Counting Faces in Polytopes

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Algebraic Combinatorixx 2011

Definitions

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$\emptyset$, P, and proper faces, ordered by inclusion

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FACE VECTOR:

$$(f_0(P), f_1(P), \ldots, f_{d-1}(P))$$

$f_i(P) = \#$ of i-dimensional faces of P

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- face vectors of 3-dimensional polytopes (Steinitz)
- face vectors of simplicial polytopes (Stanley and Billera & Lee)
- affine span (Euler's equation: Poincaré and Höhn)

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UNKNOWN:

- face vectors of polytopes of dimension 4 and higher
- face vectors of zonotopes, cubical polytopes

Flag vectors: Definition

Let $S = \{s_1, s_2, \ldots, s_k\}_< \subseteq \{0, 1, \ldots, d-1\}$

Definition

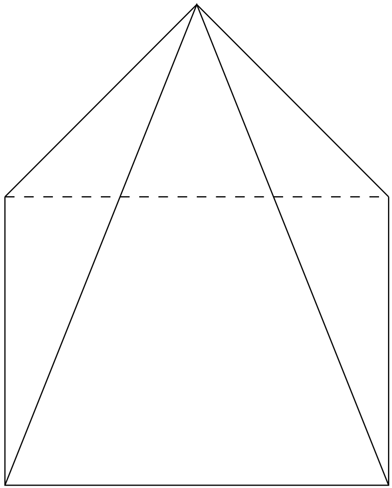
An S-flag of P is a chain

$$\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$$

with $\dim F_i = s_i$

$f_S(P) = \#$ of S-flags of P
$(f_S(P))_{S \subseteq \{0,1,\ldots,d-1\}}$ is the flag vector of P

Example



$f_\emptyset = 1$
$f_0 = 5$
$f_1 = 8$
$f_2 = 5$
$f_{01} = 16$
$f_{02} = 16$
$f_{12} = 16$
$f_{012} = 32$

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Methods: Constructions

Special polytopes:

- simplicial/simple
- zonotopes (more generally: Minkowski sums of polytopes)
- cubical polytopes
- cyclic polytopes (vertices on the moment curve)

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New polytopes out of old:

- pyramids (more generally: join of polytopes)
- bipyramids (more generally: free sum of polytopes)
- prisms (more generally: Cartesian products of polytopes)
- sewing (introducing one vertex at a time; Shemer)

Methods: Commutative Algebra

Stanley-Reisner ring

P a simplicial polytope with vertices v_1, v_2, ..., v_n
Associate with every nonface $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ the monomial $x_{i_1} x_{i_2} \cdots x_{i_k}$
Let I be the ideal generated by these monomials
The Stanley-Reisner ring is $k[x_1, x_2, \dots, x_n]/I$

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h-vector

From the Hilbert function of the Stanley-Reisner ring is extracted the *h*-vector
The *h*-vector is linearly equivalent to the face vector
The *h*-vector was first noted by Sommerville, 1927 (without knowing the algebra connection)

Methods: Toric Varieties

Associated with every rational (embedded) polytope is a toric variety
The toric variety encodes the affine dependencies of the vertices of the polytope

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Simplicial Polytopes

h-vector give ranks of cohomology of the toric variety

Rational Polytopes

h-vector gives middle perversity intersection homology Betti numbers of the toric variety
Definition extended to Eulerian posets (Stanley)
Components of general h-vector are linear functions of the flag vector
Interpretation of general h-vector for nonrational polytopes (Karu)

Methods: **cd**-index

Flag vectors modulo generalized Dehn-Sommerville equations
$\Rightarrow$ **cd**-index (Fine; Bayer & Klapper)

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Hopf Algebra setting where **cd**-index computations happen
(Ehrenborg & Readdy)

Results: Polytopes

- inequalities on face vectors of simplicial spheres come from results on Hilbert functions of Gorenstein rings (Stanley)
- all inequalities on face vectors of simplicial polytopes come from Hard Lefschetz Theorem for toric varieties (Stanley and Billera & Lee))
- inequalities on flag vectors of polytopes come from toric varieties (weaker than simplicial case)
- inequalities on flag vectors of polytopes come from nonnegativity of **cd**-index (Stanley and Karu)
- flag vector inequalities project to give face vector inequalities

Results: More on inequalities

For Polytopes

- new inequalities generated from old by combinatorial convolutions (Kalai)
- new inequalities generated from old by operations in Hopf algebra (Ehrenborg & Readdy)
- inequalities for zonotopes/central hyperplane arrangements (Billera, Ehrenborg & Readdy)

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For Posets

- conical span of flag vectors of all ranked posets (Billera & Hetyei)
- conical span of flag vectors of all Eulerian posets for low ranks (Bayer & Hetyei)

The End

Thank you.