Understanding the (total) mass of quadratic forms of fixed determinant

(Jonathan Hanke)
Understanding the (total) mass of quadratic forms of fixed determinant:

Let $Q$ be a positive definite $\mathbb{Z}$-valued quadratic form in $n$ variables. We say that two quadratic forms are equivalent over a ring $R$ if one can obtained from the other by an invertible linear change of variables over $R$, and write $Q \sim_R Q'$. We denote the $\mathbb{Z}$-equivalence class of $Q$ by $[Q]$, and define the genus of $Q$, denoted $\text{Gen}(Q)$, as all quadratic forms $Q'$ s.t. $Q' \sim_{\mathbb{Z}[p]} Q$ for all primes $p$ and $Q' \sim_{\mathbb{Q}} Q$.

We define the mass of $Q$ by

$$\text{Mass}(Q) := \sum_{[Q] \in \text{Gen}(Q)} \frac{1}{\#\text{Aut}(Q)}$$

which is known to be a finite sum of rational numbers.
Facts about the mass:

- Mass \( (Q) \) is a genus invariant that can be computed as an infinite product of local densities

\[
\text{Mass} \,(Q) = \pi \prod_v \beta_v (Q)^{-1}
\]

where

\[
\beta_v (Q) := \frac{1}{2} \lim_{\substack{u \to \mathcal{O}_v \\ u \leq M_n (\mathbb{Z}_v)}} \frac{\text{vol}(Q^{(1)}(u))}{\text{vol}(u)}
\]

and \( Q \leftrightarrow A \in \text{Sym}_n (\mathbb{Z}_v) \) defines a map

\[
M_n (\mathbb{Z}_v) \xrightarrow{\bar{\alpha}} \text{Sym}_n (\mathbb{Z}_v)
\]

\[
X \mapsto \epsilon XAX.
\]

When \( v = p \) prime, we can write \( \beta_p (Q) \) as

\[
\beta_p (Q) = \frac{1}{2} \lim_{\alpha \to \infty} \frac{\# \{ X \in M_n (\mathbb{Z}_p) \mid \epsilon XAX = A \}}{p^{\frac{n(n-1)}{2}} \cdot \alpha}.
\]

- There are explicit local formulas for \( \text{Mass} \,(Q) \), though the literature has room for improvement.
The Conway-Sloane explicit mass formula:

Suppose \( Q \) is pos. def. \( \mathbb{Z} \)-valued in \( n \geq 3 \) variables.

\[
\text{Mass}(Q) = 2 \cdot \prod_{j=1}^{n} \sqrt{j!} \left( \prod_{p} 2 \alpha_p(Q) \right)
\]

where the \( p \)-masses \( \alpha_p(Q) \) are defined by

\[
\alpha_p(Q) := \left[ \prod_{g \in \mathbb{Z}^*} M_p(Q_g) \right] \cdot \left[ \prod_{g, g' \in \mathbb{Z}^*} \left( \frac{\nu(g)}{\nu(g')^2} \right)^{n(g) n(g')} \right] \cdot 2^{n(II) - n(III)}
\]

where \( M_p(Q_g) \) is an explicit local Euler factor of a product of \( S \) and \( L \)-functions, the \( n(g) \) are the Jordan dimensions of scale \( g \), and \( n(II) \) and \( n(III) \) come from the Jordan splitting at \( p = 2 \).

What is the mass good for?

- Can study the \# of classes in \( \text{Gen}(Q) \) locally (given bounds for sizes of automorphism groups).

- Enumeration of classes in a given Genus \( \text{Gen}(Q) \).
In this talk we will be interested in the total mass of given (Hessian) determinant $D$ for positive definite quadratic forms in $n$ variables, defined by

$$TMass_n(D) := \sum_{[Q] \text{ with } \det_h(Q) = D \atop \dim(Q) = n} \frac{1}{\# \text{Aut}(Q)}$$

$$= \sum_{\text{Genus } G \text{ with } \det_h(G) = D \atop \dim(G) = n} \text{Mass}(G).$$

Main Questions:

• How can we understand $TMass_n(D)$?

• Does this have any natural structure?

• How does $TMass_n(D)$ behave as $D \to \infty$?

(Remark: The total mass can also be defined over any field $F$ by additionally specifying a signature $\sigma$.)

Why study the total mass?

• It's a coarser invariant than Mass($Q$), so it might be simpler to understand.

• It is related to the average size of the 2-torsion in class groups of $n$-monogenized cubic rings.

• It generalizes the Hurwitz class numbers for positive definite binary quadratic forms (which give an interesting weight $3/2$ modular form).

• It is related to certain Shintani Zeta functions.
To understand the total mass it is useful to adopt the language of formal series.

We define the formal Dirichlet series

\[ D_{\text{Mass},n}(s) := \sum_{D \in \mathbb{D} \cap \mathbb{N}} T_{\text{Mass},n}(D) \cdot D^{-s}. \]

**Main Question:** What can we say about the structure of \( D_{\text{Mass},n}(s) \)?

**(H.) Result #1:** If \( n \geq 3 \) is odd, then

\[ D_{\text{Mass},n}(s) = \mathcal{K}_n \left[ D_{A,n}(s) + D_{B,n}(s) \right] \]

where \( \mathcal{K}_n \) is some explicit constant and both Dirichlet series \( D_{A,n}(s) \) and \( D_{B,n}(s) \) are given as Euler products.

**(Remark:)** When \( n \) is even a similar, but more complicated result holds.
Strategy of Proof: Sum up the masses of all genera of determinant $D$, as locally as possible.

\[ TMass_n(D) = \sum_{\text{Genus } G} 2 \cdot \prod_{\text{dim}(G) = n} \beta_{\text{dim}(G)}^{-1}(D) \]

\[ = \left[ 2 \cdot \prod_{\text{K}_{\text{dim}(G)}} \beta_{\text{dim}(G)}^{-1}(D) \right] \cdot \sum_{G} \prod_{\text{dim}(G)} \frac{\beta_{\text{dim}(G)}(D)}{\beta_{\text{dim}(G)}(G)} \]

\[ = \left[ 2 \cdot \mathcal{K}_n(D) \right] \cdot \sum_{G} \prod_{\text{dim}(G)} \frac{\beta_{\text{dim}(G)}(D)}{\beta_{\text{dim}(G)}(G)} \]

Call this $\mathcal{K}_{n}(D)$

Look at these genera as tuples of local genera.

\[ = 2 \mathcal{K}_n(D) \cdot \sum_{(\epsilon_p)_{p \in \mathcal{P}}} \sum_{\text{Tuples } (\epsilon_p)_{p \in \mathcal{P}}} \prod_{p \in \mathcal{P}} \frac{\beta_{\text{dim}(G_p)}(D)}{\beta_{\text{dim}(G_p)}(G_p)} \]

\[ = 2 \mathcal{K}_n(D) \cdot \sum_{(\epsilon_p)_{p \in \mathcal{P}}} \prod_{p \in \mathcal{P}} \frac{\beta_{\text{dim}(G_p)}(D)}{\beta_{\text{dim}(G_p)}(G_p)} \]

Call this $M_{\epsilon_p}(D_p)$
We now isolate the dependence on $\varepsilon_p \ast$ by defining

\[ A_{p,n}(D_p) := M_{p,n}^+(D_p) + M_{p,n}^-(D_p) \]

\[ B_{p,n}(D_p) := M_{p,n}^+(D_p) - M_{p,n}^-(D_p) \]

\[ \Rightarrow M_{p,n}^{\varepsilon_p}(D_p) = \frac{A_{p,n}(D_p) + \varepsilon_p B_{p,n}(D_p)}{2} \]

\[ \Rightarrow T\text{Mass}_n(D) = 2 \mathcal{H}_n(D) \sum_{(\varepsilon_p)_{p \in \mathbb{P}}} \prod_{p \in \mathbb{P}} \frac{A_{p,n}(D) + \varepsilon_p B_{p,n}(D)}{2} \]

**Useful Lemma:** Let $\mu_N$ be the $N^{\text{th}}$ roots of unity, $\mathbb{P}$ a finite set, and $x_i, y_i$ be indeterminates for all $i \in \mathbb{P}$. Then

\[ \sum_{(\varepsilon_i)_{i \in \mathbb{P}}} \prod_{i \in \mathbb{P}} (x_i + \varepsilon_i y_i) = N^{\prod_{i \in \mathbb{P}} i - 1} \left[ \prod_{i \in \mathbb{P}} x_i + \frac{\nu \prod_{i \in \mathbb{P}} y_i}{i} \right] \]

with $\prod_{i \in \mathbb{P}} \varepsilon_i = 1$.

\[ \Rightarrow T\text{Mass}_n(D) = 2 \mathcal{H}_n(D) \cdot \frac{2^{\prod_{i \in \mathbb{P}} i - 1}}{2^{\prod_{i \in \mathbb{P}} 1}} \cdot \left[ \prod_{p \in \mathbb{P}} \frac{A_{p,n}(D) + \varepsilon_p B_{p,n}(D)}{A_n(D) + B_n(D)} \right] \]

\[ = \mathcal{H}_n(D) \cdot \left[ A_n(D) + B_n(D) \right]. \]

(When $n$ is odd $\mathcal{H}_n(D)$ is independent of $D$.)
Ok, but can we say anything about the Euler factors of $D_{A,n}(s)$ and $D_{B,n}(s)$?

(1.) **Result #2:** The Euler factors of $D_{A,n}(s)$ and $D_{B,n}(s)$ are rational functions in $p^{-s}$.

**Sketch of Proof:** For any fixed $n$ there are finitely many "Jordan block structures" (i.e., ordered tuples of $n_i \in \mathbb{N}$ summing to $n$), and each gives a predictable mass contribution that varies as a rational function of $p^{-s}$ as we vary the choice of scale of each block (coming from the varying "cross product" factor).

**Ex:** $n=3$, $p \neq 2$

\[
\begin{align*}
3 & \quad \rightarrow \quad L_0^{(\dim 3)} \\
2+1 & \quad \rightarrow \quad L_0^{(\dim 2)} + p^0 L_1^{(\dim 3)} \\
1+2 & \quad \rightarrow \quad L_0 + p^0 L_1^{(\dim 3)} \\
1+1+1 & \quad \rightarrow \quad L_0 + p^0 L_1 + p^{0} L_2
\end{align*}
\]
Ok, but how explicit can we make this?

(h) **Result #3:** Suppose that \( n = 3 \). Then

\[
D_{T^{\text{Mass}}, n=3}(s) = \frac{1}{48} \cdot 2^{-s} \left[ \zeta(s-1) \zeta(2s-1) - \zeta(s) \zeta(2s-2) \right].
\]

**Corollary:**

\[
T^{\text{Mass}}_{n=3}(D) = \frac{1}{48} \sum_{D=2ab^2} (ab - b^2).
\]

**Super-Sketch of Proof:** Compute \( M_{p,n=3}(D) \).

\[
p \neq 2 \implies 4 \text{ Jordan Block structures}
\]

\[
p = 2 \implies 20 \text{ "Jordan Block structures"}
\]

**Final Remarks:**

1) These results generalize to any field \( F \) where \( p = 2 \) splits completely.

2) This is essentially an explicit evaluation of a Shimura zeta function.
Thanks for staying,

and thanks again to

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Have a safe trip home!😊