Definition 1.1
A (full) lattice on a quadratic vector space \((V, b)\) over \(\mathbb{Q}\) is a subset of the shape
\[
L = \{ x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \mid x_1, \ldots, x_n \in \mathbb{Z} \}
\]
\[
= \mathbb{Z} v_1 \oplus \mathbb{Z} v_2 \oplus \cdots \oplus \mathbb{Z} v_n
\]
for some basis \(v_1, v_2, \ldots, v_n\) of \(V\).

The associated integral quadratic form is
\[
Q(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i,j} b(v_i, v_j) x_i x_j.
\]
Definition 1.2
The Gram matrix of a lattice $L$ w.r.t. a basis $v_1, \ldots, v_n$ is the symmetric $n \times n$-matrix $(b(v_i, v_j))$.
The determinant $\det L$ of $L$ is the determinant of any Gram matrix of $L$.
A quadratic lattice is called an integral lattice if $b(L, L) \subseteq \mathbb{Z}$.

Theorem 1.1 (Finiteness of Class Number)
For a given determinant $d$, the number of isometry classes of (positive definite) integral lattices with determinant $d$ is finite.

This is a consequence of reduction theory, which gives a lattice basis with $b(v_i, v_i) \leq C d^{1/n}$ for some constant $C$.

Let $p$ be a prime number. Every quadratic vector space $(V, b)$ over $\mathbb{Q}$ embeds into a quadratic vector space $(V_p, b)$ over $\mathbb{Q}_p$, its completion at the prime $p$, where $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and the natural extension $b_p : V_p \times V_p \to \mathbb{Q}_p$ is simply denoted by $b$ again.
This definition extends to $p = \infty$ with $\mathbb{Q}_\infty := \mathbb{R}$.
The (weak) local-global principle of Minkowski and Hasse for quadratic spaces says that $(V, b)$ is determined up to isomorphism by all its completions.

Similarly, a quadratic lattice $L$ embeds into its completion $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. One also sets $L_\infty = V_\infty$.
The local-global principle of Minkowski and Hasse in general does not hold for quadratic lattices. Therefore, the following notion is introduced.

Definition 1.3
Two lattices $L$ and $M$ are in the same genus if $L_p \cong M_p$ for all $p \in \mathbb{P} \cup \{\infty\}$. 

Lattices in the same genus have the same determinant. Thus:
The number \( h(G) \) of isometry classes in a genus \( G \) is finite. It is called the **class number** of the genus.

Basic task: Given a genus in terms of local data (e.g. modular decomposition, genus symbol, discriminant quadratic form on the discriminant group \( L^\# / L \)), determine a set of representatives, in particular the class number of \( G \).

**Definition 2.1 (The mass of a genus)**

Let \( L = L_1, \ldots, L_h \) be a system of representatives for a genus \( G \) of positive definite lattices of dimension \( n \). The sum of the inverses of the orders of their automorphism groups is called the **mass** of \( G \):

\[
\text{mass}(G) := \sum_{j=1}^h \frac{1}{|\text{Aut}(L_j)|}.
\]

The notion goes back to G. Eisenstein; also H.J.S Smith used it before Minkowski developed his theory.

**Theorem 2.1 (Minkowski’s mass formula)**

Let \( L = L_1, \ldots, L_h \) be a system of representatives for a genus \( G \) of positive definite lattices of dimension \( n \). The mass of \( G \) is the product of certain **representation densities** \( \alpha_p(L_p, L_p) \), where \( p \) runs over all primes, with a certain factor “at infinity”:

\[
\text{mass}(G) = \gamma(n) \prod_p \frac{1}{|\text{Aut}(L_p)|} = \gamma(n) \prod_p \alpha_p^{-1}(L_p, L_p).
\]
We want to study automorphism groups of lattices in a given genus \( G \).

**Notation:**
- \( G_0 := \{ L \in G \mid \text{Aut} L = \{ \pm \text{id} \} \}
- \( G_1 := \{ L \in G \mid \text{Aut} L \neq \{ \pm \text{id} \} \}
- \( h_0(G) := \text{card } G_0, \ h_1(G) := \text{card } G_1 \)
- \( \text{mass}(G) := m(G) \), define \( m_0(G), m_1(G) \) in the obvious way

**Obvious facts:**
- \( h(G) = h_0(G) + h_1(G) \), \( m(G) = m_0(G) + m_1(G) \)
- \( h_0(G) = 2m_0(G) \)
- \( h(G) \geq 2m(G) \)

**Theorem 2.2 (Minkowski)**

The order of finite subgroups of \( \text{GL}_n(\mathbb{Z}) \) is bounded by

\[
\bar{a}(n) := \prod_{p \leq n+1} p^{\mu(n,p)},
\]

where

\[
\mu(n,p) = \sum_{j \geq 0} \left\lfloor \frac{n}{(p-1)p^j} \right\rfloor.
\]

Minkowski obtained his bound by reducing the group modulo \( p \) (respectively modulo 4, if \( p = 2 \)).

**Example:**
- \( \bar{a}(16) = 2^{31} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \)
- \( a(h_0) = 2^{31} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \)
- \( a(2E_8) = 2^{29} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \)

Now we have an (again very crude) estimate between class number and mass in the converse direction:

\[
h(G) \leq \bar{a}(m) \cdot m(G).
\]

Notice that the bound \( \bar{a}(n) \) grows very fast (roughly like \( n^n \)). Clearly \( 2^n \cdot n! = a(I_n) \) is a lower bound.

**Theorem 2.3 (W. Magnus, 1937, H. Pfeuffer, 1971)**

For genera \( G \) of positive definite lattices of dimension \( n \geq 6 \) and determinant \( d \), one has

\[
\text{mass}(G) > 2^{-n+1} \cdot \prod_{k=1}^{n} \frac{\Gamma(k/2)}{\pi^{k/2}} \cdot d^{\frac{1}{2n}},
\]

similarly for \( 3 \leq n \leq 5 \).

Therefore, the mass, and thus also the class number \( h(G) \), goes to infinity with the dimension (very rapidly), and also with the determinant.
Theorem 2.4 (Jürgen Biermann, 1980)

For genera \( G \) of positive definite lattices in fixed dimension \( n \geq 3 \), one has

\[
\frac{h_0(G)}{h(G)} \to 1, \quad \text{if } \det G \to \infty.
\]

With \( h(G) = h_0(G) + h_1(G) \) one rewrites this as

\[
\frac{h_1(G)}{h(G)} \to 0, \quad \text{if } \det G \to \infty.
\]

“Most lattices have trivial automorphism group.”

Theorem 2.5 (Etsuko Bannai, 1988)

For the genus \( E_n \) of even or odd unimodular positive definite lattices dimension \( n \), one has

\[
\frac{m_1(E_n)}{m(E_n)} \to 0, \quad \text{if } n \to \infty.
\]

More precisely

\[
\frac{m_1(E_n)}{m(E_n)} \leq 2 \cdot \frac{(8\pi)^{n/2}}{\Gamma(n/2)} \quad \text{if } n \geq 144.
\]

Thus, “many” lattices with trivial group exist, for growing dimension.

Trying to transform Bannai’s estimate into an estimate of class numbers, using the above upper and lower bounds, leads to

\[
\frac{h_1}{h} \leq \frac{\tilde{a}(n) \cdot m_1}{2 \cdot m} \leq \tilde{a}(n) \cdot \frac{(8\pi)^{n/2}}{\Gamma(n/2)}.
\]

Since \( \tilde{a}^n > n! \), the right hand side tends to infinity.

Therefore, in order to prove that again “most lattices have trivial automorphism group”, better upper estimates for the class number \( h_1 \) of lattices with non-trivial group are needed.

We want to look at the actual distribution of (orders of) automorphism groups among all the lattices of some (arithmetically interesting) large genera.

Enumerate a set of representatives for a specified genus \( G \), following these steps:

1. Generate lattices in \( G \) by some algebraic procedure
2. Test for isometry with lattices already constructed
3. Verify the completeness of the list
Step 1 is typically handled by Kneser’s method of neighbouring lattices: $L$ and $L'$ are neighbors, if their intersection $L \cap L'$ is of index 2 in both of them. All neighbours of $L$ can be efficiently generated from (certain) classes of $L/2L$.

Step 2 is a matter of invariants (theta series, order of automorphism group, successice minima, ...) and of sophisticated algorithms for testing isometry of a given pair of lattices (improved backtracking), by Plesken and Souvignier.

Step 3 is handled best by the mass formula.

The following is joint work with Boris Hemkemeier.

Theorem 3.1 (Level 11, dimension 12)

The genus $II_{12}(11^6)$ has class number 67323. It contains precisely

- 27193 lattices with minimum 2
- 40036 lattices with minimum 4
- 94 lattices with minimum 6
- no lattice with minimum 8.

This reproves the absence of “extremal” 11-modular lattices in this genus, first shown by Nebe and Venkov using Siegel modular forms.

The automorphism groups for the genus $II_{12}(11^6)$:

recall $a(L) := |Aut(L)|$:

Among the 67323 lattices, there exist

- 16613 lattices (24.7%) with trivial group, i.e. $a(L) = 2$
- 6065 lattices for which $3 | a(L)$
- 421 lattices for which $5 | a(L)$
- 0 lattices for which $7 | a(L)$ or $13 | a(L)$
- 1 lattice for which $11 | a(L)$

Theorem 3.2 (Level 14, dimension 14)

The genus $II_{14}(7^7)$ has class number 83006. It contains precisely

- 46574 lattices with minimum 2
- 36431 lattices with minimum 4
- 1 lattice with minimum 6.

The unique extremal 7-modular lattice was not known before.
The automorphism groups for the genus $II_{14}(77)$:

**Recall:** $a(L) := |\text{Aut}(L)|$:

Among the 83006 lattices, there exist:

- 12827 lattices (15.4%) with trivial group, i.e. $a(L) = 2$
- 11797 lattices for which $3 | a(L)$
- 353 lattices for which $5 | a(L)$
- 82 lattices for which $7 | a(L)$
- 0 lattices for which $11 | a(L)$ or $13 | a(L)$

### Conclusion

- For genera of small level and dimension $12 \leq n \leq 20$, small masses ($<< 1$) occur with class numbers of several hundreds, thus only large groups.
- For many genera with larger mass ($\sim \ldots 10^4$), the class number remains computable ($\sim \ldots 10^5$), the average group order goes down to less than 10.
- In large cases, the “typical” automorphism group is a 2-group of “small” order (e.g. $\leq 64$).
- Trivial groups occur, but are not the majority; their proportion goes up from less than 1/100 to about 1/4.

### Outlook

A structure theory and a mass formula for orthogonal representations of the cyclic group $C_\ell$, $\ell$ an odd prime or $\ell = 4$ should give more precise estimates of $h_1$ and thus clarify the asymptotic behaviour of $h_1/h$.

Work in progress by Björn Hoffmann, Stefan Höppner, Timo Rosnau (PhD thesis project).


