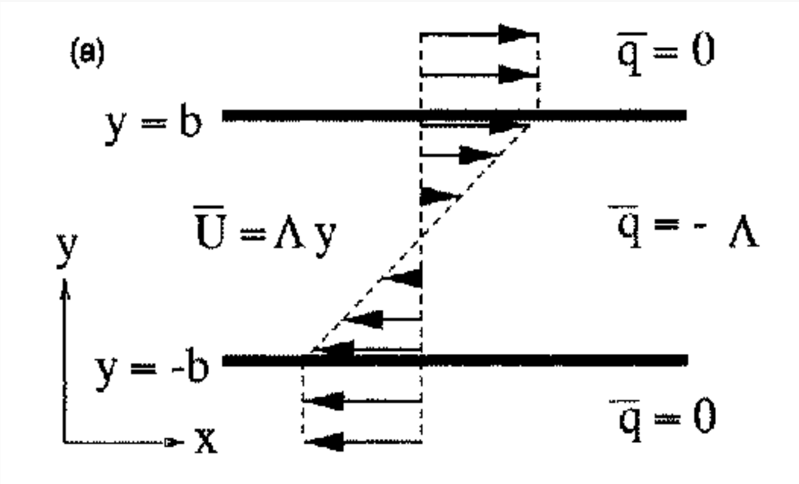
Shear flows are often found to be *unstable.*
Small perturbations grow at exponential rate.

Lord Rayleigh considered the growth of waves as the *transition to turbulence* and explained growth for the simple flow below:



$\bar{q}_y > 0$

$\bar{q}_y < 0$

Vorticity is piecewise constant ⇒ PV gradients are concentrated into two spikes: +ve at *y=b* and –ve at *y=-b.*

Rossby wave counter-propagation

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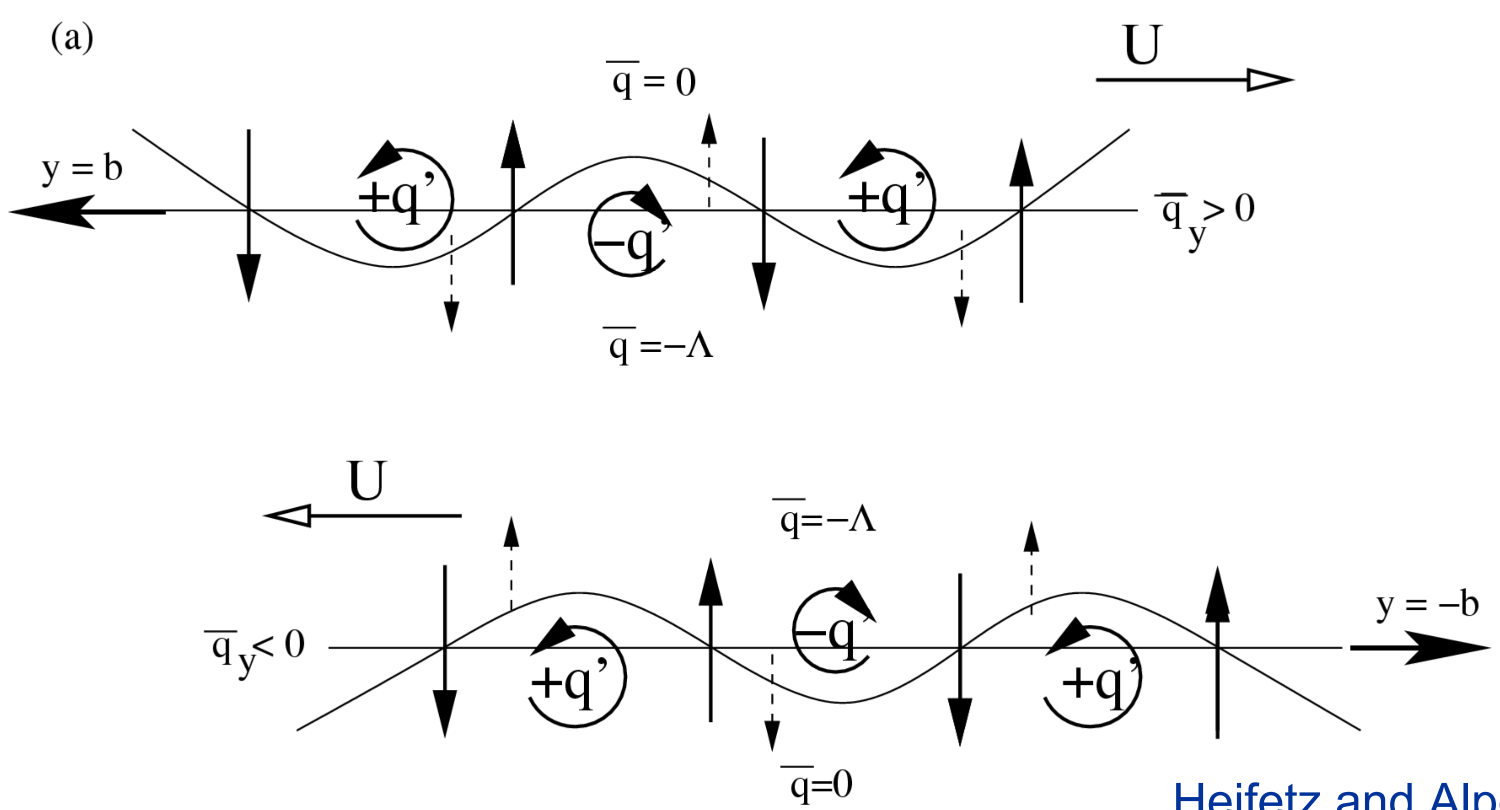


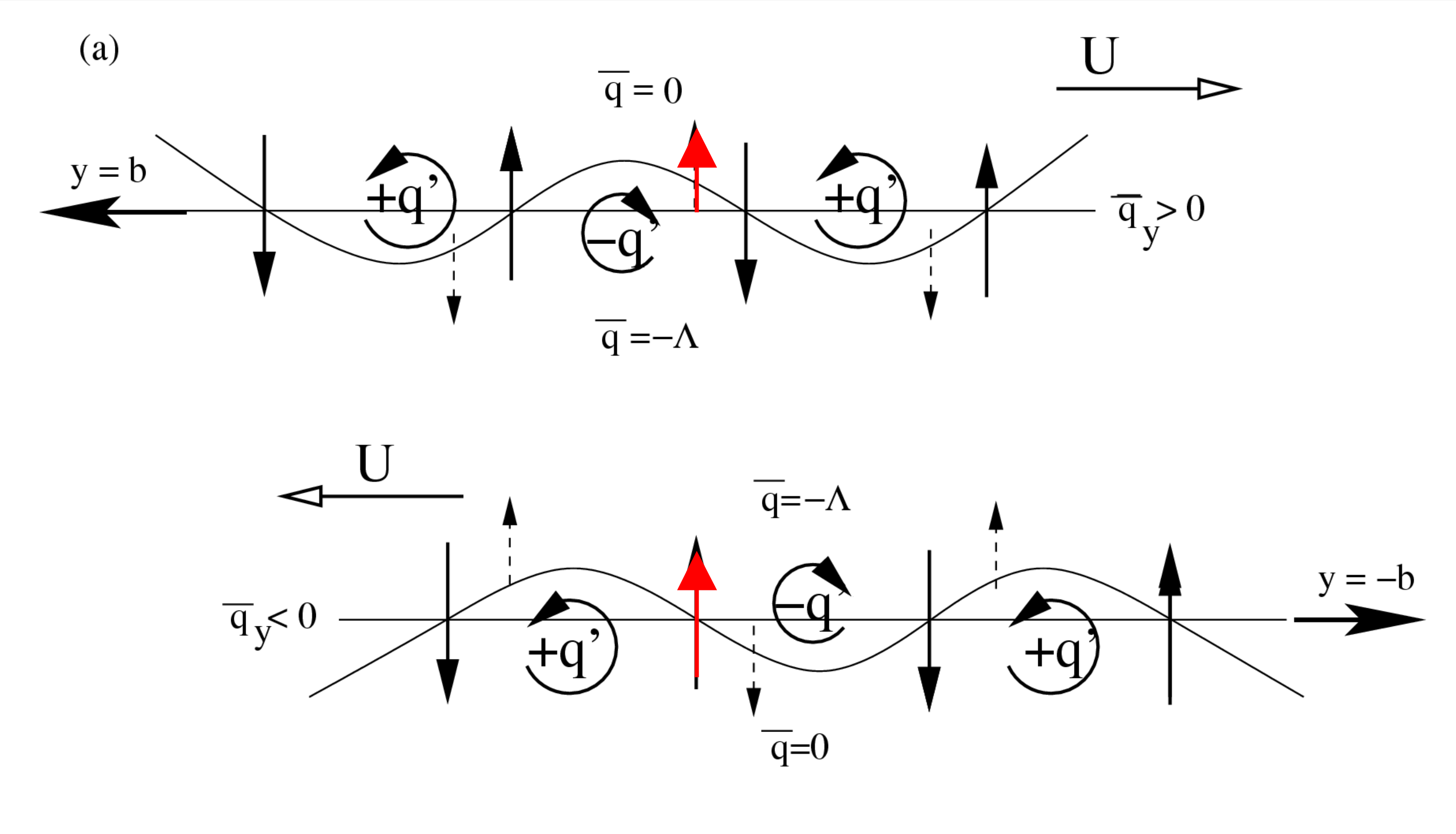
Heifetz and Alpert (1999)

At *y=b* wave propagates westwards, counter to eastward flow.

At *y=-b* wave propagates eastwards, counter to westward flow.

Counter-propagating Rossby Waves (CRWs) have similar phase speeds.

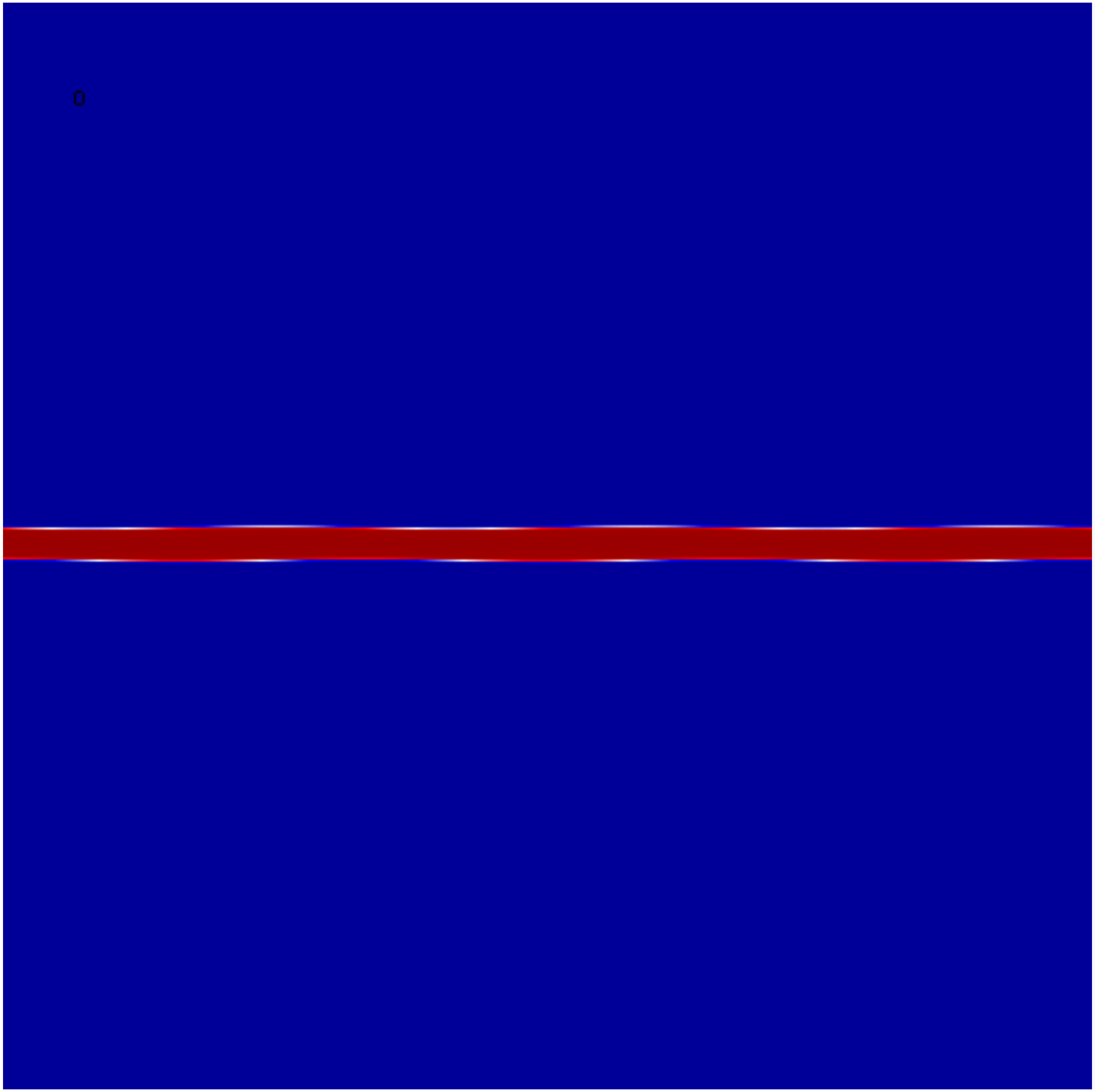
Rossby wave coupling



Northward flow induced by wave-1 is felt at home-base of wave-2.

Phase of wave-2 is such that this flow increases the –ve PV anomaly but hinders its westward propagation (reducing phase difference).

Coupling results in *mutual growth* and enables *phase-locking.*



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Shear instability associated with interaction and mutual growth of two Rossby waves

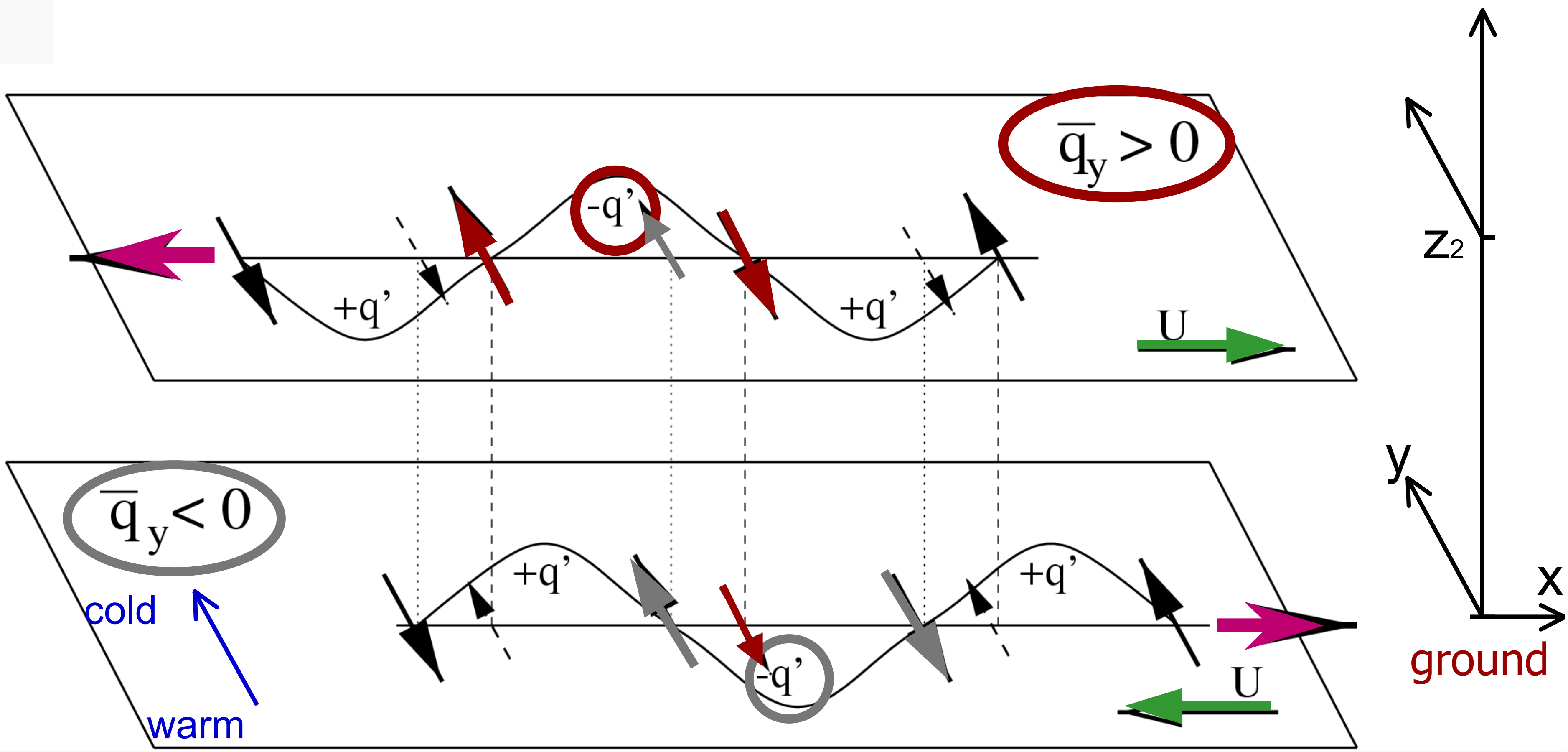
⇒ roll-up of filaments in PV

Ben Harvey,
University of Reading

Baroclinic instability theory

- Attempts to describe the growth of synoptic scale weather systems.
- Early successes using the Charney (1947), Eady (1949) and Phillips (1954) models:
  - very simple basic states
  - perturbations described by linearised quasigeostrophic eqns (small wave slopes)
- Mechanism of growth in 2-layer (Phillips) model was explained in terms of **Counter-propagating Rossby Waves (CRWs)** by Bretherton (1966).

Baroclinic instability in terms of counter-propagating Rossby waves

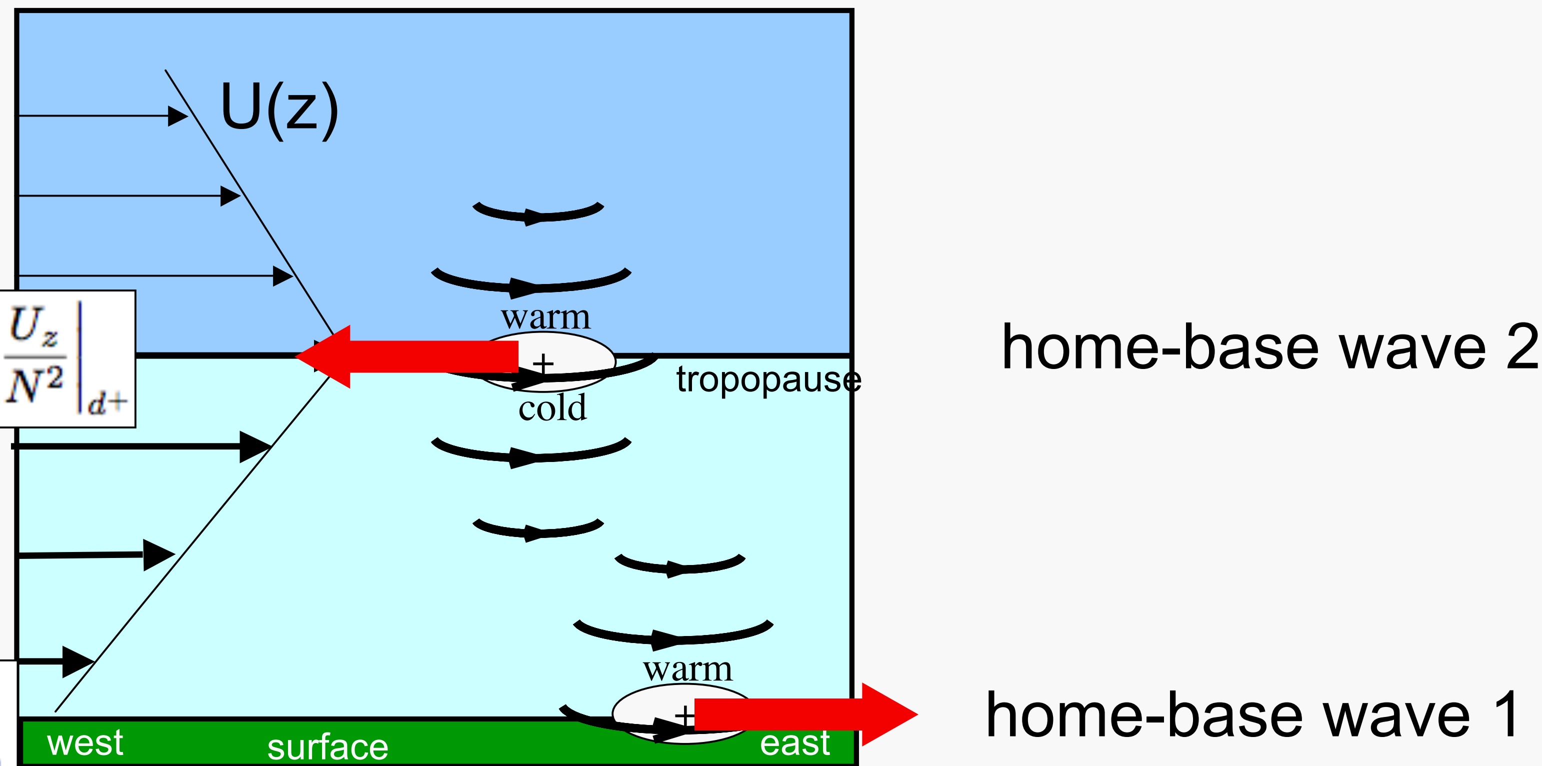


Heifetz *et al* [2004], QJ

Counter-propagating Rossby waves

- Baroclinic model of **Eady (1949)**
- Meridional PV gradient = 0 (except at ground and tropopause)
- Normal-mode growth: constructive interaction between surface and tropopause 'edge' waves = CRWs in this case

$$\frac{dQ}{dy} = \left.\frac{U_z}{N^2}\right|_{d-} - \left.\frac{U_z}{N^2}\right|_{d+}$$



$$\frac{dQ}{dy} = -\left.\frac{U_z}{N^2}\right|_{0+}$$

When Does CRW Picture Apply?

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- Parallel flow with shear.

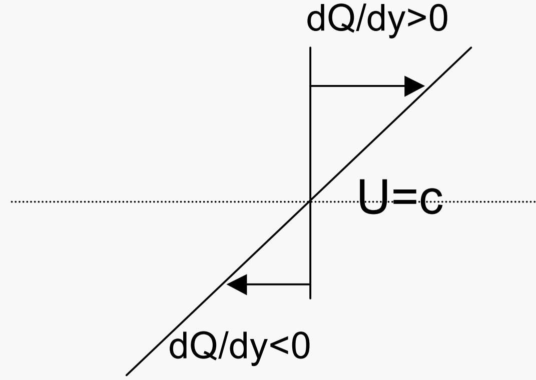
  Necessary criteria for instability:

  - *Waves propagate in opposite directions,*
  - *Wave on more +ve basic state flow has –ve propagation speed so that phase speeds of 2 waves without interaction are similar.*

- Clearly, just 2 Rossby waves exist if basic state vorticity (PV) is piecewise uniform with only 2 jumps.
- However, also applies for any unstable zonal jet *U(y,z)*
  - *Heifetz et al, QJRMS, 2004*
  - *Even primitive equations on sphere - Methven et al, QJRMS, 2005*

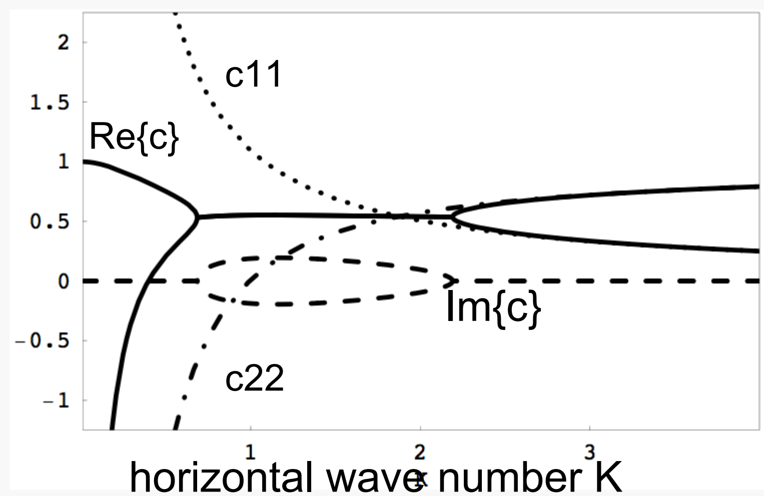
Necessary & sufficient conditions

- **Charney-Stern (1962)**
  - dQ/dy changes sign
  - => Required to get CRW mutual growth
- **Fjørtoft (1951)**
  - U and dQ/dy positively correlated
  - => Required to get CRW phase-locking

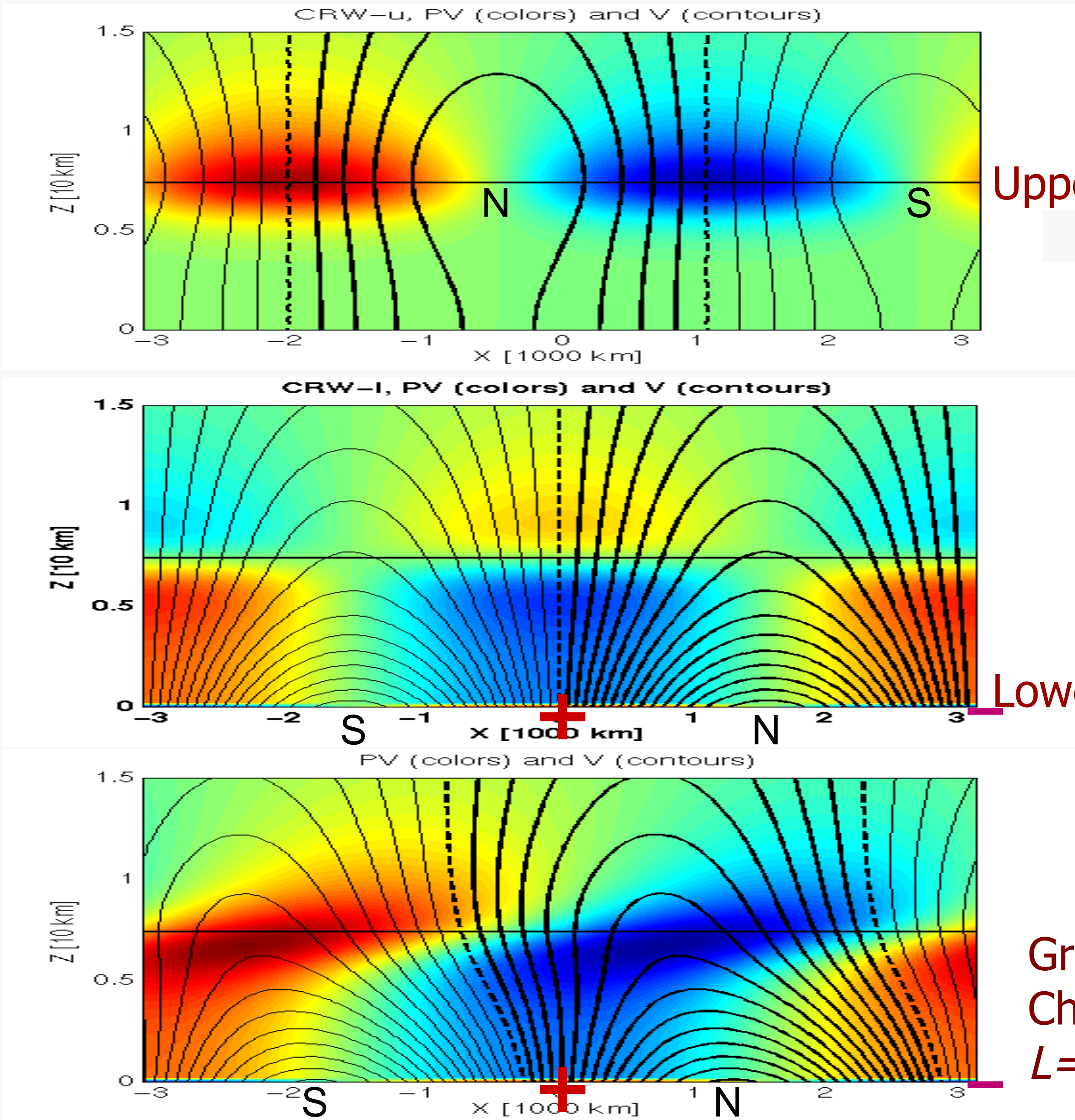


- Not all waves are unstable
  - short-wave cutoff
    - propagation << tropospheric shear
  - long-wave cutoff
    - propagation too great to phase lock

two-layer Eady dispersion relation



Charney model – constant shear but dQ/dy=β



Upper CRW

Lower CRW

Growing NM in Charney model
L=1/*k*=1000 km

Contours show meridional wind

N = northward

S = southward

Colour shading shows QGPV

Red = +ve

Blue = -ve

+/- indicate boundary temperature anomalies

Eady growth rate parameter

Eady model – constant wind shear + zero interior PV gradient + lid.

Charney model – constant wind shear + constant PV gradient (β>0).

Same result for maximum growth rate *(Lindzen and Farrell, 1987)*:

$$\sigma_{\max} \approx 0.31 \frac{f_0 \Lambda}{N} \qquad \Lambda = \frac{\partial u}{\partial z} = -\frac{1}{f_0}\frac{\partial b'}{\partial y}$$

Surprising that σ from Charney model does not involve the vorticity gradient on which the upper CRW propagates (β).

Why? A. Only place with –ve PV gradient is at ground.
Lower CRW propagates at rate ~*f*Λ*/(Nk)*

- upper CRW must have similar phase speed to phase-lock.

Return to this with moisture in second lecture.

CRW evolution equations

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- 2-wave system
  (Davies&Bishop1994, Heifetz et al, 2004)

self-propagation

interaction

$$\begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} = \mathsf{A} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \mathsf{A} = -ik \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

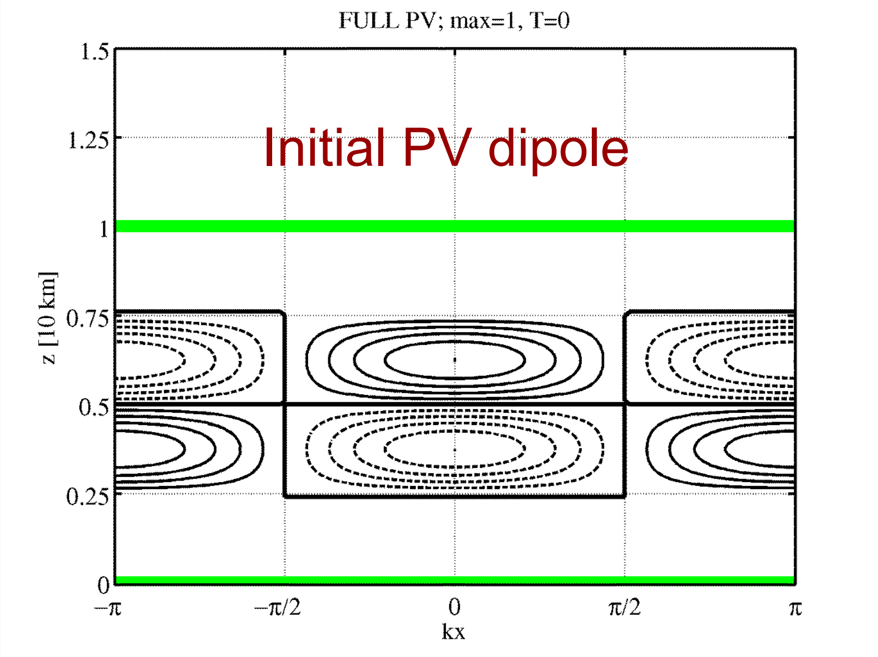
complex amplitudes
crw-L and crw-U

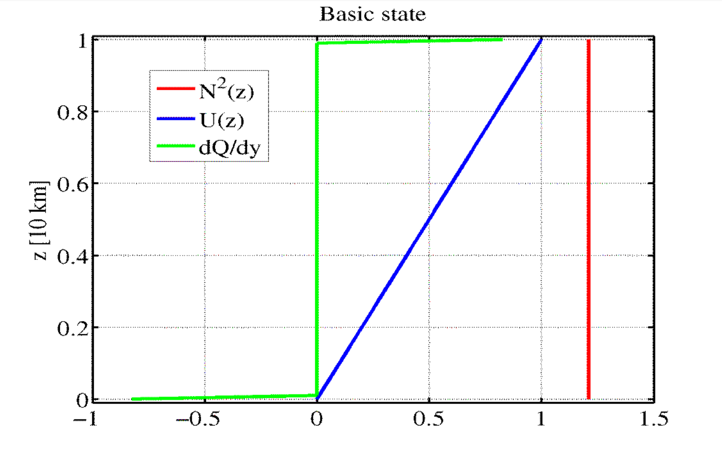
$$\alpha = ae^{i\varepsilon}$$

$$c_{ij} = \bar{u}_i \delta_{ij} - \frac{\gamma_{ij}}{k}, \quad \gamma_{ij} = \left(\frac{v_j}{q_i} \frac{\partial \bar{q}}{\partial y} \right) \Big|_{z_i}$$

- Understand (non)-modal evolution of arbitrary superpositions of growing and decaying NM: e.g. upper-level precursor
- 2 coupled non-linear ODE's for amplitude-ratio and phase-difference

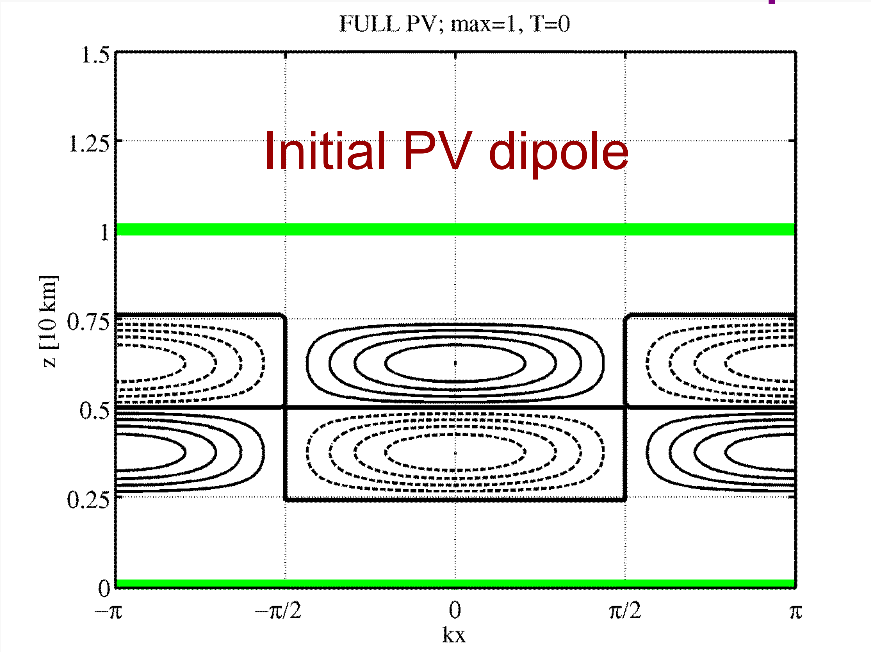
Baroclinic initial value problem - Eady model

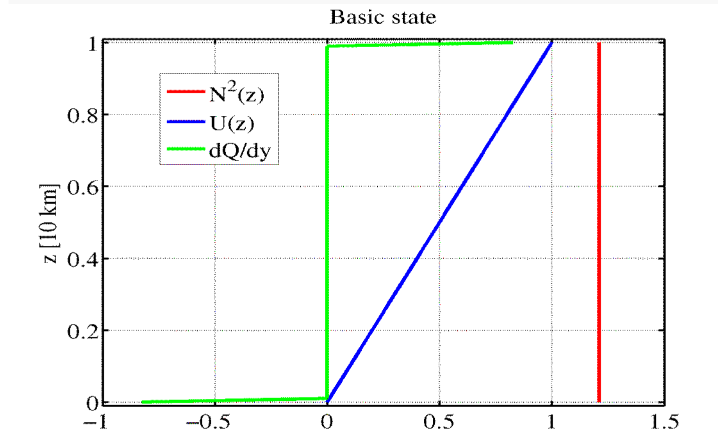


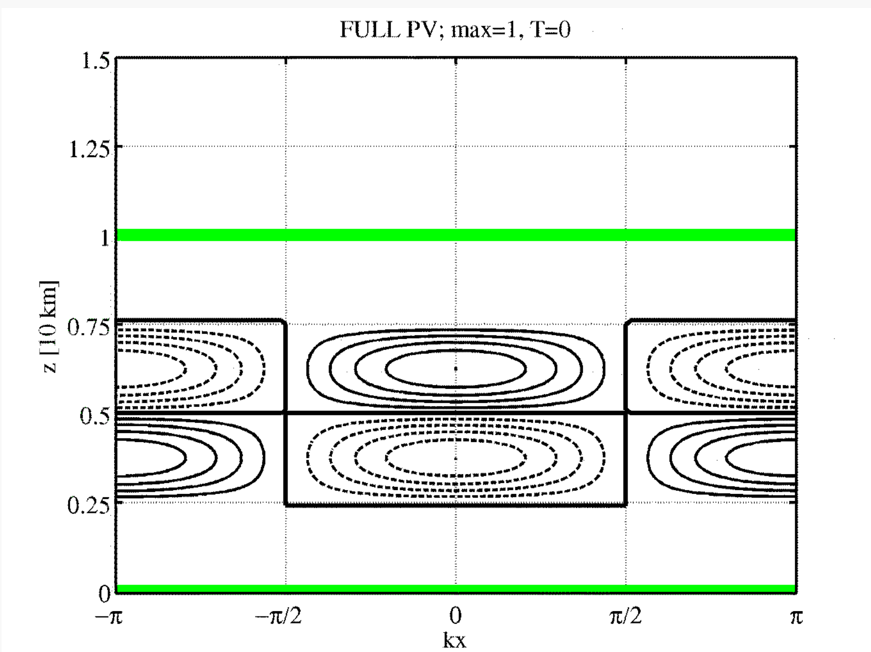


- Uniform vertical shear and static stability
- f-plane (no interior PV gradient)
- Rigid lid

Baroclinic initial value problem - Eady model





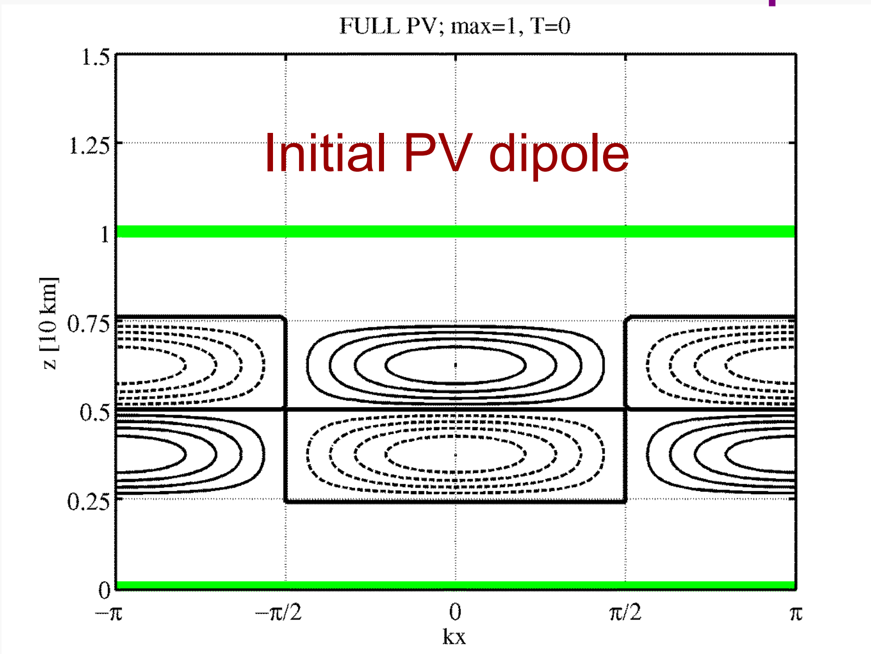


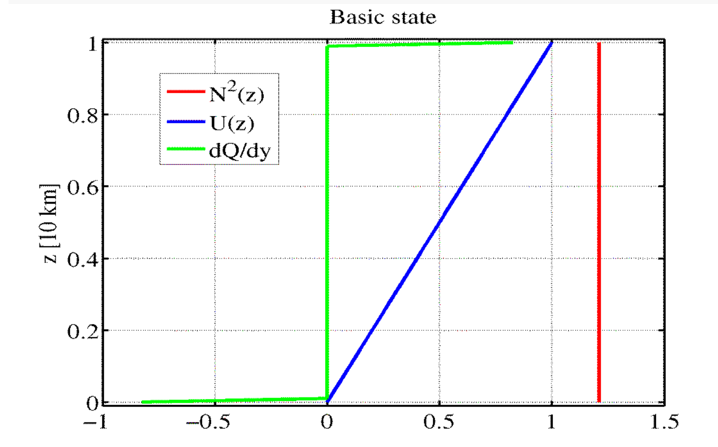
- Uniform vertical shear and static stability
- f-plane (no interior PV gradient)
- Rigid lid

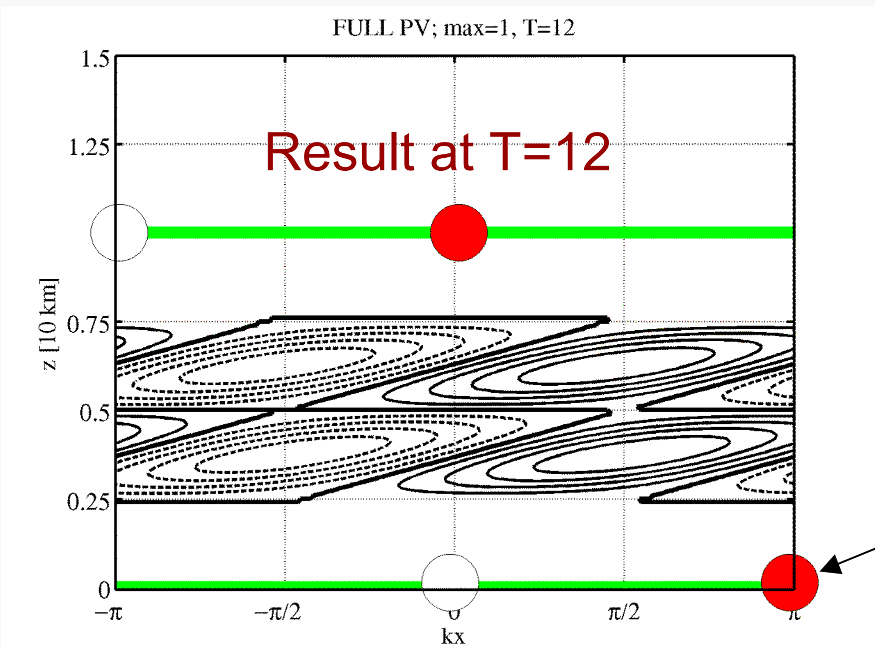
Black/red = +ve boundary PV anomaly

White = -ve boundary PV anomaly

Baroclinic initial value problem - Eady model





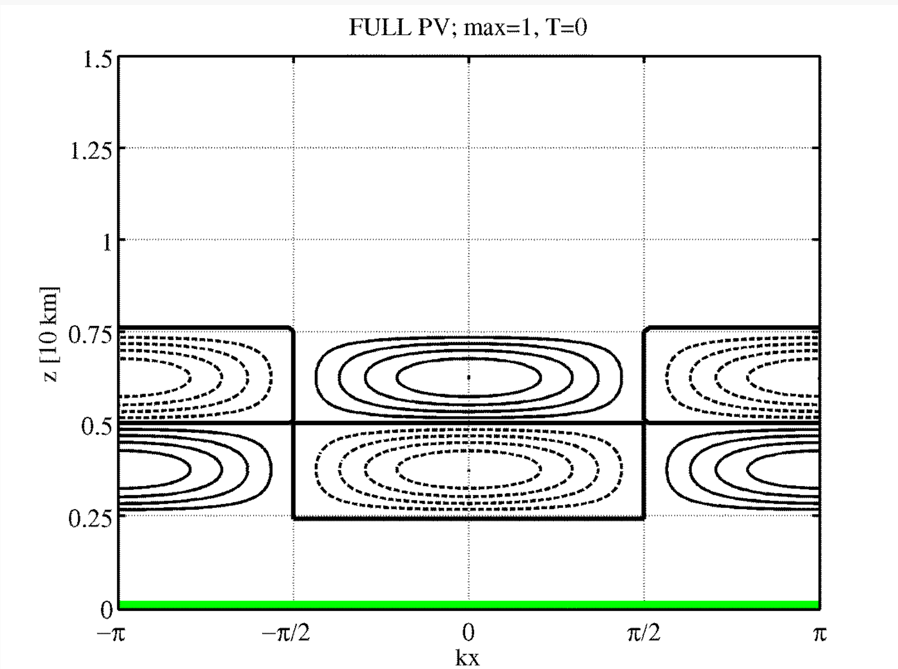


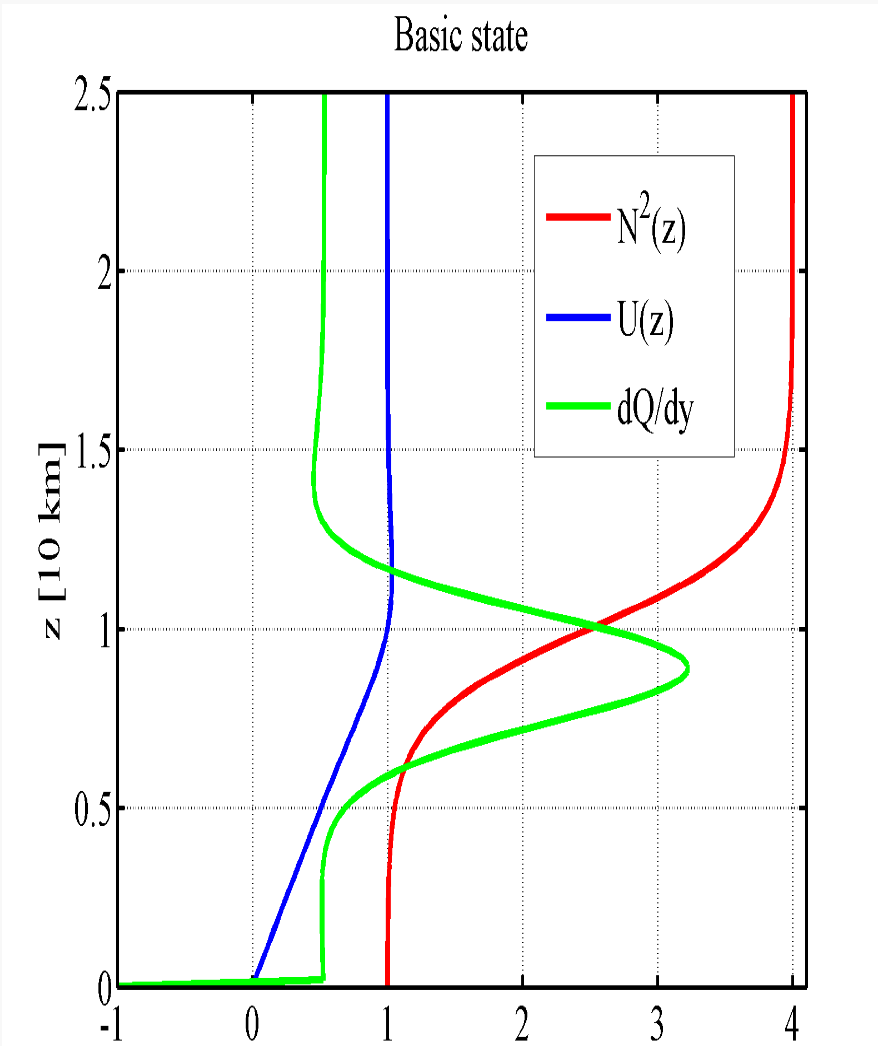
- Uniform vertical shear and static stability
- f-plane (no interior PV gradient)
- Rigid lid

Red = +ve boundary PV anomaly

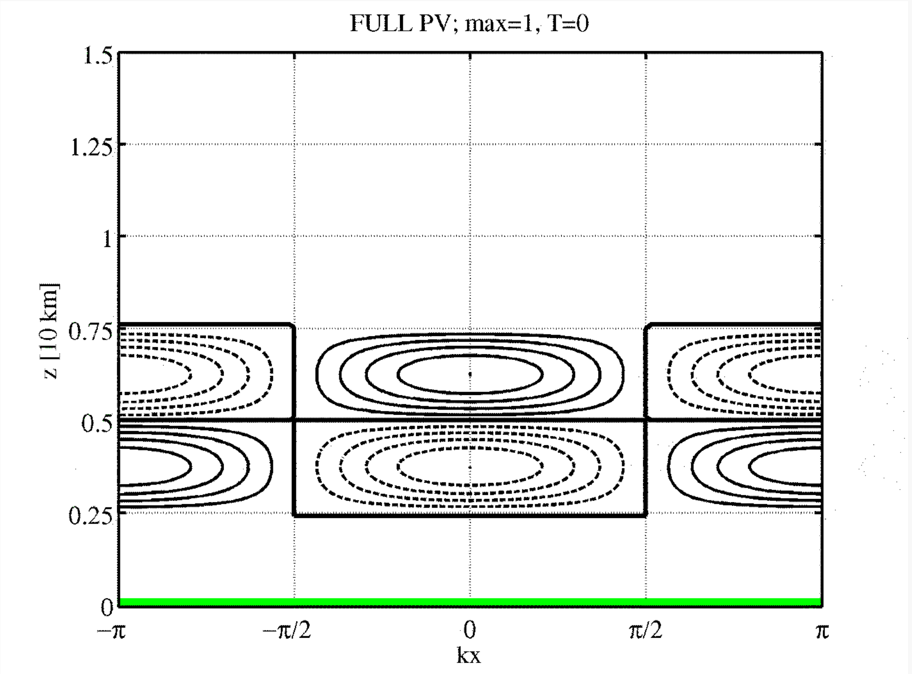
White = -ve boundary PV anomaly

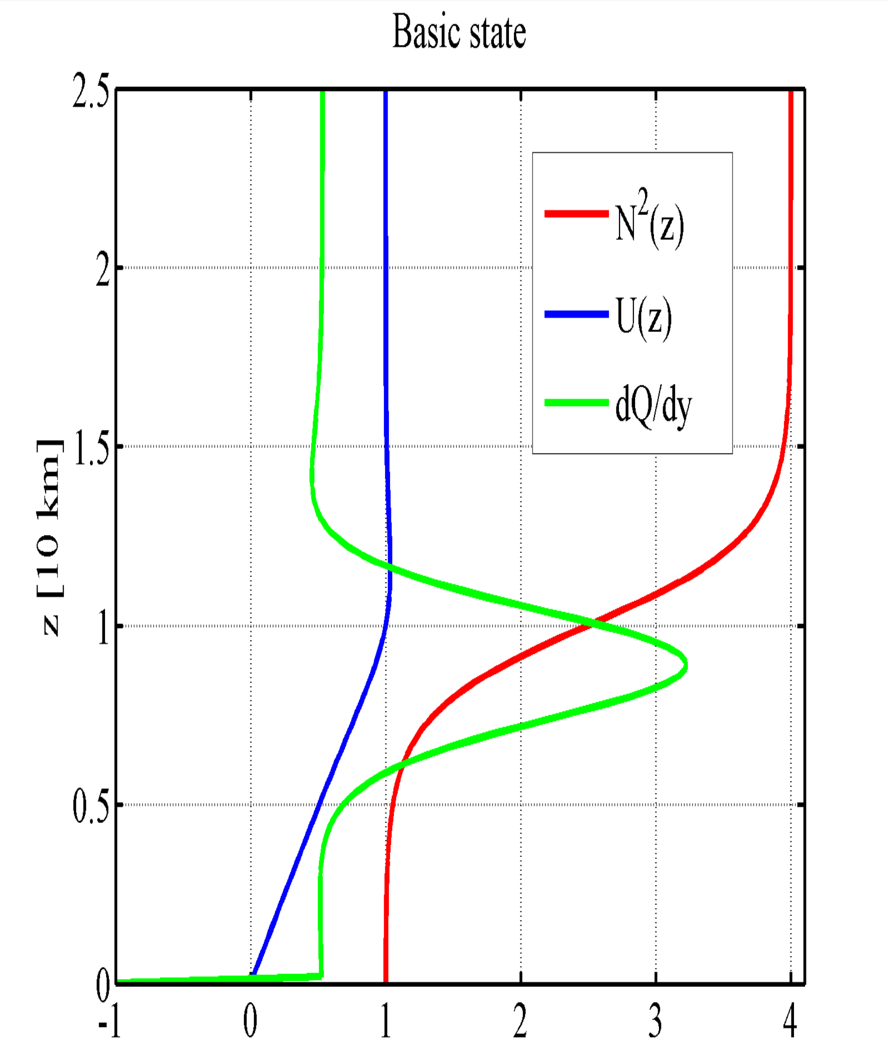
β-plane with broad tropopause (and no lid)



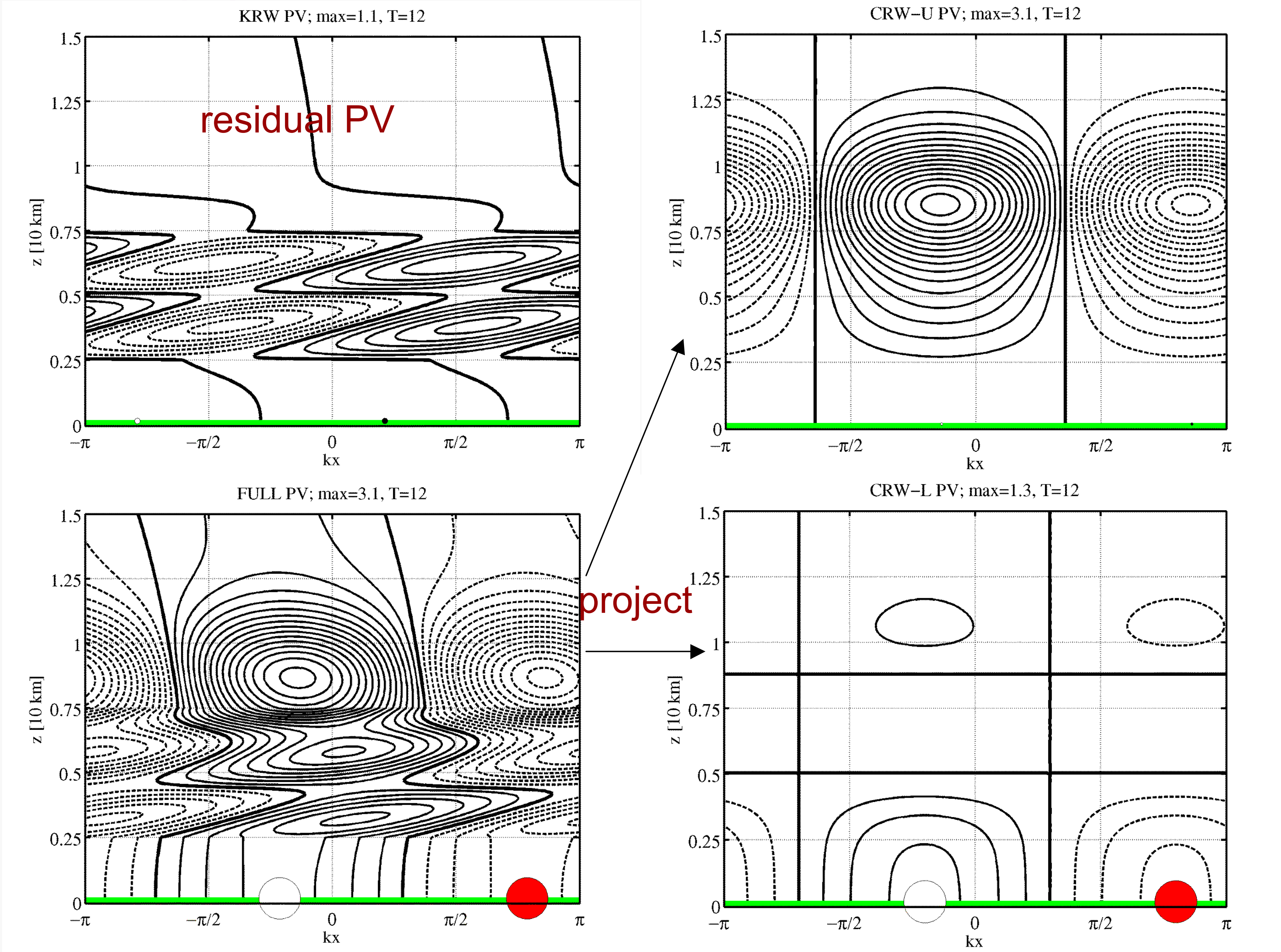


β-plane with broad tropopause (and no lid)





β-plane with broad tropopause (and no lid)



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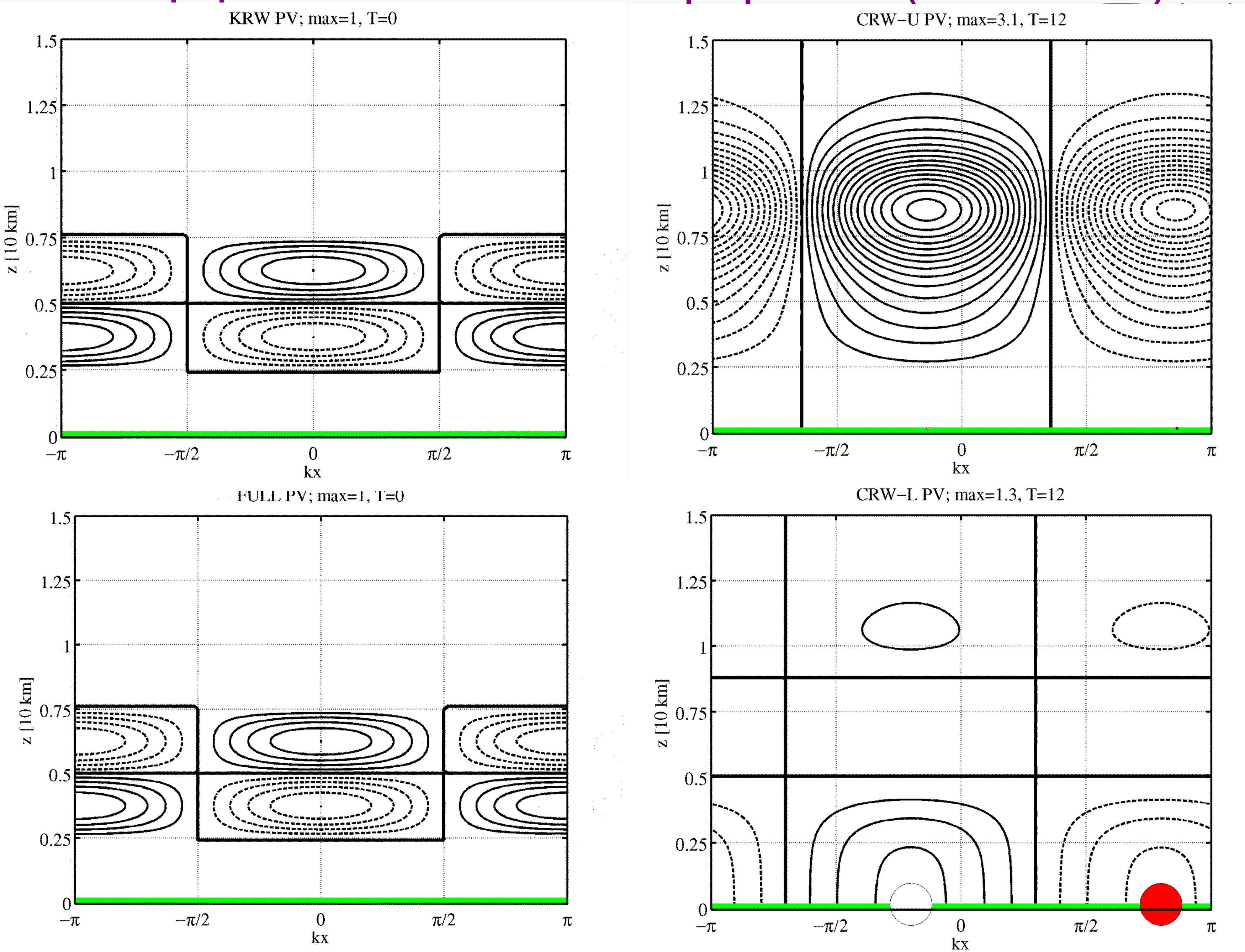
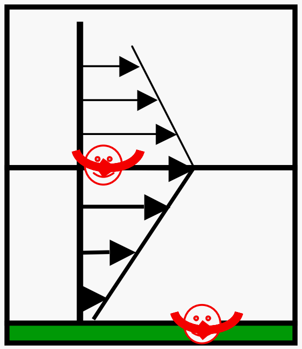


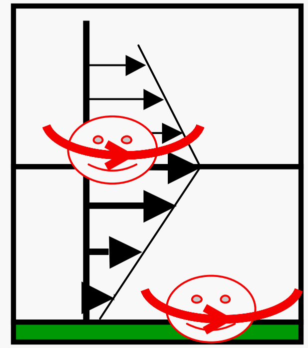
Baroclinic initial value problems: 2 CRWs + passive component

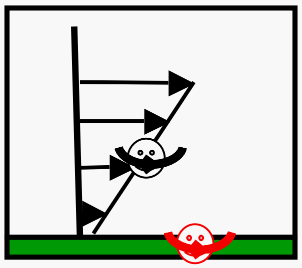
(De Vries *et al*, *JAS*, 2009)



Interaction - **two-way**

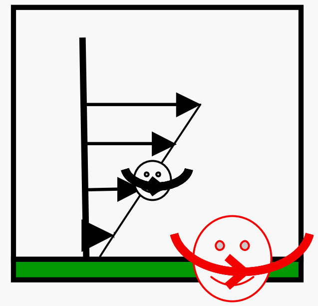
active x active => exp. growth



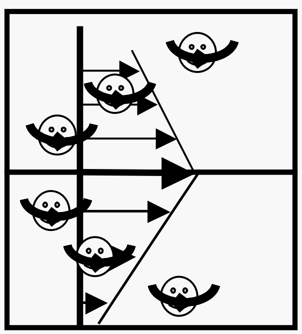


Resonance - **one-way**

passive x active => lin. growth

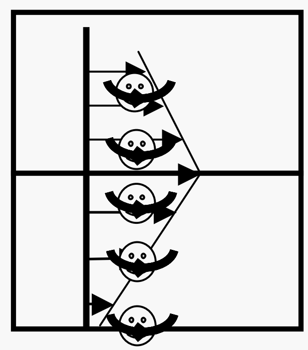


both **energy** and **enstrophy** growth



Orr-mechanism - **no-way**

passive => dep. on structure

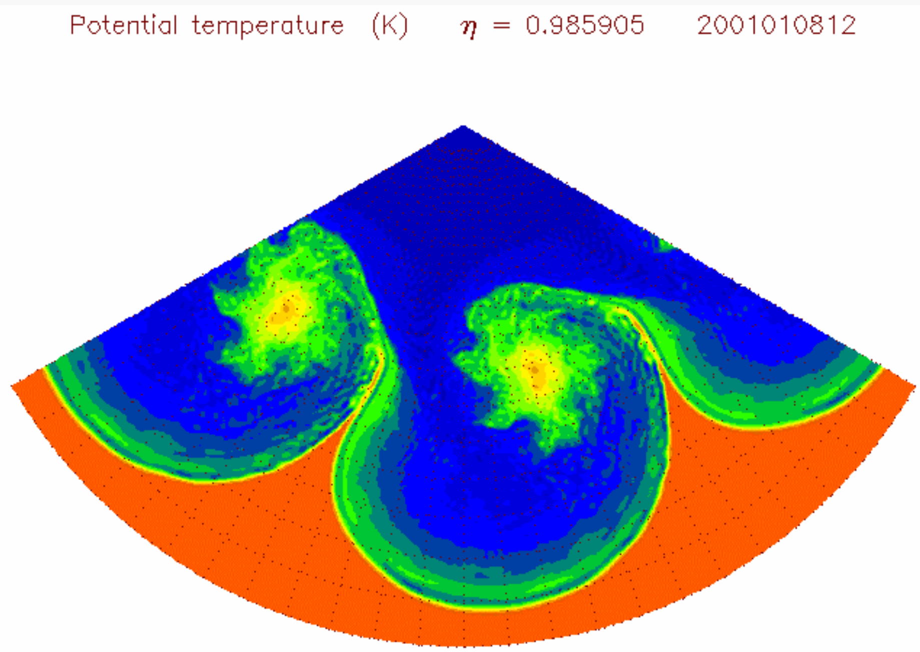


only **energy** growth

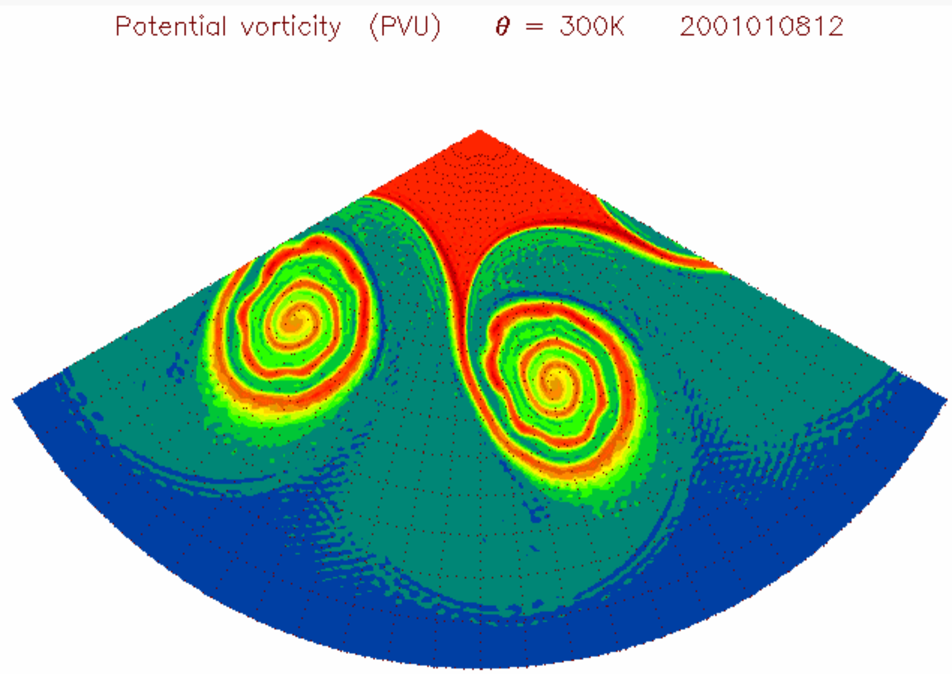
Illustration of nonlinear evolution

Idealised Baroclinic Wave Life Cycle

LC2 experiment of Thorncroft *et al* (1993), Methven (1999)



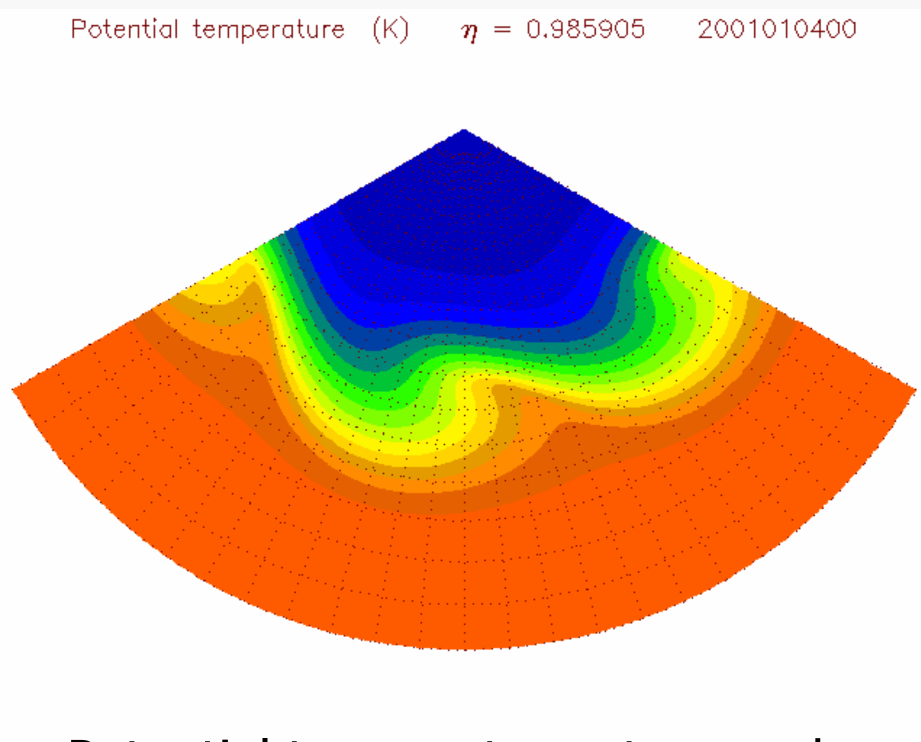
Potential temperature at ground

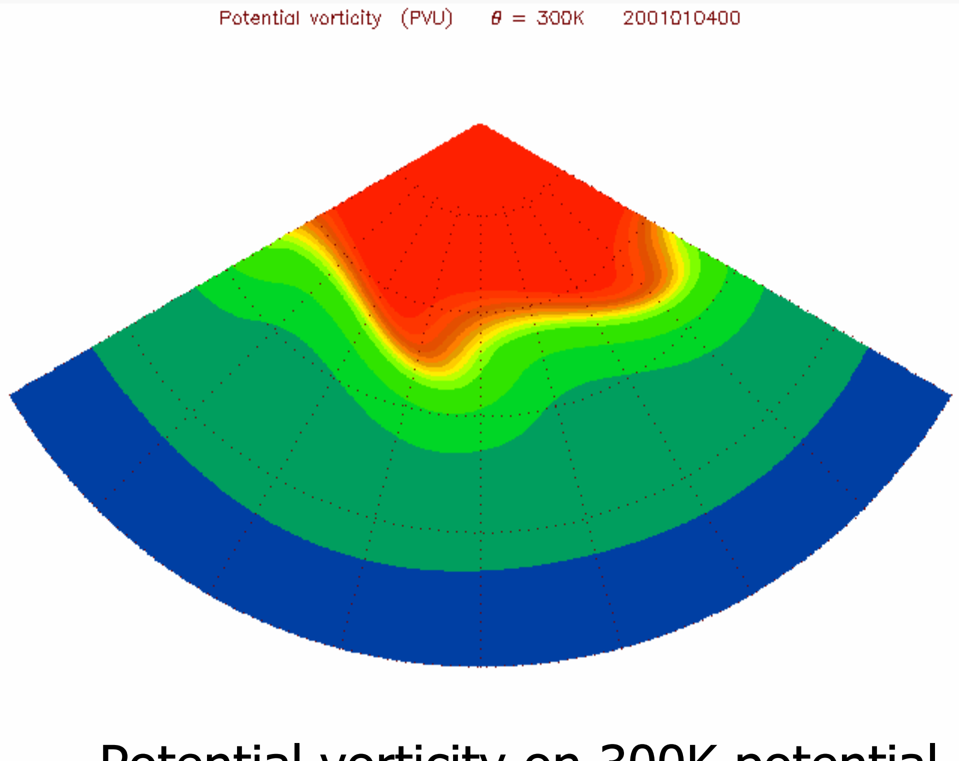


Potential vorticity on 300K potential temperature (isentropic) surface

Illustration of nonlinear evolution

Cyclonic wave breaking (LC2)





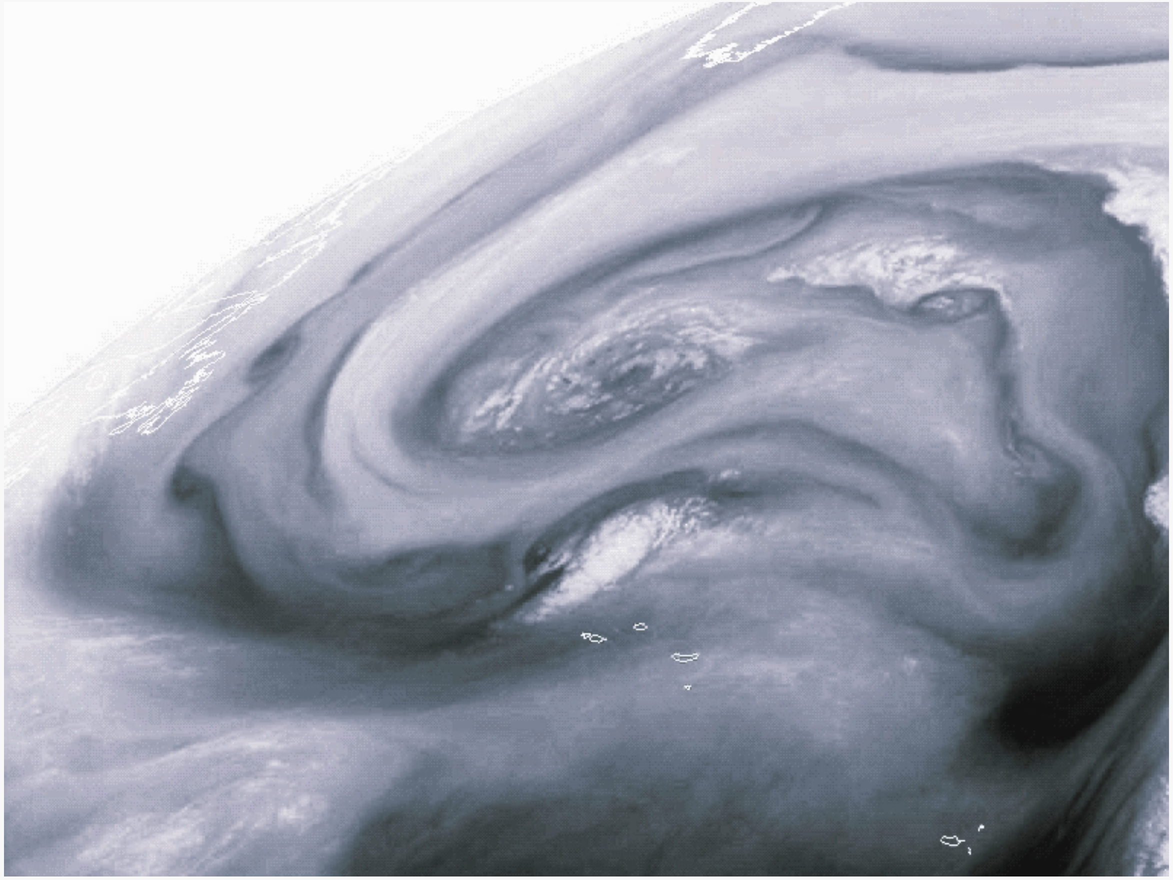
Potential temperature at ground

Lower CRW "home-base"

Potential vorticity on 300K potential temperature (isentropic) surface

Upper CRW

Big whorls have little whorls
that feed on their velocity,
and little whorls have lesser whorls and so on to viscosity.
(Lewis Fry Richardson)



- Meteosat 2nd generation captured at high temporal resolution (Eumetsat image of the month May 2006)
- Water vapour channel
  White = high cloud
  Dark = dry, clear air
- Satellite resolution (~1x1 km)

Summary of lecture

Coupling waves and shear instability

↑

Breaking Rossby waves and vortex erosion process

↑

Wave phenomena:

Gravity, Rossby, Kelvin waves

↑

Balance and potential vorticity

↑

Dynamical similarity and scale analysis

↑

Fundamental properties of fluids

8. Active Research Areas

- More accurate balance models (approximations required to obtain PV inversion relation). *Heini Wernli, Peter Lynch (Tues)*

  1) flow in tropics where geostrophic balance poor ($f \rightarrow 0$).

  2) use in numerical model and data assimilation design.

- Wave-mean flow interaction for large amplitude waves.

  1) eddy-driven jets (polar as opposed to subtropical jet).

  2) predicting Rossby wave-breaking direction.

  3) Stormtracks and blocking – prediction beyond 2 weeks.

- Non-conservative processes in waves.

  1) heating, especially latent heat release *(see second lecture)*

  2) coupling with land and ocean via boundary layer fluxes.

- High resolution forecasts (explicit convection).

  1) data assimilation and ensemble design *(lectures Weds onwards)*

  2) predictability and link with mesoscale structures *(Dale Durran)*