# Weyl's inequality and systems of forms

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Starting point:

**Theorem (Meyer 1884)**

*Let $Q \in \mathbb{Q}[X_1, \ldots, X_s]$ be an indefinite quadratic form where $s \geq 5$. Then $Q$ has a non-trivial rational zero.*

More generally, for rational quadratic forms the *Local-Global principle* holds true (Minkowski 1905): Non-trivial rational zeros exist if and only if non-trivial real and $p$-adic zeros exist.

- The condition $s \geq 5$ in Meyer's result makes sure that for all primes $p$ there are non-trivial $p$-adic zeros.
- The condition '$Q$ indefinite' makes sure that there is a non-trivial real zero.
- The 5 in the theorem is best possible.

What about systems of forms, higher degree forms?

### Theorem

*(Colliot-Thélène, Sansuc, Swinnerton-Dyer 1987) Let $Q_1, Q_2 \in \mathbb{Q}[X_1, \dots, X_s]$ be quadratic forms where $s \geq 9$. Suppose that each form in their rational pencil has rank at least 5, and that each form in their real pencil is indefinite. Then the system $Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0$ has a non-trivial rational zero $\mathbf{x}$.*

- The *pencil* of a system of forms $F_1, \dots, F_r$ is the set of all forms
$$a_1 F_1 + \ldots + a_r F_r$$
where $\mathbf{a} \neq \mathbf{0}$.
- $s \geq 9$ is needed to make sure non-trivial $p$-adic solutions exist
- 'indefinite' part of pencil condition needed to make sure non-trivial real solutions exist
- 9 is best possible

The condition rank $\geq 5$ for all forms in the rational pencil is necessary as seen by the following example (W.M. Schmidt 1982): Let

$$Q_1(X_1, \ldots, X_s) = X_1^2 + X_2^2 + X_3^2 - 7X_4^2$$

and

$$Q_2(X_1, \ldots, X_s) = X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 - \ldots - X_s^2,$$

where $s$ may be arbitrarily large. Each form in the real pencil of $Q_1, Q_2$ is indefinite, but $Q_1$ has only rank 4.
Now if $Q_1(\mathbf{x}) = 0$ for rational $\mathbf{x} \in \mathbb{Q}^s$, then necessarily

$$x_1 = \ldots = x_4 = 0.$$

Then $Q_2(\mathbf{x}) = 0$ implies that

$$x_5 = \ldots = x_s = 0.$$

Hence a lower bound on $s$ alone is not enough.

**Theorem (W.M. Schmidt 1982)**

*Let $Q_1, \ldots, Q_r \in \mathbb{Q}[X_1, \ldots, X_s]$ be quadratic forms. Suppose that*

- *each form in the rational pencil of $Q_1, \ldots, Q_r$ has rank exceeding $2r^2 + 3r$,*
- *the system $Q_1 = \ldots = Q_r = 0$ has non-singular p-adic zeros,*
- *the system $Q_1 = \ldots = Q_r = 0$ has a non-singular real zero.*

*Then the system $Q_1(\mathbf{x}) = \ldots = Q_r(\mathbf{x}) = 0$ has a non-trivial rational zero.*

Birch (1962) established a very general result: Let $F_1, \ldots, F_r \in \mathbb{Z}[X_1, \ldots, X_s]$ be forms of degree $d$, and let $V^*$ be the union of the loci of singularities of the varieties

$$F_1(\mathbf{x}) = \mu_1, \ldots, F_r(\mathbf{x}) = \mu_r.$$

Moreover, let $\mathfrak{B}$ be a box in $\mathbb{R}^s$ with sides parallel to the coordinate axes, and contained in the unit box, and let $\mathfrak{N}(P)$ be the number of integer solutions $\mathbf{x} \in \mathbb{Z}^s$ of the system

$$F_1(\mathbf{x}) = \ldots = F_r(\mathbf{x}) = 0$$

in the box $\{\mathbf{x} \in \mathbb{Z}^s \cap P\mathfrak{B}\}$. Then if

$$s > \dim V^* + r(r+1)(d-1)2^{d-1}, \tag{1}$$

then the asymptotic formula

$$\mathfrak{N}(P) = \mathfrak{J}\mathfrak{S}P^{s-rd} + O(P^{s-rd-\delta})$$

holds true.

Here $\mathfrak{J}$ is the *singular integral*, and $\mathfrak{S}$ is the *singular series*.
Interpretation of $\mathfrak{S}$ and $\mathfrak{J}$:

- $\mathfrak{S}$ is a measure for the density of $p$-adic solutions of $F_1 = \ldots = F_r = 0$,
- $\mathfrak{J}$ is a measure for the density of real solutions of $F_1 = \ldots = F_r = 0$.

Assuming that

- $F_1 = \ldots = F_r = 0$ has a *non-singular* $p$-adic solution for all primes $p$,
- $F_1 = \ldots = F_r = 0$ has a *non-singular* real solution,

one can show that

$$\mathfrak{J} > 0, \ \mathfrak{S} > 0$$

and deduces that

$$\mathfrak{N}(P) \to \infty \quad (P \to \infty).$$

Usually, $V^*$ is difficult to describe, and one would prefer a condition which is easier to handle.
Need some more notation: For a rational cubic form $C(X_1, \ldots, X_s)$, its *h-invariant* is the smallest non-negative integer $k$ such that $C$ can be written as

$$C = \sum_{i=1}^{k} Q_i L_i$$

for suitable rational quadratic forms $Q_i(X_1, \ldots, X_s)$ and rational linear forms $L_i(X_1, \ldots, X_s)$.

## Theorem (W.M. Schmidt 1982)

*Let* $Q_1, \ldots, Q_r \in \mathbb{Z}[X_1, \ldots, X_s]$ *be quadratic forms. Suppose that each form in the rational pencil of* $Q_1, \ldots, Q_r$ *has rank exceeding* $2r^2 + 3r$*. Then in the notation from above,*

$$\mathfrak{N}(P) = \mathfrak{J}\mathfrak{S}P^{s-2r} + O(P^{s-2r-\delta}).$$

*Likewise, if* $C_1, \ldots, C_r \in \mathbb{Z}[X_1, \ldots, X_s]$ *are cubic forms, such that each form in their rational pencil has h-invariant exceeding* $10r^2 + 6r$*, then*

$$\mathfrak{N}(P) = \mathfrak{J}\mathfrak{S}P^{s-3r} + O(P^{s-3r-\delta}).$$

Birch's condition (1) reads

- $s > \dim V^* + 2r^2 + 2r$ for $d = 2$,
- $s > \dim V^* + 8r^2 + 8r$ for $d = 3$,

so one might wonder if Schmidt's rank- and $h$-invariant bounds $2r^2 + 3r$ and $10r^2 + 6r$ can be relaxed to $2r^2 + 2r$ and $8r^2 + 8r$, respectively. This is indeed the case.

## Theorem (D. 201?)

*Let* $Q_1, \ldots, Q_r \in \mathbb{Z}[X_1, \ldots, X_s]$ *be quadratic forms, such that each form in their rational pencil has rank exceeding* $2r^2 + 2r$*. Then in the notation from above, the asymptotic formula*

$$\mathfrak{N}(P) = \mathfrak{J}\mathfrak{S}P^{s-2r} + O(P^{s-2r-\delta})$$

*holds true. Likewise, if* $C_1, \ldots, C_r \in \mathbb{Z}[X_1, \ldots, X_s]$ *are cubic forms, such that each form in their rational pencil has h-invariant exceeding* $8r^2 + 8r$*, then*

$$\mathfrak{N}(P) = \mathfrak{J}\mathfrak{S}P^{s-3r} + O(P^{s-3r-\delta}).$$

**Theorem (D. 2004)**

*Let p be a rational prime, and let* $Q_1, \ldots, Q_r \in \mathbb{Q}_p[X_1, \ldots, X_s]$ *be quadratic forms such that each form in their p-adic pencil has rank exceeding*

$$\begin{cases} 2r^2 & r \text{ even} \\ 2r^2 + 2 & r \text{ odd.} \end{cases}$$

*Then the system*

$$Q_1(\mathbf{x}) = \ldots = Q_r(\mathbf{x}) = 0$$

*has a non-singular p-adic solution* $\mathbf{x} \in \mathbb{Q}_p^s$.

## Corollary

*Let $Q_1, \ldots, Q_r \in \mathbb{Q}[X_1, \ldots, X_s]$ be quadratic forms. Suppose that each form in the complex pencil of $Q_1, \ldots, Q_r$ has rank exceeding $2r^2 + 2r$. Further assume that the system $Q_1 = \ldots = Q_r = 0$ has a non-singular real zero. Then the system $Q_1 = \ldots = Q_r = 0$ has a non-trivial rational zero.*

For $r = 1$ one gets back Meyer's Theorem.
The Corollary follows from the theorems on the previous two slides and the observation that the $2r^2 + 2r$ pencil condition over $\mathbb{C}$ also implies a $2r^2 + 2r$ pencil condition over $\mathbb{Q}$ as well as over all $\mathbb{Q}_p$.

The proof uses the *Hardy-Littlewood circle method* from Analytic Number Theory. Basic idea: Let

$$e(x) = e^{2\pi i x}.$$

Then for $\mathbf{n} \in \mathbb{Z}^n$, we have

$$\int_{[0,1]^r} e(\mathbf{n}\mathbf{x})\, d\mathbf{x} = \begin{cases} 1 & \text{if } \mathbf{n} = \mathbf{0} \\ 0 & \text{if } \mathbf{n} \neq \mathbf{0}. \end{cases}$$

Hence

$$\mathfrak{N}(P) = \int_{[0,1]^r} S(\boldsymbol{\alpha})\, d\boldsymbol{\alpha},$$

where $S(\boldsymbol{\alpha}) = S(\alpha_1, \ldots, \alpha_r)$ is the *exponential sum*

$$S(\boldsymbol{\alpha}) = \sum_{\mathbf{x} \in P\mathfrak{B}} e(\alpha_1 F_1(\mathbf{x}) + \ldots + \alpha_r F_r(\mathbf{x})).$$

Philosophy: If all $\alpha_i$ are 'close to a rational point', then $S(\boldsymbol{\alpha})$ can be asymptotically evaluated. Otherwise, $S(\boldsymbol{\alpha})$ is 'small'. Ideally, this gives an asymptotic formula for $\mathfrak{N}(P)$.

To keep notation simple, focus on quadratics now. Both Birch and Schmidt used the following form of Weyl's inequality.

## Lemma (Weyl's inequality for systems of quadratic forms)

*Let $0 \leq \theta < 1$, $\epsilon > 0$ and $k > 0$. Then we either (i) have*

$$S(\boldsymbol{\alpha}) \ll P^{s-k},$$

*or (ii) there are integers $a_1, \ldots, a_r, q$ such that*

$$\begin{aligned} (a_1, \ldots, a_r, q) &= 1, \\ |q\alpha_i - a_i| \ll P^{-2+r\theta} \quad (1 \leq i \leq r&), \\ 1 \leq q \leq P^{r\theta}&, \end{aligned}$$

*or (iii) we have*

$$\#\{\mathbf{x} \in P^{\theta}\mathfrak{B} : rank(\Psi_j^{(i)}(\mathbf{x})) < r\} \gg (P^{\theta})^{s-2k/\theta-\epsilon}$$

*where*

$$\Phi_j(\mathbf{a};\mathbf{x}) = \sum_{i=1}^{r} a_i \Psi_j^{(i)}(\mathbf{x}) \quad (1 \le j \le s),$$

$$\Psi_j^{(i)}(\mathbf{x}) = 2\sum_{k=1}^{s} c_{j,k}^{(i)} x_k \quad (1 \le i \le r, 1 \le j \le s),$$

$$Q_i(X_1,\dots,X_s) = \sum_{j,k=1}^{s} c_{jk}^{(i)} X_j X_k \quad (1 \le i \le r).$$

The main tool for proving Weyl's inequality is Cauchy-Schwarz' inequality. 'Differentiating' a quadratic expression yields a linear one, and this is the reason why the linear forms $\Psi$ and $\Phi$ occur.

Alternative (iii) can be given a more suitable interpretation for systems of forms.

## Lemma (Weyl's inequality for systems of quadratic forms II)

*Let $0 \leq \theta < 1$, $\epsilon > 0$ and $k > 0$. Then we either (i) have*

$$S(\boldsymbol{\alpha}) \ll P^{s-k},$$

*or (ii) there are integers $a_1, \ldots, a_r, q$ such that*

$$\begin{aligned} (a_1, \ldots, a_r, q) &= 1, \\ |q\alpha_i - a_i| \ll P^{-2+r\theta} \quad (1 \leq i &\leq r), \\ 1 \leq q &\leq P^{r\theta}, \end{aligned}$$

*or (iii) there are integers $a_1, \ldots, a_r$, not all zero, such that*

$$\mathfrak{M}(a_1, \ldots, a_r; P^\theta) \gg (P^\theta)^{s-2k/\theta-\epsilon}$$

*where*

$$\mathfrak{M}(a_1, \ldots, a_r; H) = \#\{\mathbf{x} \in \mathbb{Z}^s : \mathbf{x} \in H\mathfrak{B} \text{ and } \Phi_j(\mathbf{a}; \mathbf{x}) = 0 \ (1 \leq j \leq s)\},$$

Clearly, the larger the dimension of the span of $\Phi_1, \ldots, \Phi_s$ in the space of linear forms in $\mathbf{x}$, the smaller $\mathfrak{M}(a_1, \ldots, a_r; H)$. That dimension can be controlled by the smallest rank in the pencil of $Q_1, \ldots, Q_r$.

## Corollary

*Suppose that each quadratic form in the rational pencil of $Q_1, \ldots, Q_r$ has rank at least $m$. Then, using the notation from above, we either (i) have*

$$S(\alpha) \ll P^{s-m\theta/2},$$

*or alternative (ii) holds true.*

So alternative (iii) got eliminated. The rest is a lengthy, but straightforward application of the circle method.

Now let $A$ be a non-singular positive definite symmetric integer $n \times n$-matrix, and $B$ be a positive definite symmetric integer $m \times m$-matrix. The matrix equation

$$X^t A X = B \tag{2}$$

corresponds to the representation of a quadratic form $B$ by a quadratic form $A$. Let $N(A, B)$ be the number of integer solutions $X$ of (2).

For fixed $A$, interested in asymptotic formula for $N(A, B)$.

Case $m = 1$ has long history; $m > 1$ more difficult, also need to define what it means that $B$ is 'large enough' (in terms of $A$).

Let

$$\min B = \min_{\mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}} \mathbf{x}^t B \mathbf{x}$$

be the *first successive minimum of* $B$. We can only expect an asymptotic formula for $N(A, B)$ if $\min B$ is sufficiently large for given $A$.
In a similar way, can define second successive minimum etc.
If

$$\min B \gg (\det B)^{1/m},$$

then all successive minima of $B$ are roughly of the same size.
Using Siegel modular forms, Raghavan (1959) proved the following

**Theorem (Raghavan (1959))**

*Let $c > 0$ and $n > 2m + 2$. Then if*

$$\min B \geq c(\det B)^{1/m},$$

*then for* $\det B \gg_c 1$ *we have*

$$N(A, B) = \mathfrak{J}\mathfrak{S}(\det B)^{(n-m-1)/2} + O((\det B)^{(n-m-1)/2-\delta}).$$

Writing (2) as a system of quadratic equations, problem can also be attacked by the circle method. Dependence on $n$ gets worse, but condition on $B$ can be relaxed!

**Theorem (D., Harvey – work in progress)**

*Let $c > 0$ and suppose that*

$$\min B \geq (\det B)^c.$$

*Then there exists $N(c) \in \mathbb{N}$ such that if $n \geq N(c)$ and $\det B \gg_c 1$, then*

$$N(A, B) = \mathfrak{J}\mathfrak{S}(\det B)^{(n-m-1)/2} + O((\det B)^{(n-m-1)/2-\delta}).$$