Alternating Direction Augmented Lagrangian Algorithms for Convex Optimization

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Alternating direction augmented Lagrangian (ADAL) methods

- Alternating direction methods: go back to Peaceman, Rachford, Douglas, Gabay and Mercier, Glowinski and Marrocco, Lions and Mercier, and Passty etc.
- Augmented Lagrangian methods: Hestenes, Powell, Rockafellar

Motivation for Studying ADAL Methods

Motivation:

- Current optimization problems of interest in machine learning, data mining, medical imaging, etc., have enormous numbers of variables/constraints
- Only first-order methods are practical
- It is necessary to take advantage of the structure (e.g., sparsity) of the optimal solution

Part I: Introduction

SUM-K	SUM-2
$\min F(x) \equiv \sum_{i=1}^{K} f_i(x)$	$\min F(x) \equiv f(x) + g(x)$

- Minimize the sum of convex functions
- Assume the following problem is easy

$$\min_x \quad \tau f_i(x) + \frac{1}{2}\|x - y\|^2$$

- Examples of f_i: $\|x\|_1$, $\|x\|_2$, $\|Ax - b\|^2$, $\|X\|_*$, $-\log\det(X)$, $\|x\|_{1,2} \equiv \sum_{g \in G} \|x_g\|_2$ ......

Examples

- Compressed sensing (Lasso):

$$\min \quad \rho\|x\|_1 + \frac{1}{2}\|Ax - b\|_2^2$$

- Matrix Rank Min:

$$\min \quad \rho\|X\|_* + \frac{1}{2}\|\mathcal{A}(X) - b\|_2^2$$

- Robust PCA:

$$\min_{X,Y} \quad \|X\|_* + \rho\|Y\|_1 : X + Y = M$$

- Sparse Inverse Covariance Selection:

$$\min -\log\det(X) + \langle\Sigma, X\rangle + \rho\|X\|_1$$

- Group Lasso:

$$\min \quad \rho\|x\|_{1,2} + \frac{1}{2}\|Ax - b\|_2^2$$

Variable Splitting

$$(SUM-2) \quad \min \quad f(x)+g(x)$$

- Variable splitting

$$\begin{array}{ll} \min & f(x)+g(y) \\ \text{s.t.} & x=y \end{array}$$

- Augmented Lagrangian function:

$$\mathcal{L}(x,y;\lambda) := f(x)+g(y)-\langle \lambda, x-y\rangle + \frac{1}{2\mu}\|x-y\|^2$$

- Augmented Lagrangian Method:

$$\left\{ \begin{array}{rcl} (x^{k+1},y^{k+1}) & := & \arg\min_{(x,y)} \mathcal{L}(x,y;\lambda^k) \\ \lambda^{k+1} & := & \lambda^k - (x^{k+1}-y^{k+1})/\mu \end{array} \right.$$

Alternating Direction Augmented Lagrangian (ADAL)

- $\mathcal{L}(x, y; \lambda) := f(x) + g(y) - \langle \lambda, x - y \rangle + \frac{1}{2\mu}\|x - y\|^2$
- Solve augmented Lagrangian subproblem alternatingly

$$\begin{cases} x^{k+1} & := & \arg\min_x \mathcal{L}(x, y^k; \lambda^k) \\ y^{k+1} & := & \arg\min_y \mathcal{L}(x^{k+1}, y; \lambda^k) \\ \lambda^{k+1} & := & \lambda^k - (x^{k+1} - y^{k+1})/\mu \end{cases}$$

Symmetric ADAL

- $\mathcal{L}(x, y; \lambda) := f(x) + g(y) - \langle \lambda, x - y \rangle + \frac{1}{2\mu} \|x - y\|^2$
- Symmetric version

$$\begin{cases} x^{k+1} & := & \arg\min_x \mathcal{L}(x, y^k; \lambda^k) \\ \lambda^{k+\frac{1}{2}} & := & \lambda^k - (x^{k+1} - y^k)/\mu \\ y^{k+1} & := & \arg\min_y \mathcal{L}(x^{k+1}, y; \lambda^{k+\frac{1}{2}}) \\ \lambda^{k+1} & := & \lambda^{k+\frac{1}{2}} - (x^{k+1} - y^{k+1})/\mu \end{cases}$$

- Optimality conditions lead to (assuming f and g are smooth)

$$\lambda^{k+\frac{1}{2}} = \nabla f(x^{k+1}), \qquad \lambda^{k+1} = -\nabla g(y^{k+1})$$

Alternating Linearization Method

$$(SUM-2) \quad \min F(x) \equiv f(x) + g(x)$$

- Define

$$Q_g(u,v) := f(u) + g(v) + \langle \nabla g(v), u - v \rangle + \frac{1}{2\mu}\|u - v\|^2$$

$$Q_f(u,v) := f(u) + \langle \nabla f(u), v - u \rangle + \frac{1}{2\mu}\|u - v\|^2 + g(v)$$

- Alternating Linearization Method (ALM)

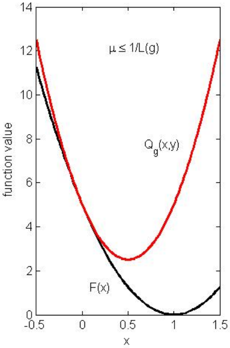
$$\begin{cases} x^{k+1} & := & \arg\min_x Q_g(x, y^k) \\ y^{k+1} & := & \arg\min_y Q_f(x^{k+1}, y) \end{cases}$$

- Gauss-Seidel like algorithm

Key Lemma for Proof

- $F(x) := f(x) + g(y)$: f and g are convex.
- $Q_g(x, y) :=$ $f(x) + g(y) + \langle \nabla g(y), x - y \rangle + \frac{1}{2\mu}\|x - y\|^2$
- $p_g(y) := \arg\min_y Q_g(x, y)$
- Key Lemma:

$$2\mu(F(x) - F(p_g(y))) \geq \|p_g(y) - x\|^2 - \|y - x\|^2$$



Complexity Bound for ALM

Theorem (Goldfarb, Ma and Scheinberg, 2009)

Assume ∇f and ∇g are Lipschitz continuous with constants $L(f)$ and $L(g)$. For $\mu \leq 1/\max\{L(f), L(g)\}$, ALM satisfies

$$F(y^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{4\mu k}$$

Therefore,

- convergence in objective value
- $O(1/\epsilon)$ iterations for an ϵ-optimal solution

- The first complexity result for splitting and alternating direction type methods
- Can we improve the complexity ?
- Can we extend this result to that ADAL method ?

Optimal Gradient Methods

$\min f(x)$ (assuming ∇f is Lipschitz continuous)

- ϵ-optimal solution $f(x) - f(x^*) \leq \epsilon$
- Classical gradient method

$$x^k = x^{k-1} - \tau_k \nabla f(x^{k-1})$$

Complexity $O(1/\epsilon)$

- Nesterov's acceleration technique (1983)

$$\begin{cases} x^k & := & y^{k-1} - \tau_k \nabla f(y^{k-1}) \\ y^k & := & x^k + \frac{k-1}{k+2}(x^k - x^{k-1}) \end{cases}$$

Complexity $O(1/\sqrt{\epsilon})$

- Optimal first-order method; best one can get

ISTA and FISTA (Beck and Teboulle, 2009)

- Assume g is smooth

$$\min \quad F(x) \equiv f(x) + g(x)$$

- Fixed Point Algorithm (Also called ISTA in compressed sensing)

$$x^{k+1} := \arg\min_x Q_g(x, x^k)$$

or equivalently

$$x^{k+1} := \arg\min_x \tau f(x) + \frac{1}{2}\|x - (x^k - \tau\nabla g(x^k))\|^2$$

- Never minimize g
- Iteration complexity: $O(1/\epsilon)$ for an ϵ-optimal solution $(F(x^k) - F(x^*) \leq \epsilon)$

ISTA and FISTA (Beck and Teboulle, 2009)

$$\min F(x) \equiv f(x) + g(x)$$

- Fast ISTA (FISTA)

$$\begin{cases} x^k & := \quad \arg\min_x \tau f(x) + \frac{1}{2}\|x - (y^k - \tau\nabla g(y^k))\|^2 \\ t_{k+1} & := \quad \left(1 + \sqrt{1 + 4t_k^2}\right)/2 \\ y^{k+1} & := \quad x^k + \frac{t_k - 1}{t_{k+1}}(x^k - x^{k-1}) \end{cases}$$

Complexity $O(1/\sqrt{\epsilon})$

Fast Alternating Linearization Method

- ALM

$$\begin{cases} x^{k+1} & := & \arg\min_x Q_g(x, y^k) \\ y^{k+1} & := & \arg\min_y Q_f(x^{k+1}, y) \end{cases}$$

- Accelerate ALM in the same way as FISTA
- Fast Alternating Linearization Method (FALM)

$$\begin{cases} x^k & := & \arg\min_x Q_g(x, z^k) \\ y^k & := & \arg\min_y Q_f(x^k, y) \\ w^k & := & (x^k + y^k)/2 \\ t_{k+1} & := & \left(1 + \sqrt{1 + 4t_k^2}\right)/2 \\ z^{k+1} & := & w^k + \frac{1}{t_{k+1}}(t_k(y^k - w^{k-1}) - (w^k - w^{k-1})) \end{cases}$$

- computational effort at each iteration is almost unchanged
- both f and g must be smooth; however, both are minimized

Complexity Bound for FALM

$$\min F(x) \equiv f(x) + g(x)$$

Theorem (Goldfarb, Ma and Scheinberg, 2009)

Assume ∇f and ∇g are Lipschitz continuous with constants $L(f)$ and $L(g)$. For $\mu \leq 1/\max\{L(f), L(g)\}$, FALM satisfies

$$F(y^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{\mu(k+1)^2}$$

Therefore,

- convergence in objective value
- $O(1/\sqrt{\epsilon})$ iterations for an ϵ-optimal solution
- Optimal first-order method

ALM with skipping steps

At k-th iteration of ALM-S:

- $x^{k+1} := \arg\min_x \mathcal{L}_\mu(x, y^k; \lambda^k)$
- If $F(x^{k+1}) > \mathcal{L}_\mu(x^{k+1}, y^k; \lambda^k)$, then $x^{k+1} := y^k$
- $y^{k+1} := \arg\min_y Q_f(y, x^{k+1})$
- $\lambda^{k+1} := \nabla f(x^{k+1}) - (x^{k+1} - y^{k+1})/\mu$

Note that only one function is required to be smooth.

Complexity Bound for ALM-S

Theorem (Goldfarb, Ma and Scheinberg, 2010)

Assume ∇f is Lipschitz continuous. For $\mu \leq 1/L(f)$, the iterates y^k in ALM-S satisfy:

$$F(y^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{2\mu(k + k_s)}, \forall k,$$

where k_s is the number of iterations until the k-th for which $F(x^{k+1}) \leq \mathcal{L}_\mu(x^{k+1}, y^k; \lambda^k)$. Thus $O(1/\epsilon)$ iterations to obtain an ϵ-optimal solution.

Similar algorithm can be designed for FALM with $O(1/\sqrt{\epsilon})$ complexity and only one function is required to be smooth.

Conjecture: Complexity result for ADAL and fast ADAL (FADAL)

Theorem (Conjectured)

Assume both ∇f and ∇g are Lipschitz continuous. For $\mu \leq 1/\max\{L(f), L(g)\}$, ADAL and FADAL need $O(1/\epsilon)$ and $O(1/\sqrt{\epsilon})$ iterations, respectively, to obtain an ϵ-optimal solution.

No proof currently known.

Basis for possible proof

- Let $A := \partial f$, $B := \partial g$ and the operator

$$S := (I - \mu A)(I + \mu A)^{-1}(I - \mu B)(I + \mu B)^{-1}$$

- The k-th iteration of ALM can be written as

$$v^{k+1} = S \circ v^k.$$

  where $v^k = (I + \mu B)y^k$, for all k.

- We can verify that at the k-th iteration for ADAL, the following relation holds

$$v^{k+1} = \frac{1}{2}(I + S) \circ v^k$$

Fast Generalized Alternating Direction Augmented Lagrangian (FGADAL)

Choose μ and a sequence $\theta_k = \min\{1, \frac{4/\rho}{k+2}\}$ and $0 < \rho \leq 2$.

$$\begin{cases} x^k = arg\min_x \mathcal{L}_\mu(x, y^k; \lambda^k) \\ \lambda^{k+\frac{1}{2}} = \lambda^k - \frac{\rho-1}{\mu}(x^k - y^k) \\ z^k = x^k + \theta_k(\frac{2}{\rho}\theta_{k-1}^{-1} - 1)[x^k - x^{k-1} - (1-\frac{\rho}{2})(y^k - y^{k-1}) \\ \quad +(\mu\lambda^{k+\frac{1}{2}} - \mu\lambda^{k-\frac{1}{2}}) - (1-\frac{\rho}{2})(\mu\lambda^k - \mu\lambda^{k-1})] \\ y^{k+1} = arg\min_y \mathcal{L}_\mu(x^{k+1}, y; \lambda^{k+\frac{1}{2}}) \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \frac{1}{\mu}(x^{k+1} - y^{k+1}) \end{cases}$$

- If $\rho = 2$, FGADAL reduces to FALM.
- If $\rho = 1$, FGADAL is a fast version of ADAL.
- No proof of complexity currently known.

SUM-K

From P.L.Lions and B.Mercier's 1979 paper on operator splitting

- Generalization from 2 to K is possible, but
- Convergence proof for $K \geq 3$ is difficult

$$\min \quad F(x) \equiv f(x) + g(x) + h(x)$$

- Define

$$Q_{gh}(u, v, w) := f(u) + g(v) + \langle \nabla g(v), u - v \rangle + \|u - v\|^2 / 2\mu$$
$$+ h(w) + \langle \nabla h(w), u - w \rangle + \|u - w\|^2 / 2\mu.$$

$$\begin{cases} x^{k+1} & := & \arg\min Q_{gh}(x, y^k, z^k) \\ y^{k+1} & := & \arg\min Q_{fh}(x^{k+1}, y, z^k) \\ z^{k+1} & := & \arg\min Q_{fg}(x^{k+1}, y^{k+1}, z) \end{cases}$$

- However, no complexity results for Gauss-Seidel like algorithm!

MSA: A Jacobi Type Algorithm

$$\min \quad F(x) \equiv f(x) + g(x) + h(x)$$

- Multiple Splitting Algorithm (MSA)

$$\begin{cases} x^{k+1} & := & \arg\min Q_{gh}(x, w^k, w^k) \\ y^{k+1} & := & \arg\min Q_{fh}(w^k, y, w^k) \\ z^{k+1} & := & \arg\min Q_{fg}(w^k, w^k, z) \\ w^{k+1} & := & (x^{k+1} + y^{k+1} + z^{k+1})/3 \end{cases}$$

- Jacobi type algorithm
- Can be done in parallel
- We have a complexity result!

Complexity Bound for MSA

$$\min F(x) \equiv f(x) + g(x) + h(x)$$

Theorem (Goldfarb and Ma, 2009)

Assume ∇f, ∇g and ∇h are Lipschitz continuous with constants $L(f)$, $L(g)$ and $L(h)$. For $\mu \leq 1/\max\{L(f), L(g), L(h)\}$, MSA satisfies

$$\min\{F(x^k), F(y^k), F(z^k)\} - F(x^*) \leq \frac{\|x_0 - x^*\|^2}{\mu k}.$$

Therefore,

- convergence in objective value
- $O(1/\epsilon)$ iterations for an ϵ-optimal solution

Fast Multiple Splitting Algorithm

Fast Multiple Splitting Algorithm (FaMSA)

$$
\left\{
\begin{array}{lcl}
x^k & := & \arg\min Q_{gh}(x, w_x^k, w_x^k) \\
y^k & := & \arg\min Q_{fh}(w_y^k, y, w_y^k) \\
z^k & := & \arg\min Q_{fg}(w_z^k, w_z^k, z) \\
w^k & := & (x^k + y^k + z^k)/3 \\
t_{k+1} & := & \left(1 + \sqrt{1 + 4t_k^2}\right)/2 \\
w_x^{k+1} & := & w^k + \frac{1}{t_{k+1}}[t_k(x^k - w^k) - (w^k - w^{k-1})] \\
w_y^{k+1} & := & w^k + \frac{1}{t_{k+1}}[t_k(y^k - w^k) - (w^k - w^{k-1})] \\
w_z^{k+1} & := & w^k + \frac{1}{t_{k+1}}[t_k(z^k - w^k) - (w^k - w^{k-1})]
\end{array}
\right.
$$

Complexity Bound for FaMSA

$$\min F(x) \equiv f(x) + g(x) + h(x)$$

Theorem (Goldfarb and Ma, 2009)

Assume ∇f, ∇g and ∇h are Lipschitz continuous with constants $L(f)$, $L(g)$ and $L(h)$. For $\mu \leq 1/\max\{L(f), L(g), L(h)\}$, FaMSA satisfies

$$\min\{F(x^k), F(y^k), F(z^k)\} - F(x^*) \leq \frac{4\|x_0 - x^*\|^2}{\mu(k+1)^2}$$

Therefore,

- convergence in objective value
- $O(1/\sqrt{\epsilon})$ iterations for an ϵ-optimal solution
- optimal first-order method

Comparison of ALM/FALM and MSA/FaMSA

ALM/FALM

- Gauss-Seidel like algorithms
- expected to be faster than MSA/FaMSA since the information from current iteration is used
- complexity results for (SUM-2), no results for (SUM-K) when $K \geq 3$
- only one function needs to be smooth

MSA/FaMSA

- Jacobi like algorithms
- can be done in parallel
- complexity results for (SUM-K) for any $K \geq 2$

Comparison on compressed sensing model with $\rho = 0.01$

solver	obj in k-th iteration				cpu (iter)*
	200	500	800	1000	
FALM-S	9.726599e+4	9.341282e+4	9.182962e+4	9.121742e+4	24.3 (51)
FALM	9.516208e+4	9.186355e+4	9.073086e+4	9.028790e+4	23.1 (51)
FISTA	9.752858e+4	9.372093e+4	9.233719e+4	9.178455e+4	26.0 (69)
ALM-S	1.107103e+5	1.042869e+5	1.021905e+5	1.013128e+5	208.9 (531)
ALM	1.116683e+5	1.047410e+5	1.025611e+5	1.016589e+5	208.1 (581)
ISTA	1.079721e+5	1.040666e+5	1.025107e+5	1.018068e+5	196.8 (510)
SALSA	1.132676e+5	1.054600e+5	1.031346e+5	1.021898e+5	223.9 (663)
SADAL	1.068386e+5	1.021905e+5	1.004005e+5	9.961905e+4	113.5 (332)

* to achieve $F(x) \leq 1.04e + 5$

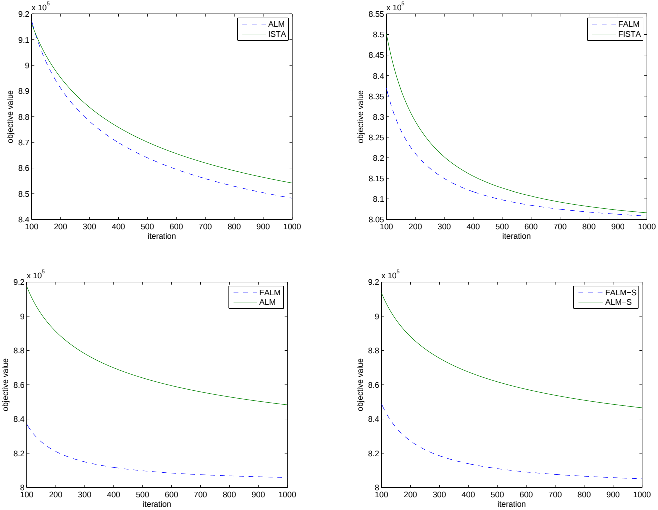


Figure: comparison of the algorithms

Application: Robust PCA

- Robust PCA:

$$\min_{X,Y \in \mathbb{R}^{m \times n}} \quad \{\text{rank}(X) + \rho \|Y\|_0 : X + Y = M\}$$

Recently it has been shown that under suitable conditions on the rank of X and the sparsity of Y, for ρ in a suitable range this generally NP-hard problem can be solved by solving the convex optimization problem

$$\min_{X,Y \in \mathbb{R}^{m \times n}} \quad \{\|X\|_* + \rho \|Y\|_1 : X + Y = M\}$$

ALM and FALM for Robust PCA

- Robust PCA: $f(X) = \|X\|_*, g(Y) = \rho\|Y\|_1$

$$\min_{X,Y \in \mathbb{R}^{m \times n}} \quad \{\|X\|_* + \rho\|Y\|_1 : X + Y = M\}$$

- Subproblem wrt X (a matrix shrinkage operator, corresponds to an SVD):

$$X^{k+1} := \arg\min_X f(X) + g(Y^k) + \langle \gamma_g(Y^k), M - X - Y^k \rangle + \|X + Y^k - M\|_F^2/2\mu$$

- Subproblem wrt Y (a vector shrinkage operator):

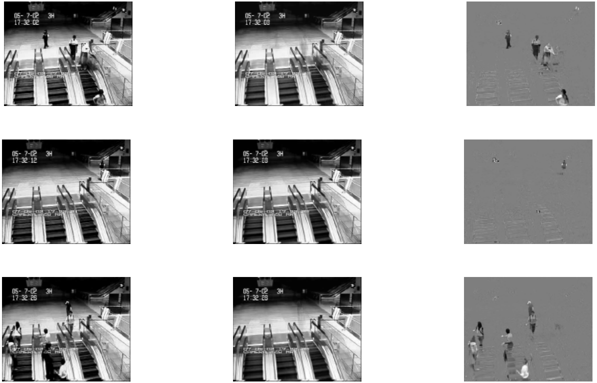
$$Y^{k+1} := \arg\min_Y f(X^{k+1}) + \langle \gamma_f(X^{k+1}), M - X^{k+1} - Y \rangle + \|X^{k+1} + Y - M\|_F^2/2\mu + g(Y)$$

- Smoothed $f(X)$ and $g(Y)$: subgradient $\gamma_f(X^k) = \nabla f(X^k)$ and subgradient $\gamma_g(Y^k) = \nabla g(Y^k)$

Surveillance video

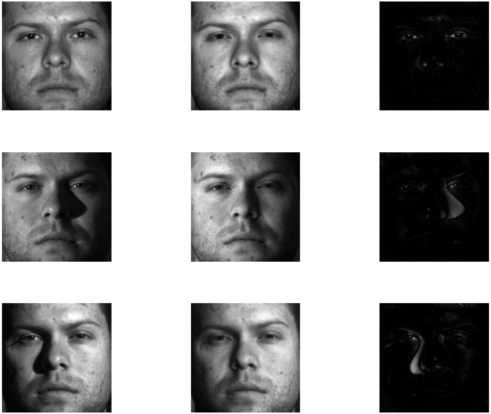
- 200 images with size 144×176, so $M \in \mathbb{R}^{25344 \times 200}$
- 43 SVDs, CPU time: 04:03.
- MATLAB code runs on a Dell Precision 670 workstation with an Intel Xeon(TM) 3.4GHZ CPU and 6GB of RAM.

Surveillance video



- 300 images with size 130×160, so $M \in \mathbb{R}^{20800\times 300}$
- 45 SVDs, CPU time: 05:53.

Shadow and specularities removal from face images



- 65 images with size 200×200, so $M \in \mathbb{R}^{40000 \times 65}$
- 42 SVDs, CPU time: 01:39

Video denoising

- 300 colored images with size 144×176, so $M \in \mathbb{R}^{25344\times 900}$
- 42 SVDs, CPU time: 01:00:18

ALM-S for Sparse Inverse Covariance Selection

- SICS: $f(X) = -\log\det(X) + \langle \Sigma, X \rangle, \quad g(Y) = \rho \|Y\|_1$
- Subproblem wrt X (corresponds to an eigenvalue decomposition):

$$X^{k+1} := \arg\min_X f(X) + g(Y^k) - \langle \Lambda^k, X - Y^k \rangle + \|X - Y^k\|_F^2 / 2\mu$$

- Subproblem wrt Y (a vector shrinkage operator):

$$Y^{k+1} := \arg\min_Y f(X^{k+1}) + \langle \nabla f(X^{k+1}), Y - X^{k+1} \rangle + \|Y - X^{k+1}\|_F^2 / 2\mu + g(Y)$$

Numerical Results: Synthetic Data

- Create data: create sparse matrix $U \in \mathbb{R}^{n \times n}$ with nonzero entries equal to -1 or 1 with equal probability.
- Compute $S := (U * U^\top)^{-1}$ as the true covariance matrix. Hence, S^{-1} is sparse.
- We then draw $p = 5n$ iid vectors, $Y_1, \ldots, Y_p$, from the Gaussian distribution $\mathcal{N}(\mathbf{0}, S)$ by using the *mvnrnd* function in MATLAB.
- $S := \frac{1}{p} \sum_{i=1}^{p} Y_i Y_i^\top$.
- We compare ALM with PSM (Duchi et.al.2008) and VSM (Lu 2009)
- Termination: $Dgap \leq 10^{-3}$

Numerical Results: Synthetic Data

	ALM			PSM			VSM		
n	iter	Dgap	CPU	iter	Dgap	CPU	iter	Dgap	CPU
$\rho = 0.1$									
200	300	8.70e-4	13	1682	9.99e-4	38	857	9.97e-4	37
500	220	5.55e-4	84	861	9.98e-4	205	946	9.98e-4	377
1000	180	9.92e-4	433	292	9.91e-4	446	741	9.97e-4	1928
2000	200	6.13e-5	3110	349	1.12e-3	3759	915	1.00e-3	16085
$\rho = 0.5$									
200	140	9.80e-4	6	6106	1.00e-3	137	1000	9.99e-4	43
500	100	1.69e-4	39	903	9.90e-4	212	1067	9.99e-4	425
1000	100	9.28e-4	247	489	9.80e-4	749	1039	9.95e-4	2709
2000	160	4.70e-4	2529	613	9.96e-4	6519	1640	9.99e-4	28779
$\rho = 1.0$									
200	180	4.63e-4	8	7536	1.00e-3	171	1296	9.96e-4	57
500	140	4.14e-4	55	2099	9.96e-4	495	1015	9.97e-4	406
1000	160	3.19e-4	394	774	9.83e-4	1172	1310	9.97e-4	3426
2000	240	9.58e-4	3794	1158	9.35e-4	12310	2132	9.99e-4	37406

Numerical Results: Real Data

Data on gene expression networks (Li and Toh, 2010): *(1) Lymph node status; (2) Estrogen receptor; (3) Arabidopsis thaliana; (4) Leukemia; (5) Hereditary breast cancer.*

	ALM			PSM			VSM		
n	iter	Dgap	CPU	iter	Dgap	CPU	iter	Dgap	CPU
587	60	9.41e-6	35	178	9.22e-4	64	467	9.78e-4	273
692	80	6.13e-5	73	969	9.94e-4	531	953	9.52e-4	884
834	100	7.26e-5	150	723	1.00e-3	662	1097	7.31e-4	1668
1255	120	6.69e-4	549	1405	9.89e-4	4041	1740	9.36e-4	8568
1869	160	5.59e-4	2158	1639	9.96e-4	14505	3587	9.93e-4	52978

Contributions and Future Work

Our contributions

- New alternating direction augmented Lagrangian, alternating linearization and multiple splitting methods
- Optimal first-order methods
- First complexity results for splitting and alternating direction methods (including Peaceman-Rachford method)

Current and Future Work

- Extension of ALM/FALM, MSA/FaMSA to constrained problems
- Line search variants
- Extension of MSA/FaMSA to nonsmooth problems
- Applications in many fields such as Medical Imaging, Machine Learning, Model Selection, Optimal acquisition basis selection (radar), etc.

Current Work: Constrained Problems

- Stable Robust PCA (SRPCA): Here the the elements of the matrix M are assumed to have noise.

$$\min_{X,Y \in \mathbb{R}^{m \times n}} \{\text{rank}(X) + \rho \|Y\|_0 : \|X + Y - M\|_F \leq \sigma, \}$$

  As in the RPCA problem, under suitable conditions on the rank of X and the sparsity of Y, for ρ in a suitable range solving (SRPCA) can be accomplished by solving

$$\min_{X,Y \in \mathbb{R}^{m \times n}} \{\|X\|_* + \rho \|Y\|_1 : \|X + Y - M\|_F \leq \sigma\}$$

  We have developed ISTA/FISTA and ALM/FALM algorithms for SRPCA that require only a modest increase in the work over that required to solve RPCA
- Overlapping Group Lasso: Here the groups are allowed to overlap,resulting in additional linear constraints. We have developed ISTA/FISTA and ALM/FALM algorithms for this problem.

Current Work: FISTA with line search

Given μ_0 and $0 < \beta < 1$. Cycle to find μ_k and t_k

$x^k = \arg\min_y Q_f(y^k, y)$
Find the smallest $i_k \geq 0$ such that
$\mu_k = \beta^{i_k}\mu_0$ and $F(x^k) \leq Q_f(y^k, x^k)$
$t_{k+1} := \frac{1+\sqrt{1+4\theta_k t_k^2}}{2}, \quad \theta_k := \frac{\mu_k}{\mu_{k+1}}$
$\mu_k t_k^2 \geq \mu_{k+1} t_{k+1}(t_{k+1} - 1)$
$y^{k+1} := x^k + \frac{t_k - 1}{t_{k+1}}(x^k - x^{k-1})$

$\Downarrow$

$$F(x^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{2\mu_k t_k^2}$$

$$\mu_k t_k^2 \geq \frac{\beta k^2}{4L} \Rightarrow F(x^k) - F(x^*) \leq \frac{2L\|x^0 - x^*\|^2}{\beta k^2}$$