Modular representation theory of symmetric groups and p-combinatorics

Christine Bessenrodt

Leibniz Universität Hannover

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Representations of finite groups

Let G be a finite group, K a field (large enough).

Aims:

- Classify irreducible (and indecomposable) representations

$$\rho : G \to GL(V)\,,\ V \text{ a finite dimensional } K\text{-vector space.}$$

- Decompose representations into irreducible ones.
- Understand relations between representations.

Ordinary representation theory: $\mathrm{Char}\, K = 0$ or $\mathrm{Char}\, K \nmid |G|$

p-**modular representation theory:** $\mathrm{Char}\, K = p \mid |G|$

Ordinary and modular theory: *p*-blocks of characters

For $x \in G$: $\widehat{x^G} = \sum_{y \in x^G} y$, the **class sum** to x.

The set of class sums is a basis of $Z(\mathbb{C}G)$.

The **central character** $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$ to $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$:

$$\omega_\chi(\widehat{x^G}) = \frac{|x^G|\chi(x)}{\chi(1)} \quad \text{for all } x \in G.$$

Then $\omega_\chi(\widehat{x^G}) \in R =$ the ring of algebraic integers..

Let p be a prime, $p \in \wp$ maximal ideal of R. Let $\chi, \psi \in \mathrm{Irr}_{\mathbb{C}}(G)$.

$$\chi \sim_p \psi :\Leftrightarrow \omega_\chi(\widehat{x^G}) \equiv \omega_\psi(\widehat{x^G}) \mod \wp \quad \forall x \in G$$

The $\sim_p$ equivalence classes are the ***p*-blocks of *G***.

Character table of S_5

cycle type	1^5	$1^3 2$	$1^2 3$	14	12^2	23	5
length	1	10	20	30	15	20	24
$\mathbf{1} = [5]$	1	1	1	1	1	1	1
$[41]$	4	2	1	0	0	-1	-1
$[32]$	5	1	-1	-1	1	1	0
$[31^2]$	6	0	0	0	-2	0	1
$[2^2 1]$	5	-1	-1	1	1	-1	0
$[21^3]$	4	-2	1	0	0	1	-1
$\mathrm{sgn} = [1^5]$	1	-1	1	-1	1	-1	1

Central characters of S_5

cycle type	1^5	$1^3 2$	$1^2 3$	14	12^2	23	5
length	1	10	20	30	15	20	24
$\omega_{[5]}$	1	10	20	30	15	20	24
$\omega_{[41]}$	1	5	5	0	0	-5	-6
$\omega_{[32]}$	1	2	-4	-6	3	4	0
$\omega_{[31^2]}$	1	0	0	0	-5	0	4
$\omega_{[2^2 1]}$	1	-2	-4	6	3	-4	0
$\omega_{[21^3]}$	1	-5	5	0	0	5	-6
$\omega_{[1^5]}$	1	-10	20	-30	15	-20	24

Modulo 3:

cycle type	1^5	$1^3 2$	$1^2 3$	14	12^2	23	5
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***p*-blocks of defect 0** (*p*-cores)
Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$; then: $\{\chi\}$ is a p-block $\Leftrightarrow \chi(1)_p = |G|_p$.
In this case: $\chi(x) = 0$ for all p-singular $x \in G$.

Characters and group structure

Applications of block theory: classification of finite simple groups.

Let p be a prime.
A finite group G is ***p*-nilpotent**, if it has a normal subgroup N such that

$$p \nmid |N| \quad \text{and} \quad G/N \text{ is a } p\text{-group.}$$

Example. S_3 is 2-nilpotent, but not 3-nilpotent.

Theorem (Thompson 1970)
If $p \mid \chi(1)$, for all non-linear $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, then G is p-nilpotent.

Characters and block structure

Generalization of p-nilpotent groups: **nilpotent** p-blocks.

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Conjecture (Malle-Navarro)

The p-block B of G is nilpotent if and only if all $\chi \in B$ of height 0 have the same degree.

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Theorem (Malle-Navarro 2011)

Let G be quasi-simple, B a p-block which is neither a spin block of the double cover of the alternating group, nor a quasi-isolated block of an exceptional group of Lie type for p a bad prime. Then the conjecture holds for B.

For the symmetric groups and a prime p:

p-blocks $\leftrightarrow$ p-core partitions

Degree computation for irreducible characters:

hook formula

Malle-Navarro: not adequate for the purpose ...

New relative degree formula:

factor the character degrees along their p-core degrees.

The Hook Formula

Theorem (Frame, Robinson, Thrall 1954)

Let $\prod \mathcal{H}(\lambda)$ *be the product of all hook lengths in* $\lambda \vdash n$. *Then*

$$[\lambda](1) = \frac{n!}{\prod \mathcal{H}(\lambda)} .$$

Example

Let $\lambda = (5, 4, 4, 2, 2) \vdash 17$.

9	8	5	4	1
7	6	3	2	
6	5	2	1	
3	2			
2	1			

$$[\lambda](1) = \frac{17!}{9\cdot8\cdot5\cdot4\cdot1\cdot7\cdot6\cdot3\cdot2\cdot6\cdot5\cdot2\cdot1\cdot3\cdot2\cdot2\cdot1} = 1.361.360$$

d-cores

Let $d \in \mathbb{N}$. For a partition λ, denote by $\lambda_{(d)}$ its ***d*-core**, obtained by removing as many d-hooks as possible.

Example

Let $d = \mathbf{5}$, $\lambda = (5,4,4,2,2) \vdash 17$. Then $\lambda_{(5)} = (3,1,1,1,1)$:

9	8	**5**	4	1
7	6	3	2	
6	**5**	2	1	
3	2			
2	1			

9	**5**	4	3	1
7	3	2	1	
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Removal process may be described by the d-quotient $\lambda^{(d)}$, a d-tuple of partitions.

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1		

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Remark. $[\lambda]$ is of height 0 $\Leftrightarrow [\lambda](1)_p = [\lambda_{(p)}](1)_p = |\lambda_{(p)}|_p$.

Theorem (Malle-Navarro: relative degree formula)

Let p be a prime, $\lambda \vdash n$, $\lambda_{(p)} \vdash r$. Let S be a symbol associated to the p-quotient $\lambda^{(p)}$, b_i the number of beads on the i^{th} runner of the p-abacus for $\lambda_{(p)}$, $c_i = pb_i + i - 1$. Then

$$[\lambda](1) = \frac{n!}{r!} \frac{1}{\prod_{h \text{ hook of } S} |p\ell(h) + c_{i(h)} - c_{j(h)}|} [\lambda_{(p)}](1) \, .$$

Note on the proof: In his work on unipotent character degrees of general linear groups (1995), Malle used p-symbols as labels, defined hooks (and associated lengths) in p-symbols and proved a ‘hook formula’ for the unipotent degrees. Its specialization at $q = 1$ is crucial for the relative degree formula.

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B.-Gramain-Olsson: Generalized hook lengths in symbols and partitions, arXiv 1101.5067

Useful tool: β-sets

Any finite subset $X = \{a_1, \ldots, a_s\}_>$ of $\mathbb{N}_0$ is a **β-set**.

This is a **β-set for the partition** $\lambda = p(X)$ with parts the positive numbers among

$$a_i - (s - i),\ i = 1, \ldots, s.$$

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- For the shifts $X^{+k} = \{a + k \mid a \in X\} \cup \{k - 1, \ldots, 1, 0\}$ we have: $p(X) = p(X^{+k})$.
- The set of first column hook lengths of λ is a β-set for λ.

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- The set of first column hook lengths of λ is a β-set for λ.

A ***d*-hook of X** is a pair $(a, b) \in \mathbb{N}_0^2$ with

$$a \in X,\ b < a,\ b \notin X \text{ and } a - b = d.$$

Removal of this d-hook from X: replace a by b
($\leftrightarrow$ removal of a d-hook from $\lambda = p(X)$).

The d-abacus

Place the elements of X as beads on an abacus with d runners!

Example

$X = \{\mathbf{11}, \mathbf{8}, \mathbf{6}, \mathbf{2}, \mathbf{0}\}$ is a β-set of $p(X) = \lambda = (7, 5, 4, 1) \vdash 17$.
Fix $d = 3$. The 3-abacus representation for X and its 3-core:

0	1	**2**
3	4	5
6	7	**8**
9	10	**11**

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3	4	5		**3**	4	**5**
6	7	**8**		6	7	**8**
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3-core $C_3(X) = \{8, 5, 3, 2, 0\}$

$$c_3(X) = p(C_3(X)) = p(\{8, 5, 3, 2, 0\}) = (4, 2, 1, 1) = \lambda_{(3)}$$

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Remarks.

- Easy computation of d-core.
- d-core independent of removal process!

d-symbols

A ***d*-symbol** is a d-tuple of β-sets $S = (X_0, \ldots, X_{d-1})$.

We have a bijection

$$\begin{array}{rccc} s_d : & \{\beta\text{-sets}\} & \to & \{d\text{-symbols}\} \\ & X & \mapsto & (X_0^{(d)}, \ldots, X_{d-1}^{(d)}), \end{array}$$

where $X_j^{(d)} = \{k \in \mathbb{N}_0 \mid kd + j \in X\}$, $j = 0, \ldots, d-1$.

A **hook** of S: $(a, b, i, j) \in \mathbb{N}_0^4$ with $i, j \in \{0, \ldots, d-1\}$, $a \in X_i$, $b \notin X_j$, and either $a > b$, or $a = b$ and $i > j$.

$H(S)$ = the set of all hooks of S.

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$H(S)$ = the set of all hooks of S.

Remark. There are canonical bijections between the hooks in X, $\lambda = p(X)$ and $S = s_d(X)$.

Example

β-set $X = \{11, 8, 6, 2, 0\}$ for $p(X) = \lambda = (7, 5, 4, 1) \vdash 17$.
Let $d = 3$; 3-abacus representation for X and $S = s_3(X)$:

$$s_3 : \begin{array}{ccc} 0 & 1 & 2 \\ \hline \mathbf{0} & 1 & \mathbf{2} \\ 3 & 4 & 5 \\ \mathbf{6} & 7 & \mathbf{8} \\ 9 & 10 & \mathbf{11} \end{array} \mapsto \begin{array}{ccc} 0 & 1 & 2 \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ 1 & 1 & 1 \\ \mathbf{2} & 2 & \mathbf{2} \\ 3 & 3 & \mathbf{3} \end{array}$$

$$S = (\{2, 0\}, \emptyset, \{3, 2, 0\})$$

Example: hook $(11, 4)$ in $X \leftrightarrow$ hook $(3, 1, 2, 1)$ in S.

Balanced quotients

Let $S = (X_0, \ldots, X_{d-1})$ be a d-symbol.

S is **balanced**, if $|X_0| = \ldots = |X_{d-1}|$ and $0 \notin X_i$ for some i.

The **balanced quotient** of S is the unique *balanced* d-symbol

$$Q(S) = (X'_0, \ldots, X'_{d-1}) \text{ with } p(X'_i) = p(X_i) \text{ for all } i.$$

The **core** of S is the d-symbol $C(S)$ with i^{th} component

$$\{|X_i| - 1, \ldots, 1, 0\},\ i = 0, \ldots, d-1.$$

If $X = s_d^{-1}(S)$, the **balanced d-quotient of X** is the β-set

$$Q_d(X) = s_d^{-1}(Q(S))$$

and the **d-quotient partition of $\lambda = p(X)$** is

$$q_d(X) = p(Q_d(X))\,.$$

Example

Let $S = s_3(X) = (\{2,0\}, \emptyset, \{3,2,0\})$.

Associated partitions: $((1), \emptyset, (1,1))$.

Balanced quotient of S: $Q(S) = (\{2,0\}, \{1,0\}, \{2,1\})$.

$$Q(S): \begin{array}{ccc} \mathbf{0} & \mathbf{0} & 0 \\ 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & 2 & \mathbf{2} \\ 3 & 3 & 3 \end{array} \quad \xmapsto{s_3^{-1}} \quad Q_3(X): \begin{array}{ccc} \mathbf{0} & \mathbf{1} & 2 \\ 3 & \mathbf{4} & \mathbf{5} \\ \mathbf{6} & 7 & \mathbf{8} \\ 9 & 10 & 11 \end{array}$$

$$q_3(X) = p(Q_3(X)) = p(\{8,6,5,4,1,0\}) = (3,2,2,2)$$

Note: $|q_3(X)| + |c_3(X)| = 9 + 8 = 17 = |p(X)|$.

Connections between a β-set X, its associated d-symbol $S = s_d(X)$ and associated partition $\lambda = p(X)$:

$$
\begin{array}{ccccc}
q_d(X) & \xleftarrow{\;p\;} & Q_d(X) & \xrightarrow{\;s_d\;} & Q(S) \\
\uparrow{\scriptstyle quot} & & \uparrow{\scriptstyle quot} & & \uparrow{\scriptstyle quot} \\
\lambda = p(X) & \xleftarrow{\;p\;} & X & \xrightarrow{\;s_d\;} & S \\
\downarrow{\scriptstyle core} & & \downarrow{\scriptstyle core} & & \downarrow{\scriptstyle core} \\
\lambda_{(d)} = c_d(X) & \xleftarrow{\;p\;} & C_d(X) & \xrightarrow{\;s_d\;} & C(S)
\end{array}
$$

Note that $q_d(X)$ is *not* the usual d-quotient for λ!

What are we trying to do about the relative degree formula?

Example

As before: $\lambda = (7,5,4,1)$, $X = \{11,8,6,2,0\}$, $d = 3$.

$S = (\{2,0\}, \emptyset, \{3,2,0\})$, $(x_0, x_1, x_2) = (2,0,3)$.

3-core and 3-quotient partitions to λ:

$$\lambda_{(3)} = (4,2,1,1),\ q_3(X) = (3,2,2,2)\,.$$

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Hook diagrams for λ, $\lambda_{(3)}$:

$$\begin{array}{ccccccc} 10 & 8 & 7 & 6 & 4 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & & \\ 5 & 3 & 2 & 1 & & & \\ 1 & & & & & & \end{array} \qquad \begin{array}{cccc} 7 & 4 & 2 & 1 \\ 4 & 1 & & \\ 2 & & & \\ 1 & & & \end{array}$$

What are we trying to do about the relative degree formula?

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Hook diagrams for λ, $\lambda_{(3)}$:

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7	5	4	3	1		
5	3	2	1			
1						

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2			
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3-core and 3-quotient partitions to λ:

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Hook diagrams for λ, $q_3(X)$:

$$\begin{array}{ccccccc} 10 & 8 & 7 & 6 & 4 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & & \\ 5 & 3 & 2 & 1 & & & \\ 1 & & & & & & \end{array} \quad \overset{?}{\longleftrightarrow} \quad \begin{array}{ccc} 6 & 5 & 1 \\ 4 & 3 & \\ 3 & 2 & \\ 2 & 1 & \end{array}$$

Let $S = (X_0, \ldots, X_{d-1})$ be a d-symbol.
We consider only the hooks between runners i and j:

$$\begin{aligned} H_{ij}(S) &= \{(a,b,i,j) \mid (a,b,i,j) \in H(S)\}, \\ H_{\{ij\}}(S) &= H_{ij}(S) \cup H_{ji}(S)\,. \end{aligned}$$

For $\ell \geq 0$ we define the ℓ-level section

$$H_{ij}^{\ell}(S) = \{(a,b,i,j) \in H_{ij}(S) \mid a - b = \ell\}\,.$$

Hook correspondence in symbols

Theorem

Let S be a d-symbol with balanced quotient Q and core C.
For all i, j, we have bijective multiset correspondences

$$H_{\{ij\}}(S) \to H_{\{ij\}}(Q) \cup H_{\{ij\}}(C) \, ,$$

with control on the level sections.
We glue these bijections together to a universal bijection

$$\omega_S : H(S) \to H(Q) \cup H(C) \, .$$

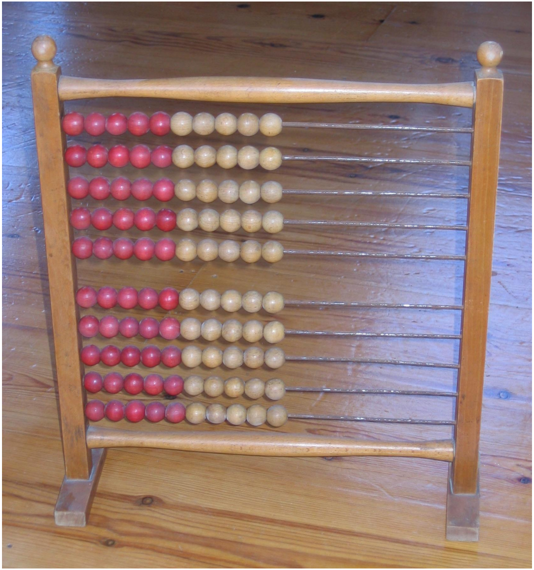
Remark. For $S = (X_0, \ldots, X_{d-1})$, the differences $|X_i| - |X_j|$ are crucial for controlling the correspondence of the level sections.

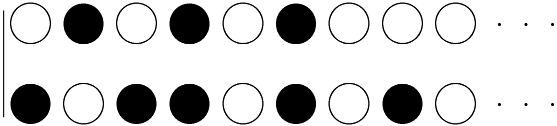
Theorem. Let S, Q, C be as above, $i \neq j$, $\Delta = |X_i| - |X_j| \geq 0$. When $\Delta > 0$, we have the following equalities:

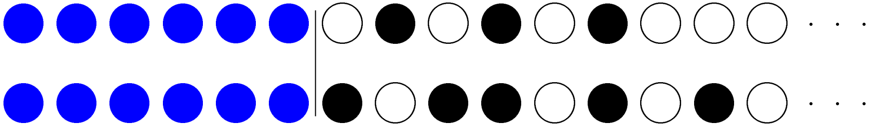
- For all $\ell > \Delta$: $|H_{ij}^{\ell}(S)| = |H_{ij}^{\ell-\Delta}(Q)|$.
- For all $\ell > \Delta$: $|H_{ji}^{\ell-\Delta}(S)| = |H_{ji}^{\ell}(Q)|$.
- For all $0 < \ell < \Delta$: $|H_{ij}^{\ell}(S)| = |H_{ji}^{\Delta-\ell}(Q)| + |H_{ij}^{\ell}(C)|$.
- For $\ell = \Delta$: $|H_{ij}^{\Delta}(S)| = \begin{cases} |H_{ij}^{0}(Q)| = |H_{\{ij\}}^{0}(Q)| & \text{if } i > j \\ |H_{ji}^{0}(Q)| = |H_{\{ij\}}^{0}(Q)| & \text{if } i < j \end{cases}$.
- For $\ell = 0$:
  $$|H_{ji}^{\Delta}(Q)| + |H_{ij}^{0}(C)| = \begin{cases} |H_{ij}^{0}(S)| = |H_{\{ij\}}^{0}(S)| & \text{if } i > j \\ |H_{ji}^{0}(S)| = |H_{\{ij\}}^{0}(S)| & \text{if } i < j \end{cases}.$$
- $|H_{ij}^{\Delta}(S)| + |H_{\{ij\}}^{0}(S)| = |H_{ji}^{\Delta}(Q)| + |H_{\{ij\}}^{0}(Q)| + |H_{ij}^{0}(C)|$.

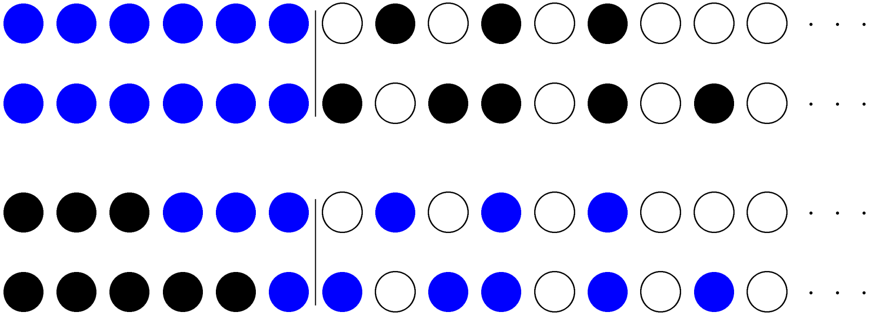
When $\Delta = 0$, we have

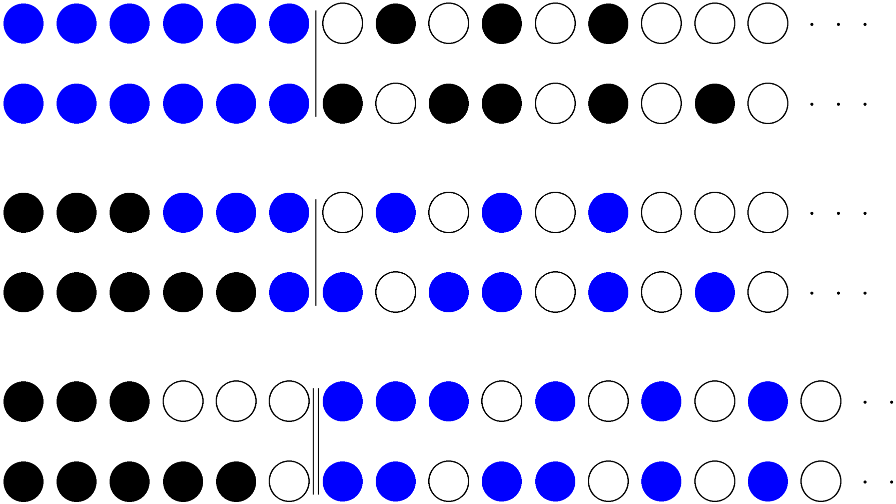
- $|H_{ij}^{\ell}(S)| = |H_{ij}^{\ell}(Q)|$, $H_{ij}^{\ell}(C) = \emptyset$, for all $\ell \geq 0$.

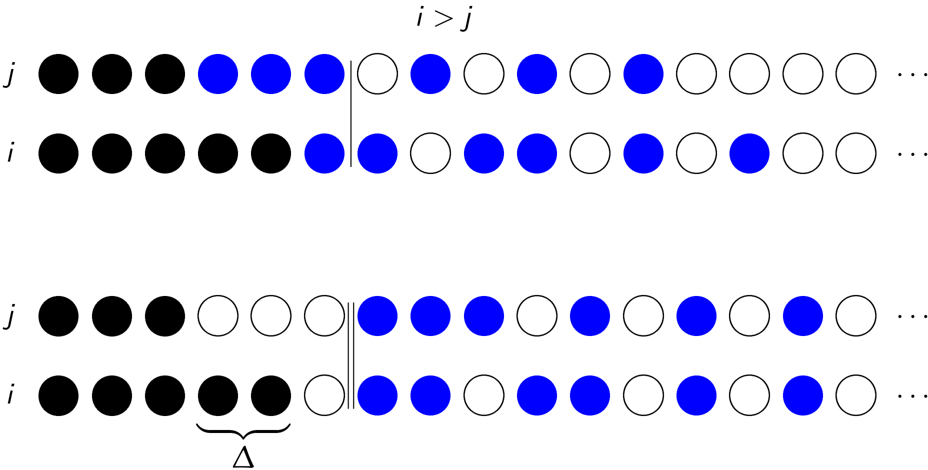






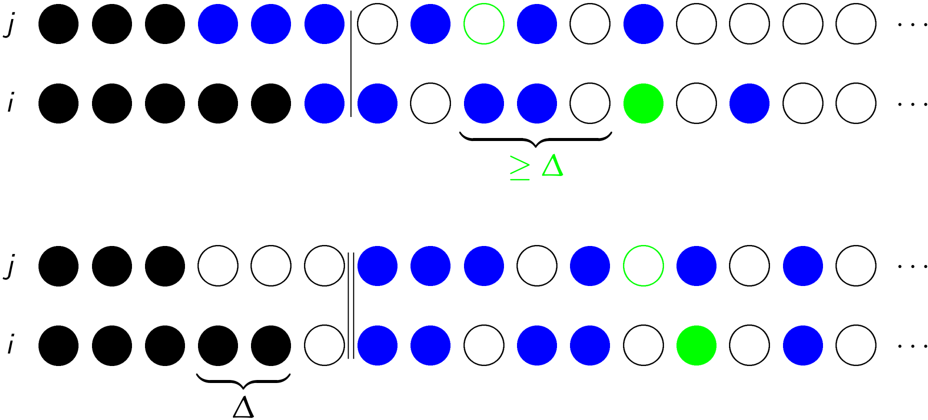


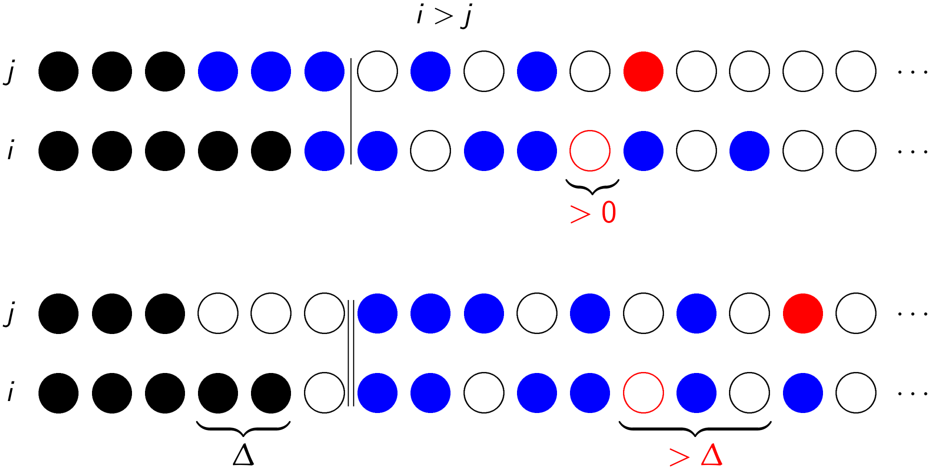




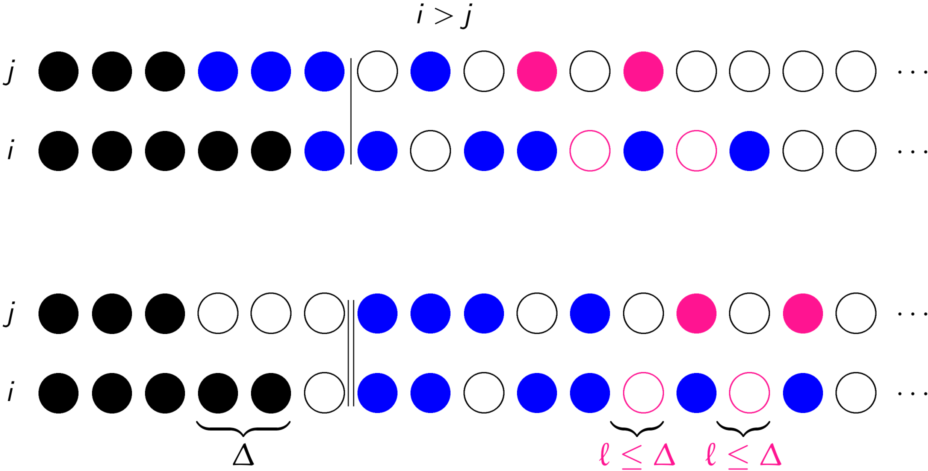
$i > j$
j
i
j
i
Δ

$i > j$



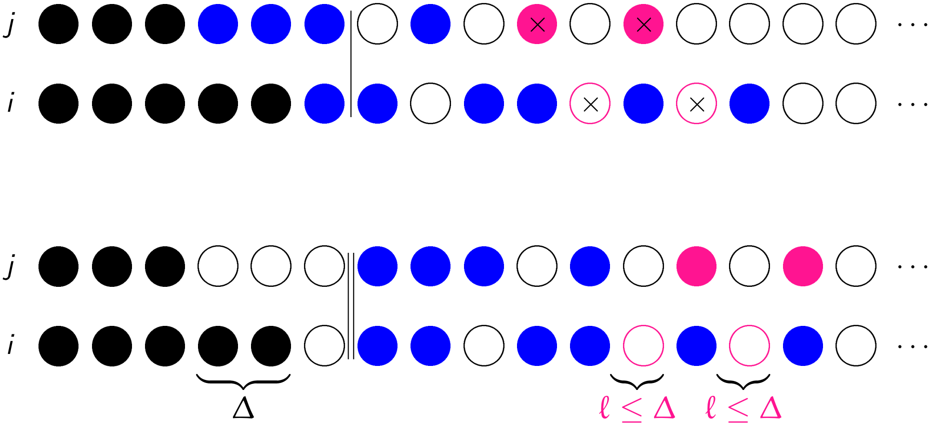


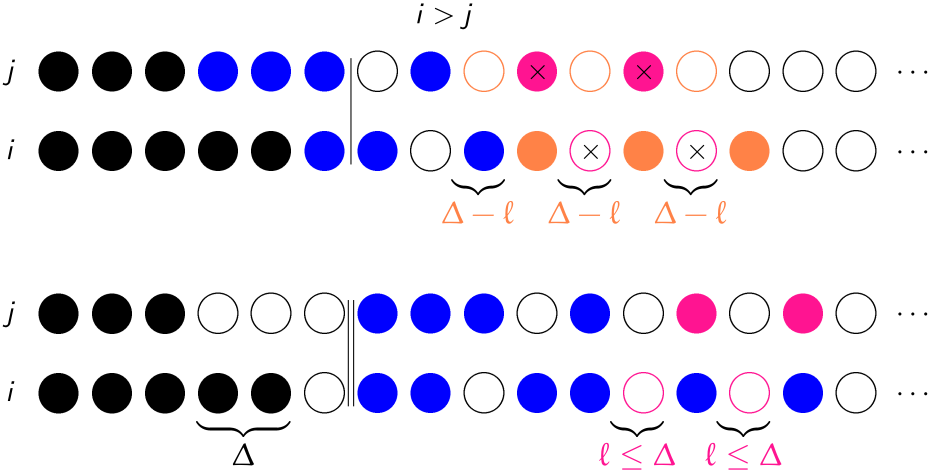
$i > j$
j
i
> 0
j
i
Δ
$> \Delta$



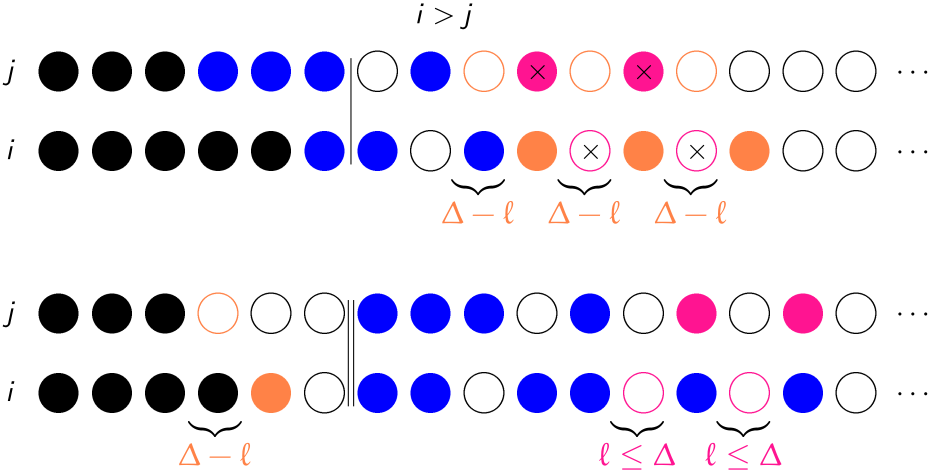
$i > j$
j
i
j
i
Δ
$\ell \leq \Delta$
$\ell \leq \Delta$

$i > j$





$i > j$
j
i
$\Delta - \ell$
$\Delta - \ell$
$\Delta - \ell$
j
i
Δ
$\ell \leq \Delta$
$\ell \leq \Delta$



i > j
j
i
Δ − ℓ
Δ − ℓ
Δ − ℓ
j
i
Δ − ℓ
ℓ ≤ Δ
ℓ ≤ Δ

Let $H = \{(a, b, i, j) \mid a \geq b \text{ and } i > j \text{ if } a = b\}$.

Consider **(generalized) hook length functions** $h : H \to \mathbb{R}$ s.t. **the value $h(a, b, i, j)$ depends only on $\ell = a - b$, i and j.**

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the value $h(a, b, i, j)$ depends only on $\ell = a - b$, i and j.

Important hook length functions for d-symbols:
d-hook data tuple:

$$\delta = (c_0, c_1, \ldots, c_{d-1}; k),\ c_0, \ldots, c_{d-1}, k \in \mathbb{R},\ k \geq 0.$$

δ-length of $(a, b, i, j) \in H$:

$$h^\delta(a, b, i, j) = k(a - b) + c_i - c_j \,.$$

For any d-symbol S, the multiset of generalized hook lengths is

$$\mathcal{H}^\delta(S) = \{h^\delta(a, b, i, j) \mid (a, b, i, j) \in H(S)\}.$$

Important special choices for applications:

- $\delta = (0, 1, \ldots, d-1; d)$ the **partition d-hook data tuple**.

  Then the δ-length of a hook of S equals the usual hook length $a - b$ of the corresponding hook (a, b) of X.

- $\delta = (0, 0, \ldots, 0; 1)$ the **minimal d-hook data tuple**.

  Then the δ-length of long hooks $(a > b)$ in S coincides with the hook length in symbols as defined by Malle, and the short hooks $(a = b)$ have δ-length 0.

The Meta-Theorem

Theorem

Let $S = (X_0, X_1, \ldots, X_{d-1})$ be a d-symbol, $x_i = |X_i|$.

Let Q be its balanced quotient, C be its core.

Let $\delta = (c_0, c_1, \ldots, c_{d-1}; k)$ be a d-hook data tuple, and set $\delta_S = (c_0 + x_0 k, c_1 + x_1 k, \ldots, c_{d-1} + x_{d-1} k; k)$.

Then we have the multiset equality

$$\mathcal{H}^{\delta}(S) = \mathcal{H}^{\delta}(C) \cup \overline{\mathcal{H}}^{\delta_S}(Q),$$

where $\overline{\mathcal{H}}^{\delta_S}(Q) = \{\overline{h}^{\delta_S}(z) \mid z \in H(Q)\}$
is the multiset of all modified δ_S-lengths *of hooks in Q.*

Modified hook lengths

We assume that i, j are such that $\Delta = x_i - x_j \geq 0$.

Let $\quad H_{ij}^{\ell} = \{(a, b, i, j) \in H \mid a - b = \ell\}$.

Then for $z \in H_{\{ij\}}$ we define

$$\overline{h}^{\delta_S}(z) = \begin{cases} h^{\delta_S}(z) & \text{if } z \in H_{ij} \cup H_{ji}^{>\Delta}, \text{ or } z \in H_{ji}^{\Delta} \text{ if } i < j \\ -h^{\delta_S}(z) & \text{otherwise} \end{cases}$$

Crucial property w.r.t. the universal bijection ω_S:

$$h^{\delta}(z) = \begin{cases} h^{\delta}(\omega_S(z)) & \text{if} \quad \omega_S(z) \in H(C) \\ \overline{h}^{\delta_S}(\omega_S(z)) & \text{if} \quad \omega_S(z) \in H(Q) \end{cases}$$

Application for partitions

Theorem

Let $d \in \mathbb{N}$, λ a partition, X a β-set for λ, $x_i = |X_i^{(d)}|$.

Let $q_d(X)$ be the d-quotient partition of X.

For $z \in H(q_d(X))$ with hand and foot d-residue i and $j+1$, respectively, let

$$\overline{h}(z) = h(z) + (x_i - x_j)d\ .$$

Let $\overline{\mathcal{H}}(q_d(X))$ be the multiset of all $\overline{h}(z)$, $z \in H(q_d(X))$.

Then we have the multiset equality

$$\mathcal{H}(\lambda) = \mathcal{H}(\lambda_{(d)}) \cup \mathrm{abs}(\overline{\mathcal{H}}(q_d(X))$$

where $\mathrm{abs}(\overline{\mathcal{H}}(q_d(X)) = \{|m| \mid m \in \overline{\mathcal{H}}(q_d(X))\}$.

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Theorem

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Corollary Generalization of the Malle-Navarro formula.
In particular, the Malle-Navarro formula **is** the hook formula!

Example

As before: $\lambda = (7, 5, 4, 1)$, $X = \{11, 8, 6, 2, 0\}$, $d = 3$.

$S = (\{2, 0\}, \emptyset, \{3, 2, 0\})$, $(x_0, x_1, x_2) = (2, 0, 3)$.

3-core and 3-quotient partitions to λ:

$$\lambda_{(3)} = (4, 2, 1, 1),\ q_3(X) = (3, 2, 2, 2)\,.$$

Hook diagrams for λ, $\lambda_{(3)}$, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

7	4	2	1
4	1		
2			
1			

6	5	1
4	3	
3	2	
2	1	

Hook diagrams for λ, $q_3(X)$:

$$\begin{array}{ccccccc} 10 & 8 & 7 & 6 & 4 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & & \\ 5 & 3 & 2 & 1 & & & \\ 1 & & & & & & \end{array} \quad \overset{?}{\longleftrightarrow} \quad \begin{array}{ccc} 6 & 5 & 1 \\ 4 & 3 & \\ 3 & 2 & \\ 2 & 1 & \end{array}$$

Hook diagrams for λ, $q_3(X)$:

$$\begin{matrix} 10 & 8 & \color{magenta}{7} & 6 & \color{magenta}{4} & \color{magenta}{2} & \color{magenta}{1} \\ 7 & 5 & \color{magenta}{4} & 3 & \color{magenta}{1} & & \\ 5 & 3 & \color{magenta}{2} & 1 & & & \\ \color{magenta}{1} & & & & & & \end{matrix} \quad \overset{?}{\longleftrightarrow} \quad \begin{matrix} 6 & 5 & 1 \\ 4 & 3 & \\ 3 & 2 & \\ 2 & 1 & \end{matrix}$$

Consider the 3-residue diagram of $q_3(X)$.

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 0 & \\ 1 & 2 & \\ 0 & 1 & \end{matrix}$$

Hook diagrams for λ, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

$$\overset{?}{\longleftrightarrow}$$

6	5	1
4	3	
3	2	
2	1	

Modify the length of each hook in $q_3(X)$ by $3(x_i - x_j)$ according to residues i and $j+1$ of its hand and foot.

				i
	0	1	2	**2**
	2	0		**0**
	1	2		**2**
	0	1		**1**
j	**2**	**0**	**1**	

Hook diagrams for λ, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

$\overset{?}{\longleftrightarrow}$

6	5	1
4	3	
3	2	
2	1	

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Recall: $(x_0, x_1, x_2) = (2, 0, 3)$.

				i
	0	1	2	**2**
	2	0		**0**
	1	2		**2**
	0	1		**1**
j	**2**	**0**	**1**	

$3x_i \backslash -3x_j$	**−9**	**−6**	**0**
9	6	5	1
6	4	3	
9	3	2	
0	2	1	

Hook diagrams for λ, $q_3(X)$:

10	8	7	6	4	2	1
7	5	4	3	1		
5	3	2	1			
1						

$\overset{?}{\longleftrightarrow}$

6	5	1
4	3	
3	2	
2	1	

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$3x_i \backslash -3x_j$	$\mathbf{-9}$	$\mathbf{-6}$	$\mathbf{0}$
$\mathbf{9}$	6	5	1
$\mathbf{6}$	4	3	
$\mathbf{9}$	3	2	
$\mathbf{0}$	2	1	

$\longrightarrow$

6	8	10
1	3	
3	5	
−7	−5	

Hook diagrams for λ, $q_3(X)$:

$$\begin{array}{ccccccc} 10 & 8 & 7 & 6 & 4 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & & \\ 5 & 3 & 2 & 1 & & & \\ 1 & & & & & & \end{array} \quad \overset{?}{\longleftrightarrow} \quad \begin{array}{ccc} 6 & 5 & 1 \\ 4 & 3 & \\ 3 & 2 & \\ 2 & 1 & \end{array}$$

Finally, take absolute values!

$$\begin{array}{ccc} 6 & 8 & 10 \\ 1 & 3 & \\ 3 & 5 & \\ -7 & -5 & \end{array} \quad \to \quad \begin{array}{ccc} 6 & 8 & 10 \\ 1 & 3 & \\ 3 & 5 & \\ 7 & 5 & \end{array}$$

Hook diagrams for λ, $q_3(X)$:

$$\begin{matrix} 10 & 8 & 7 & 6 & 4 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & & \\ 5 & 3 & 2 & 1 & & & \\ 1 & & & & & & \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} 6 & 5 & 1 \\ 4 & 3 & \\ 3 & 2 & \\ 2 & 1 & \end{matrix}$$

Finally, take absolute values!

$$\begin{matrix} 6 & 8 & 10 \\ 1 & 3 & \\ 3 & 5 & \\ -7 & -5 & \end{matrix} \quad \to \quad \begin{matrix} 6 & 8 & 10 \\ 1 & 3 & \\ 3 & 5 & \\ 7 & 5 & \end{matrix}$$

Generalizations

Symbols were introduced by Lusztig (1977) as labels for characters of classical groups; generalized notions of ℓ-cores, (ℓ, e)-cores etc. for symbols.

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Theorem

Let $S = (X_0, X_1, \ldots, X_{d-1})$ be a d-symbol, $\delta = (0, \ldots, 0; 1)$, $\ell \in \mathbb{N}$. Let C be the ℓ-core and Q the balanced ℓ-quotient of S. Then we have a multiset equality for the δ-lengths of hooks in S:

$$\mathcal{H}^{\delta}(S) = \mathcal{H}^{\delta}(C) \cup abs(\mathcal{H}^{\delta_{\ell,S}}(Q))$$

where $abs(\mathcal{H}^{\delta_{\ell,S}}(Q))$ is the multiset of all $|h^{\delta_{\ell,S}}(z)|$, $z \in H(Q)$, $\delta_{\ell,S}$ a modified $d\ell$-hook data tuple.