Polyhedral Reduction of Humbert forms over a totally real number field

Takao Watanabe
(joint work with Hayashi and Yano)

Osaka University

Notations

Let

$\mathbf{k}$ = a totally real algebraic number field of degree r,

∞ = the set of archimedean places of $\mathbf{k}$,

$\mathbf{k}_\sigma$ = the completion of $\mathbf{k}$ at $\sigma \in \infty$,

$\mathbf{k}_\mathbf{R} = \mathbf{k} \otimes_\mathbf{Q} \mathbf{R} = \prod_{\sigma\in\infty} \mathbf{k}_\sigma \cong \mathbf{R}^r$,

For $x = (x_\sigma)_{\sigma\in\infty} \in \mathbf{k}_\mathbf{R}$, the trace of x is defined by

$$\mathrm{Tr}_{\mathbf{k}_\mathbf{R}}(x) = \sum_{\sigma} x_\sigma .$$

Humbert forms

Let

$M_n(\mathsf{k}_\mathrm{R})$ = the space of all $n \times n$ matrices with entries in k_R,

$GL_n(\mathsf{k}_\mathrm{R})$ = the unit group of $M_n(\mathsf{k}_\mathrm{R}) = \prod_{\sigma\in\infty} GL_n(\mathsf{k}_\sigma)$,

$H_n = \{a \in M_n(\mathsf{k}_\mathrm{R}) \ : \ {}^t a = a\} \cong \mathrm{Sym}_n(\mathrm{R})^{\oplus r}$,

$P_n = \{{}^t gg \ : \ g \in GL_n(\mathsf{k}_\mathrm{R})\} \subset H_n(\mathsf{k}_\mathrm{R})$,

An element of P_n is called an Humbert form and denoted by $a = (a_\sigma)_{\sigma\in\infty}$. Each component a_σ is a positive definite real symmetric matrix.

Purpose

Let $\mathbf{o}$ be the ring of integers of k.

We fix a projective $\mathbf{o}$-module $\Lambda \subset \mathsf{k}^n$ of rank n.

Λ is viewed as a lattice in $\mathsf{k}_{\mathbf{R}}^n$ by a natural inclusion $\mathsf{k}^n \hookrightarrow \mathsf{k}_{\mathbf{R}}^n$.

The discrete subgroup

$$GL(\Lambda) = \{\gamma \in GL_n(\mathsf{k}_{\mathbf{R}}) \ : \ \gamma\Lambda = \Lambda\}$$

acts on P_n by

$$(a, \gamma) \mapsto a \cdot \gamma = {}^t\gamma a \gamma\,.$$

Our purpose is to construct a polyhedral fundamental domain of $P_n/GL(\Lambda)$.

Brief history

- Voronoï (1908) gave a polyhedral reduction of GL_n over $\mathbb{Q}$, i.e., of $GL_n(\mathbb{R})/GL_n(\mathbb{Z})$.
- Köcher (1960) extended Voronoï's reduction theory to self-dual homogeneous cones. In particular, Köecher's theory covers a polyhedral reduction of $GL_n(\mathbf{k}_\mathbb{R})/GL_n(\mathbf{o})$, i.e., the case of $\Lambda = \mathbf{o}^n$.
- Theory of perfect forms plays an important role in polyhedral reduction. Ong (1986), Leibak (2005), Gunnells and Yasaki (2010) studied perfect forms over $\mathbf{k}$.

Minimum function and minimal vectors

The minimum function $\mathbf{m} = \mathbf{m}_{\mathbf{\Lambda}} : P_n \longrightarrow \mathbf{R}_{\geq 0}$ is defined by

$$\mathbf{m}(a) = \min_{0 \neq x \in \Lambda} (a, x^t x) \,,$$

where

$$(a, x^t x) = \mathbf{Tr}_{\mathbf{k}_{\mathbf{R}}}(\mathrm{tr}(a \cdot x^t x)) = \mathbf{Tr}_{\mathbf{k}_{\mathbf{R}}}({}^t x a x) \,.$$

For $a \in P_n$, put

$$S(a) = S_{\Lambda}(a) = \{x \in \Lambda \;:\; (a, x^t x) = \mathbf{m}(a)\} \,.$$

$S(a)$ is a finite subset.

Λ-perfection

Definition 1

$a \in P_n$ is said to be Λ-perfect if $\{x^t x \ : \ x \in S(a)\}$ spans H_n as an $\mathbf{R}$-vector space. Namely

$$\dim \operatorname{Span}\{x^t x \ : \ x \in S(a)\} = r \cdot \frac{n(n+1)}{2}$$

Ryshkov polyhedron

The domain

$$K_1 = K_1(\mathbf{m}) = \{a \in P_n \ : \ \mathbf{m}(a) \geq 1\}\,.$$

is a closed convex subset in P_n.

For a non-empty finite subset $S \subset \Lambda \setminus \{0\}$, we put

$$\mathcal{F}_S = \{a \in \partial K_1 \ : \ S \subset S(a)\}\,.$$

Ryshkov polyhedron

The domain

$$K_1 = K_1(\mathbf{m}) = \{a \in P_n \ : \ \mathbf{m}(a) \geq 1\}\,.$$

is a closed convex subset in P_n.

For a non-empty finite subset $S \subset \Lambda \setminus \{0\}$, we put

$$\mathcal{F}_S = \{a \in \partial K_1 \ : \ S \subset S(a)\}\,.$$

Proposition 1

K_1 is a locally finite polyhedron, i.e., the intersection of K_1 and any polytope is a polytope. $\mathcal{F}_S$ gives a face of K_1 if $\mathcal{F}_S \neq \emptyset$. Conversely, any face of K_1 is of the form $\mathcal{F}_S$ for some S.

Geometric properties of Λ-perfect forms

Let $\partial^0 K_1$ be the set of all vertices of K_1.

Geometric properties of Λ-perfect forms

Let $\partial^0 K_1$ be the set of all vertices of K_1.

Theorem 2 (Hayashi–W–Yano–Okuda for general k)

1. $a \in \partial^0 K_1$ *iff* a *is* Λ*-perfect with* $m(a) = 1$.
2. *If* $a \in \partial^0 K_1$*, then* $a \in GL_n(\mathsf{k})$.
3. $\partial^0 K_1 / GL(\Lambda)$ *is a finite set.*
4. *For* $a, b \in \partial^0 K_1$*, there exists a finite sequence of vertices* $a_0, \cdots, a_t \in \partial^0 K_1$ *such that* $a_0 = a$*,* $a_t = b$ *and* a_{i+1} *is adjacent to* a_i *for* $i = 0, \cdots, t-1$*, i.e.,*

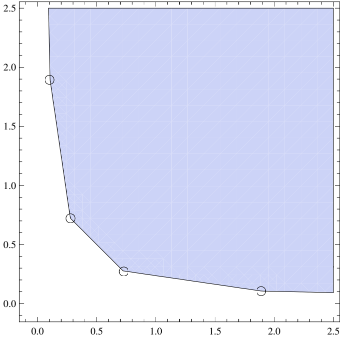
$$\overline{a_i a_{i+1}} = \{\lambda a_i + (1-\lambda) a_{i+1} \ : \ 0 \leq \lambda \leq 1\}$$

is a 1-dimensional face of ∂K_1.

Example

The case of $\mathbf{k} = \mathbb{Q}(\sqrt{5})$, $n = 1$ and $\Lambda = \mathfrak{o}$.

In this case, $P_1 = \mathbb{R}^2_{>0} \supset K_1$ is given by



$$a = \left(\frac{1}{2} - \frac{\sqrt{5}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}\right), \qquad \sharp(\partial^0 K_1 / GL(\Lambda)) = 1$$

Rational closure of P_n

Let P_n^- be the closure of P_n in H_n.
For $a \in P_n^-$, the radical of a is defined by

$$\mathrm{rad}(a) = \{x \in \mathsf{k}_{\mathrm{R}}^n \ : \ (a, x^t x) = 0\}\,.$$

Let

$$\Omega_{\mathsf{k}} = \{a \in P_n^- \ : \ (\mathrm{rad}(a) \cap \mathsf{k}^n) \otimes_{\mathrm{Q}} \mathrm{R} = \mathrm{rad}(a)\}.$$

We have $P_n \subsetneqq \Omega_{\mathsf{k}} \subsetneqq P_n^-$.

Rational closure of P_n

Let P_n^- be the closure of P_n in H_n.
For $a \in P_n^-$, the radical of a is defined by

$$\mathrm{rad}(a) = \{x \in \mathsf{k}_{\mathrm{R}}^n \;:\; (a, x^t x) = 0\}\,.$$

Let

$$\Omega_{\mathsf{k}} = \{a \in P_n^- \;:\; (\mathrm{rad}(a) \cap \mathsf{k}^n) \otimes_{\mathrm{Q}} \mathrm{R} = \mathrm{rad}(a)\}.$$

We have $P_n \subsetneq \Omega_{\mathsf{k}} \subsetneq P_n^-$.

Proposition 2

$$\Omega_{\mathsf{k}} = \left\{ \sum_i \lambda_i (x_i{}^t x_i) \;:\; \lambda_i \in \mathrm{R}_{\geq 0},\; x_i \in \mathsf{k}^n \right\}.$$

Subdivision of Ω_k by perfect cones

For $a \in \partial^0 K_1$, put

$$D_a = \text{the closed cone generated by } \{x^t x \ : \ x \in S(a)\}.$$

If $a \neq b$, then $D_a \cap \mathrm{Int}(D_b) = \emptyset$.

Subdivision of Ω_k by perfect cones

For $a \in \partial^0 K_1$, put

$$D_a = \text{the closed cone generated by } \{x^t x \;:\; x \in S(a)\}\,.$$

If $a \neq b$, then $D_a \cap \mathrm{Int}(D_b) = \emptyset$.

Proposition 3

For any $a \in \Omega_k \setminus \{0\}$, there exists $b_0 \in \partial^0 K_1$ such that

$$\inf_{b \in K_1} (a, b) = (a, b_0)\,,$$

and then $a \in D_{b_0}$.

Therefore, we have

$$\Omega_k = \bigcup_{b \in \partial^0 K_1} D_b\,.$$

Polyhedral reduction of $\Omega_k/GL(\Lambda)$

Ω_k is stable by the action of $GL(\Lambda)$.
Let $\{b_1, \cdots, b_t\}$ be a set of representatives of $\partial^0 K_1/GL(\Lambda)$.
For each i, Γ_i denotes the stabilizer of b_i in $GL(\Lambda)$, i.e.,

$$\Gamma_i = \{\gamma \in GL(\Lambda) \ : \ b_i \cdot \gamma = b_i\},$$

which is a finite subgroup and stabilizes D_{b_i}.

Polyhedral reduction of $\Omega_k/GL(\Lambda)$

Ω_k is stable by the action of $GL(\Lambda)$.
Let $\{b_1, \cdots, b_t\}$ be a set of representatives of $\partial^0 K_1/GL(\Lambda)$.
For each i, Γ_i denotes the stabilizer of b_i in $GL(\Lambda)$, i.e.,

$$\Gamma_i = \{\gamma \in GL(\Lambda) \ : \ b_i \cdot \gamma = b_i\},$$

which is a finite subgroup and stabilizes D_{b_i}.

Theorem 3

$$\Omega_k/GL(\Lambda) = \bigcup_{i=1}^{t} D_{b_i}/\Gamma_i .$$

The case of $n = 1$

If $n = 1$ and $\Lambda = \mathsf{o}$, then

$$\Omega_{\mathsf{k}} \setminus \{0\} = \mathsf{k}_{\mathbf{R}}^{+} := \{(\alpha_\sigma)_{\sigma\in\infty} \in \mathsf{k}_{\mathbf{R}} \ : \ \alpha_\sigma > 0 \text{ for all } \sigma \in \infty\}$$

and $GL(\Lambda) = E_{\mathsf{k}}$ the unit group of o. The action of E_{k} on $\mathsf{k}_{\mathbf{R}}^{+}$ is given by $x \cdot u = u^2 x$ for $(x, u) \in \mathsf{k}_{\mathbf{R}}^{+} \times E_{\mathsf{k}}$.
Let $\{b_1, \cdots, b_t\}$ be a set of representatives of $\partial^0 K_1/E_{\mathsf{k}}$. Since $\Gamma_i = \{\pm 1\}$ trivially acts on D_{b_i}, we have

$$\mathsf{k}_{\mathbf{R}}^{+}/E_{\mathsf{k}} = E_{\mathsf{k}}^{2}\backslash \mathsf{k}_{\mathbf{R}}^{+} = \bigcup_{i=1}^{t} D_{b_i}^{*}\,, \quad \text{where } D_{b_i}^{*} = D_{b_i} \setminus \{0\}\,.$$

Namely, a fundamental domain of $E_{\mathsf{k}}^{2}\backslash \mathsf{k}_{\mathbf{R}}^{+}$ decomposes into a union of perfect cones. This result is viewed as a refeinment of Shintani's unit theorem for E_{k}^{2}.

Example: the case of $\mathsf{k} = \mathbf{Q}(\sqrt{d})$, $n = 1$ and $\Lambda = \mathfrak{o}$

Let

$d \geq 2$ be a square free positive integer,

$\mathsf{k} = \mathbf{Q}(\sqrt{d})$ a real quadratic field,

$\tau =$ the Galois involution of $\mathsf{k}/\mathbf{Q}$,

$\omega = \sqrt{d}$ if $d \equiv 2, 3 \mod 4$ or $(1 + \sqrt{d})/2$ if $d \equiv 1 \mod 4$,

$\epsilon =$ a fundamental unit with $\epsilon^2 < 1$.

In the case of $n = 1$ and $\Lambda = \mathfrak{o} = \mathbf{Z}[\omega]$, the Ryshkov polyhedron K_1 is a convex domain in $\mathsf{k}_{\mathbf{R}}^{+} = \mathbf{R}_{>0}^{2}$ with infinite vertices.

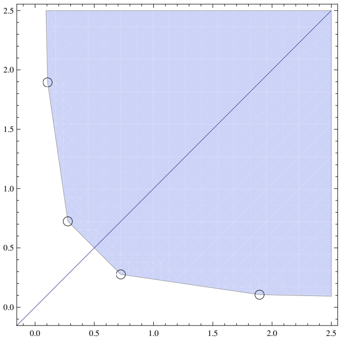
For $a \in \partial^0 K_1$, the equivalent class of a is given by

$$\{\epsilon^{2n} a \ : \ n \in \mathbf{Z}\}.$$

Example: the case of $\mathsf{k} = \mathbb{Q}(\sqrt{d})$, $n = 1$ and $\Lambda = \mathbf{o}$

It is easy to check

- $\boldsymbol{K_1}$ is invariant by $\boldsymbol{\tau}$, i.e., $\boldsymbol{K_1}$ is symmetric with respect to the diagonal line $\boldsymbol{\ell} = \mathbf{R}_{>0}(1, 1)$.
- If $a \in \partial^0 \boldsymbol{K_1}$, then $a \in \mathsf{k}^+_{\mathbf{R}} \cap \mathsf{k}$ and $\boldsymbol{\tau}(a) \in \partial^0 \boldsymbol{K_1}$.
- There is no $\mathbf{o}$-perfect form on $\boldsymbol{\ell}$.



Example: the case of $\mathsf{k} = \mathbb{Q}(\sqrt{d})$, $n = 1$ and $\Lambda = \mathbf{o}$

Let $t_{\mathsf{k}} = \sharp(E^2_{\mathsf{k}} \backslash \partial^0 K_1)$ be the class number of **o**-perfect unary forms.

Example: the case of $\mathbf{k} = \mathbb{Q}(\sqrt{d})$, $n = 1$ and $\Lambda = \mathbf{o}$

Let $t_{\mathbf{k}} = \sharp(E_{\mathbf{k}}^2 \backslash \partial^0 K_1)$ be the class number of $\mathbf{o}$-perfect unary forms.

When $d < 10000$, there exist 154 d such that $t_{\mathbf{k}} = 1$, for example, d = 2, 3, 5, 10, 13, 15, 21, 26, 29, 35, 53, 77, 82, 85, 122, 143, 165, 170, 173, 195, 221, 226, 229, 255, 285, 290, 293, 323, 357, 362, 365, 399, 437, 443, 445, 483, 530, 533, 626, 629, 730, 733, 842, 899, 957, 962, 965, ...

Example: the case of $\mathbf{k} = \mathbb{Q}(\sqrt{d})$, $n = 1$ and $\Lambda = \mathbf{o}$

Let $t_{\mathbf{k}} = \sharp(E_{\mathbf{k}}^2 \backslash \partial^0 K_1)$ be the class number of $\mathbf{o}$-perfect unary forms.

When $d < 10000$, there exist 154 d such that $t_{\mathbf{k}} = 1$, for example, d = 2, 3, 5, 10, 13, 15, 21, 26, 29, 35, 53, 77, 82, 85, 122, 143, 165, 170, 173, 195, 221, 226, 229, 255, 285, 290, 293, 323, 357, 362, 365, 399, 437, 443, 445, 483, 530, 533, 626, 629, 730, 733, 842, 899, 957, 962, 965, ...

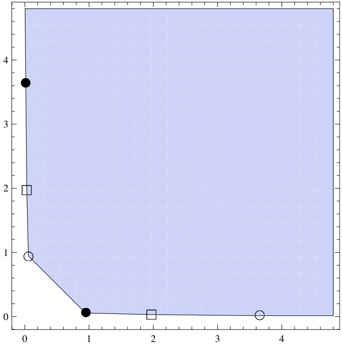
Recently, Dan Yasaki proved the following theorem.

Theorem 4 (Yasaki)

There exists infinitely many quadratic fields $\mathbf{k}$ *such that* $t_{\mathbf{k}} = 1$.

Example of k with $t_{\mathsf{k}} = 3$

$\mathsf{k} = \mathbf{Q}(\sqrt{17})$. In this case, $t_{\mathsf{k}} = 3$.



$$\circ = \left(\frac{1}{2} - \frac{11\sqrt{17}}{102}, \frac{1}{2} + \frac{11\sqrt{17}}{102}\right).$$