Frenet-Serret and the Estimation of Curvature and Torsion

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BIRS

Joint work with Kang-Rae Kim (Korea University), Ja-Yong Koo (Korea University) and Michael Pierrynowski (McMaster University)
Outline

Preliminaries

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Statistical Estimation
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Statistical Estimation
Application to Biomechanics
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where $'$ indicates differentiation and $\times$ denotes cross product
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Frenet-Serret frame satisfies the differential equation

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\begin{pmatrix}
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N' \\
B'
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Frenet-Serret formulas
Curvature and Torsion

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torsion characterizes the non-planarity of a three-dimensional space curve
consider \( h_i = f(t_i) + \varepsilon_i \) using data \((t_i, h_i)\) and assuming \( \varepsilon_i \) are errors, \( i = 1, \ldots, n \)
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then form \( s(t; \beta) = \sum_{j=1}^{K} \beta_j B_j(t) \)
Statistical Estimation

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estimate $\beta$ by

$$\hat{\beta} = \arg \min_{\beta} \sum_{i=1}^{n} \left( h_i - \beta^\top B(t_i) \right)^2,$$
Statistical Estimation

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the estimator of \( f \) is defined by \( \hat{f} = \sum_{j=1}^{K} \hat{\beta}_j B_j \)
since we do not know which basis functions are useful in fitting, we adopt a model selection procedure as follows:
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select the model having the minimum Akaike information criterion (AIC) statistic
let \( \alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top \) be the fitted curve
Estimation of curvature and torsion

let $\alpha_n(t) = (\alpha_{1,n}(t), \alpha_{2,n}(t), \alpha_{3,n}(t))^\top$ be the fitted curve

define $\alpha_n^{(\ell)}(t) = (\alpha_{1,n}^{(\ell)}(t), \alpha_{2,n}^{(\ell)}(t), \alpha_{3,n}^{(\ell)}(t))^\top$ be the $\ell$–th derivative
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$\hat{\kappa}_n(t) = \frac{\|\alpha'_n(t) \times \alpha''_n(t)\|}{\|\alpha'_n(t)\|^3}$ is the curvature estimator
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$\hat{\kappa}_n(t) = \frac{\|\alpha_n'(t) \times \alpha_n''(t)\|}{\|\alpha_n'(t)\|^3}$ is the curvature estimator

$\hat{\tau}_n(t) = -\frac{\langle \alpha_n'(t) \times \alpha_n''(t), \alpha_n'''(t) \rangle}{\|\alpha_n'(t) \times \alpha_n''(t)\|^2}$ is the torsion estimator
Knee data

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