The unreasonable effectiveness of tensor product.

Renaud Coulangeon, Université Bordeaux 1

based on a joint work with Gabriele Nebe

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- $\min(L \otimes M) = \min L \cdot \min M$? NO in general (one has to consider *non-split* vectors $\sum_{i=1}^{t} x_i \otimes y_i$ for $t > 1$).

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Remark : If one considers the similar problem for the tensor product of (Hermitian) lattices over the ring of integers of an imaginary quadratic field, explicit examples with

$$\min \left(L \otimes_{\mathcal{O}_K} M\right) < \min L \min M$$

are relatively easy to construct in small dimension.

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Definition

A (resp. L) is **perfect** if

$$\mathrm{Span}\,\{XX', X \in S(A)\} = \mathrm{S_n}(\mathbb{R}).$$

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Proof : set $\ell = \dim L$, $m = \dim M$. Kitaoka's result implies that the minimal vectors of $L \otimes M$ are *split*. Consequently, setting $r_{L\otimes M} = \dim \mathrm{Span}\,\{(X \otimes Y)(X \otimes Y)'\ ,\ X \otimes Y \in S(L \otimes M)\}$ one has

$$r_{L\otimes M} \leq \frac{\ell(\ell+1)}{2}\frac{m(m+1)}{2} < \frac{\ell m(\ell m+1)}{2}.$$

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In particular, there is no hope to obtain extremal modular lattices in this way.

Tensor product of Hermitian lattices

$K/\mathbb{Q}$ an imaginary quadratic field, with ring of integers $\mathcal{O}_K$.

$\mathcal{D}_{K/\mathbb{Q}}$ (resp. $\mathfrak{d}_K$) its different (resp. discriminant).

$V \simeq K^m$ endowed with a positive definite Hermitian form h.

L a Hermitian lattice *i.e.*

$$L = \mathfrak{a}_1 e_1 \oplus \cdots \oplus \mathfrak{a}_m e_m,$$

where $\{e_1, \ldots, e_m\}$ is a K-basis of $V \simeq K^m$ and the $\mathfrak{a}_i$s are fractional ideals in K.

The *discriminant* of a pseudo-basis $\{e_1, \ldots, e_m\}$ is $\det\left(h(e_i, e_j)\right)$.

For any $1 \leq r \leq m = \operatorname{rank}_{\mathcal{O}_K} L$ we define $d_r(L)$ as the minimal discriminant of a free $\mathcal{O}_K$-sublattice of rank r of L. In particular, one has $d_1(L) = \min(L) := \min\{h(v, v) \mid 0 \neq v \in L\}$.

The (Hermitian) dual of a Hermitian lattice L is defined as

$$L^{\#} = \{y \in V \mid h(y, L) \subset \mathcal{O}_K\}\,.$$

By restriction of scalars, an $\mathcal{O}_K$-lattice of rank m can be viewed as a $\mathbb{Z}$-lattice of rank $2m$, with inner product defined by

$$x \cdot y = \mathrm{Tr}_{K/\mathbb{Q}}\, h(x, y).$$

The dual L^* of L with respect to that inner product is linked to $L^{\#}$ by

$$L^* = \mathcal{D}_{K/\mathbb{Q}}^{-1} L^{\#}.$$

The minimum of L, viewed as an ordinary $\mathbb{Z}$-lattice, is twice its "Hermitian" minimum $d_1(L)$.

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The following proposition allows for an estimation of the minimal Hermitian norm of a tensor product $L \otimes_{\mathcal{O}_K} M$:

Proposition

Let L and M be Hermitian lattices. Then for any vector $z \in L \otimes_{\mathcal{O}_K} M$ of rank r one has

$$h(z, z) \geq r\, d_r(L)^{1/r} d_r(M)^{1/r}. \tag{1}$$

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Proof : Arithmetic-geometric mean inequality. □

An extremal unimodular lattice in dimension 72

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The *Barnes lattice* P_b is a Hermitian lattice of rank 3 over $\mathbb{Z}[\alpha]$, with Hermitian Gram matrix

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Fact :

1. $d_1(P_b) = 2$.
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3. $d_3(P_b) = 1$.

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The (Hermitian) minimum of the lattices R_i is either 3 or 4. The number of vectors of norm 3 in R_i is equal to the representation number of P_i for the sublattice P_b. In particular $\min(R_i) = 4$ *if and only if the Hermitian Leech lattice P_i does not contain a sublattice isomorphic to P_b.*

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Proof : One checks easily that $d_1(R_i) = 2$ and $d_2(R_i) = \dfrac{12}{7}$. Together with the values of $d_1(P_b)$ and $d_2(P_b)$ computed before, it shows that vectors of rank 1 and 2 have Hermitian norm at least 4. As for vectors of rank 3, one checks easily that they have norm at least 3, and the case of equality is analysed via the previous proposition. □

To summarize, one has, for each of the nine Hermitian structures $P_1, \ldots, P_9$ of the Leech lattice over $\mathbb{Z}[\alpha]$, the following alternative :

- either P_i contains a sublattice isometric to P_b, in which case $R_i := P_b \otimes_{\mathbb{Z}[\alpha]} P_i$ is not extremal ($\min R_i = 3$)
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Question : can one find a more direct argument to prove that one of the P_i, say P_1, does not contain any sublattice isometric to P_b while the eight others do ?

Slopes of lattices, tensor product of semi-stable lattices.

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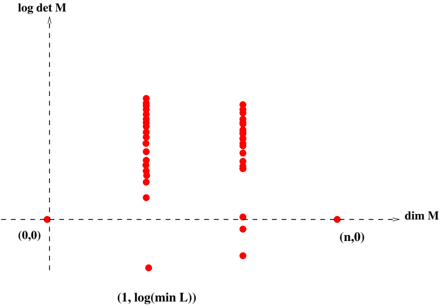

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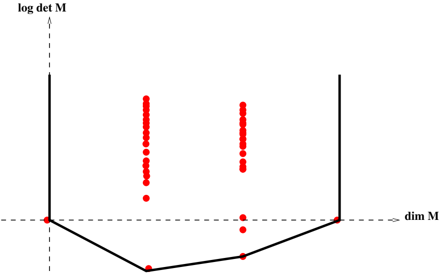

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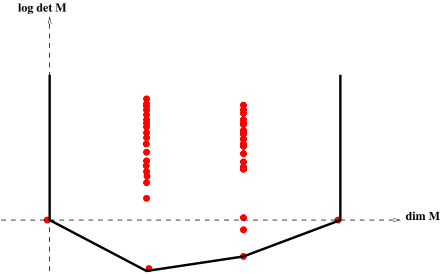

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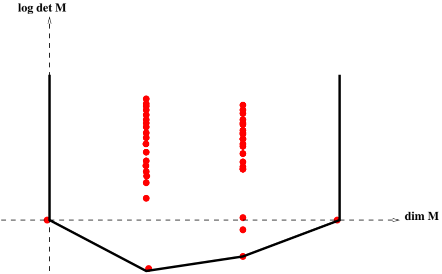

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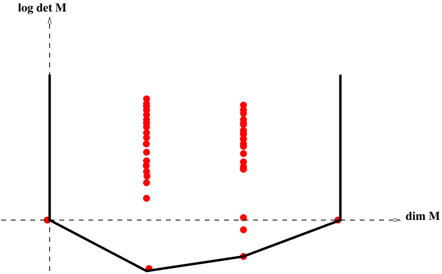

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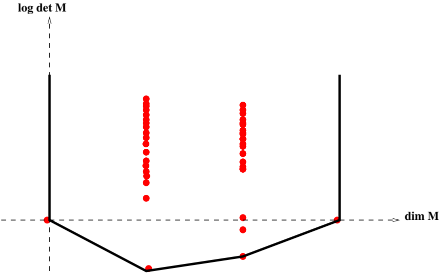

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$$\text{minimal slope} = \min_{M \subset L} \frac{\log\det M}{\dim M} = \log\min_k (d_k L)^{1/k}$$

($d_k L$ = minimal determinant of k-dimensional sublattices of L)

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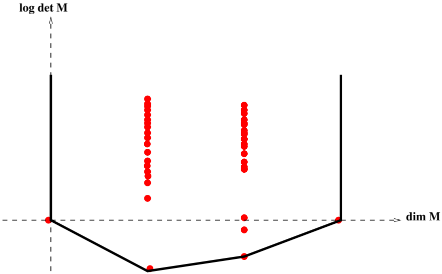
- 

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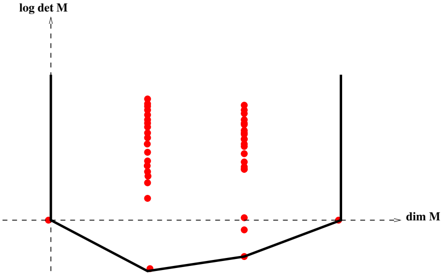
- 

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There exists a unique sublattice M_0 of L such that

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When $\mu(L) = \det L$ (*i.e.* $M_0 = L$), we say that L is *semi-stable*.

Conjecture (Bost)

For any lattices L and M, one has

$$\mu(L \otimes M) = \mu(L)\mu(M).$$

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- For further information on this conjecture, see Yves André *On nef and semistable hermitian lattices, and their behaviour under tensor product* `http://arxiv.org/abs/1008.1553`