Discrete Morse Theory (for Posets)

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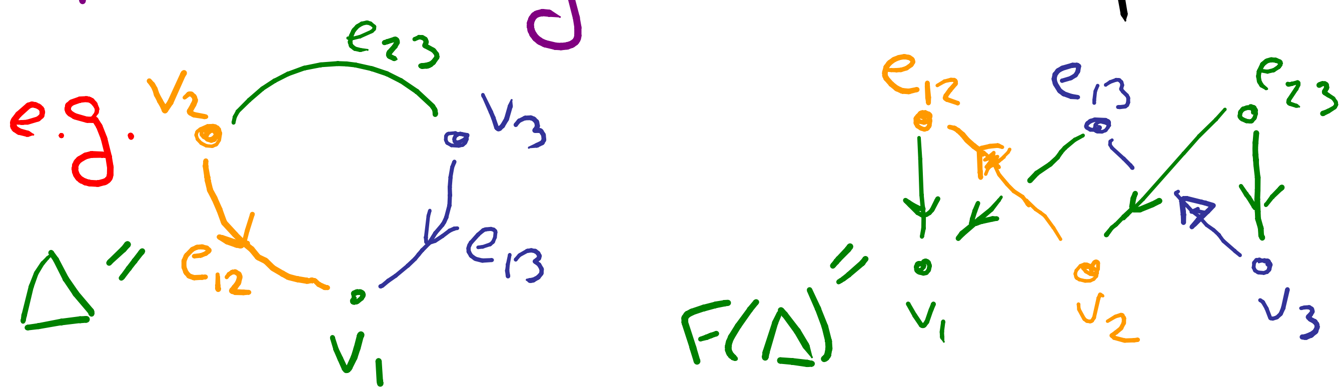
Discrete Morse Theory
(introduced by Robin Forman)

Given simplicial complex Δ, one constructs an "acyclic matching" aka "Morse matching" on its face poset

e.g. 

s.t. directed graph with matching edges oriented upward has no directed cycles.

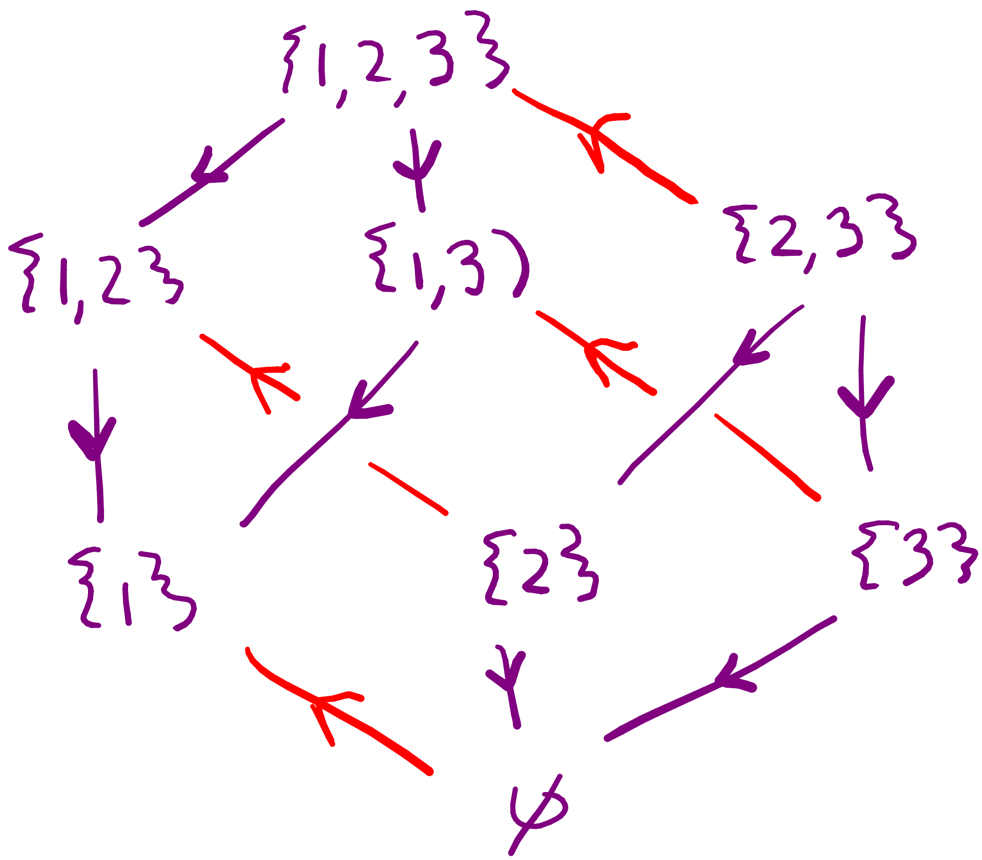
Theorem (Forman): Δ is homotopy equivalent to a CW complex Δ^M comprised of the unmatched elements, the so-called "critical cells"

e.g. 

First Example

Δ = simplex

$F(\Delta)$ = Boolean algebra



- Match by adding 1 to set
- Directed cycles would alternate up and down since forbidden
- Acyclic since no arrows delete 1

Types of Things one can try to Prove by Discrete Morse Theory

1. Δ is collapsible (and hence contractible)
   - suffices to find complete acyclic matching on $F(\Delta)$

Warning: Some contractible Δ are not collapsible, e.g. "dunce cap"

2. Δ is homotopy equivalent to a wedge of j-dim'l spheres

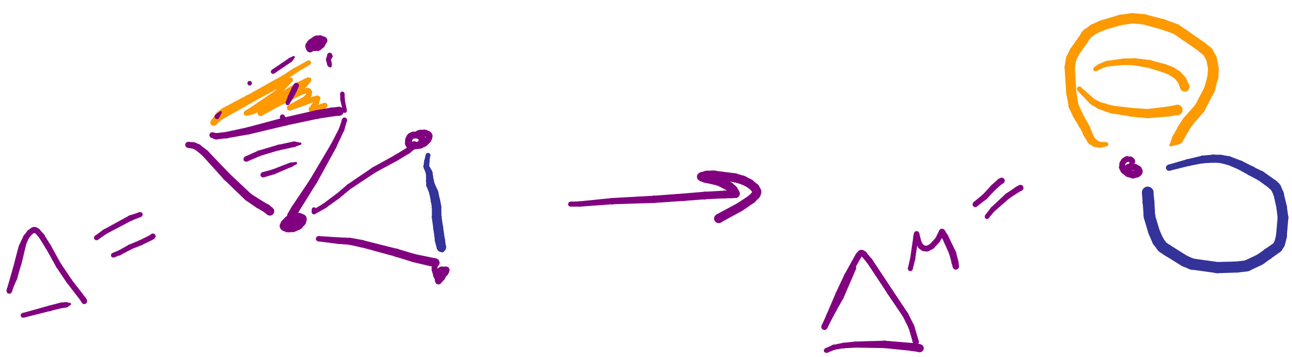
- suffices to find acyclic matching such that all critical cells are j-dimensional

3. Δ is simply connected with $\tilde{H}_i(\Delta;\mathbb{Z})=0$ for $i<j$

- suffices to find acyclic matching with all critical cells in dimensions $\geq j$

4. Δ homotopy equiv. to wedge of spheres of various dimensions

- if all critical cells are facets (so all attach only to base point in Δ^M)



5. $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^M)$

$= -1 + \#\,0\text{-cells} - \#\,1\text{-cells} + \#\,2\text{-cells} - \cdots$

$= -1 + \beta_0 - \beta_1 + \beta_2 - \cdots$

which can be easier to compute for Δ^M

For Posets: $\mu_P(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^M(x,y))$

6. Δ is homotopy equivalent to a wedge of even dimensional spheres (and has homology concentrated in even dimensions)

- if all critical cells are even dimensional

Formalities (of discrete Morse theory)

A discrete Morse function is a function $f: \Delta \to \mathbb{R}$ assigning real numbers to the faces of a simplicial complex or cells of (regular) cell complex s.t. for each $\sigma^{(p)}$ ← notation for cell being p-dimensional

1. $\left| \{ \tau^{(p+1)} \mid \sigma^{(p)} \leq \overline{\tau^{(p+1)}} \ \& \ f(\sigma) \geq f(\tau) \} \right| \leq 1$

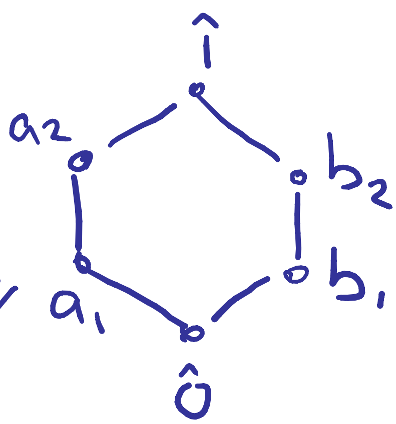
& 2. $\left| \{ \mu^{(p-1)} \mid \mu^{(p-1)} \leq \overline{\sigma^{(p)}} \ \& \ f(\mu) \geq f(\sigma) \} \right| \leq 1$

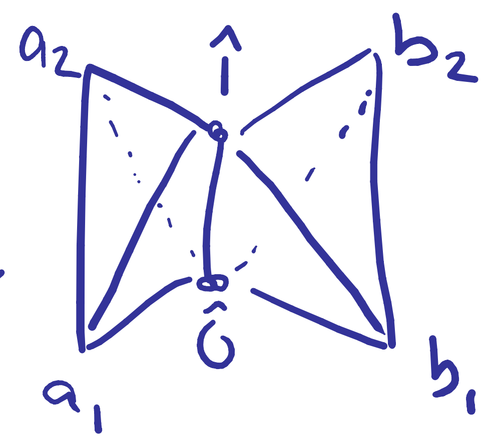
Exercise: Prove for any shelling order $F_1, \ldots, F_k$ on the facets of a simplicial complex Δ that there is an acyclic matching on each poset $F(\overline{F_j} \setminus \bigcup_{i<j} \overline{F_i})$ whose union is an acyclic matching on $F(\Delta)$ with critical cells the "homology facets" of the shelling, i.e. those F_j closing off spheres.

Hint: Prove for any filtration $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k$ of simplicial complexes that a union of acyclic matchings on posets $F(\Delta_j \setminus \Delta_{j-1})$ is acyclic.

Qn: How to use this for computing Möbius functions and other poset topology?

Def'n: The order complex (or nerve) of a poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \dots < v_i$ in P

e.g. $P =$ 

$\Delta(P) =$ 

Key Property (Hall; popularized by Rota):

$$\mu_P(x,y) = \tilde{\chi}(\underbrace{\Delta_P(x,y)}_{\substack{\text{order complex for} \\ \{z \in P \mid x < z < y\}}}) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{2-faces} \dots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \dots$$

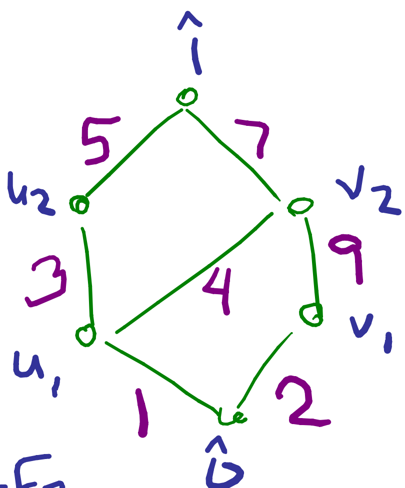
A Discrete Morse Theory Approach to Poset Order Complexes

(partly joint with Eric Babson)

Step 1: Any edge labeling (or chain labeling) on poset P induces lexicographic order $F_1, \ldots, F_m$ on maximal faces (facets) of $\Delta(P)$

e.g.

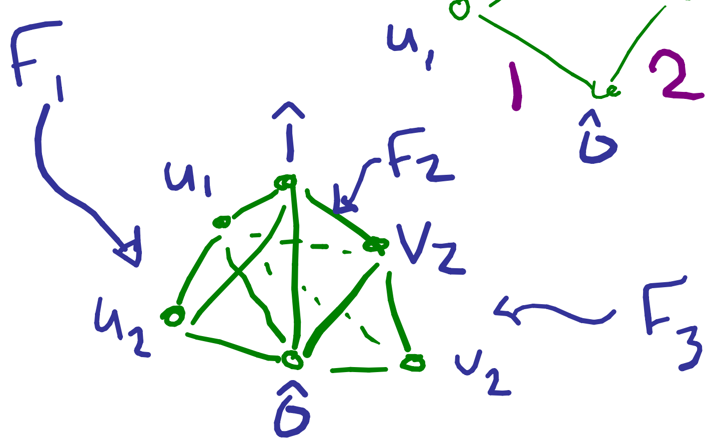
$P =$



$F_1 = 135$

$F_2 = 147$

$F_3 = 297$



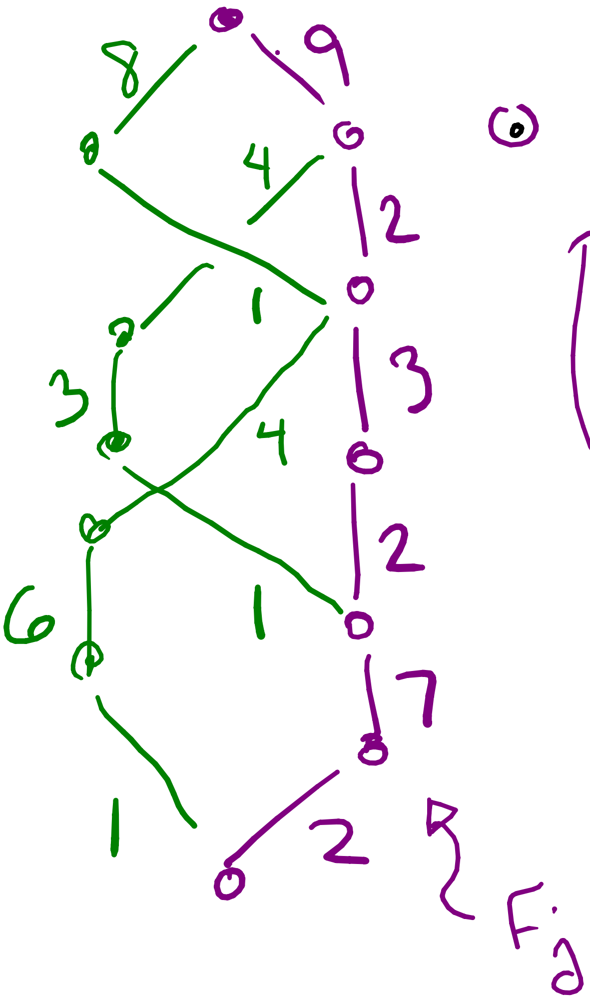
Step 2: Morse matching on each $\overline{F}_j \setminus (\cup_{i<j} \overline{F}_i)$ s.t.

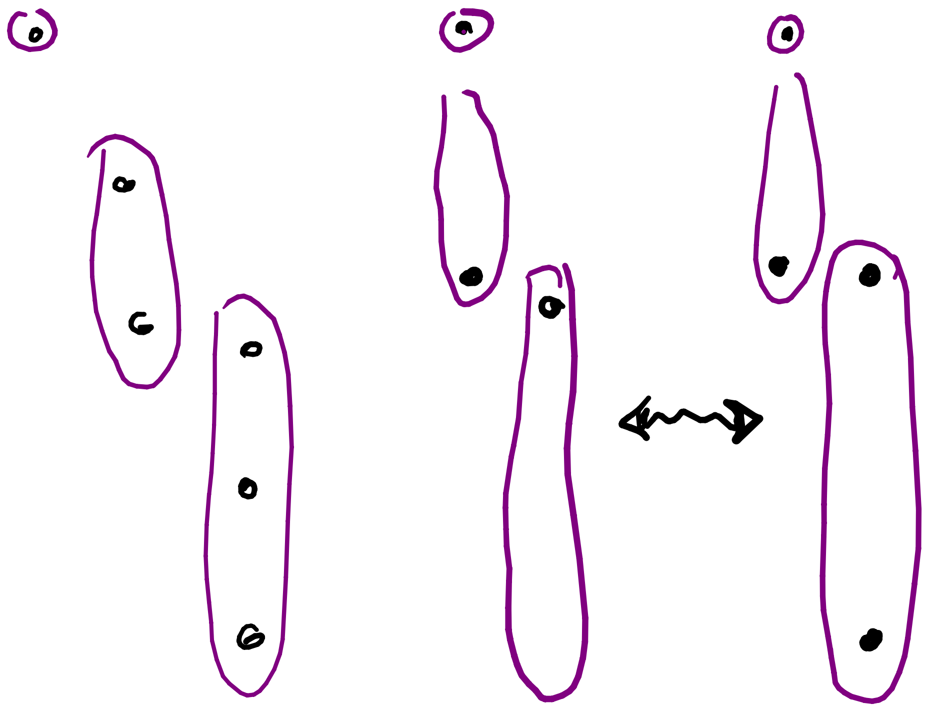
(1) Each $\overline{F}_j \setminus (\cup_{i<j} \overline{F}_i)$ has 0 or 1 unmatched (critical) faces

(2) Union of matchings is Morse matching for $\Delta(P)$

Theorem (Babson-H, 2005) Every edge labeling on any finite poset gives rise to lexicographic discrete Morse function with "few" critical cells (0 or 1 crit. cell per facet attachment)

Acyclic matching idea



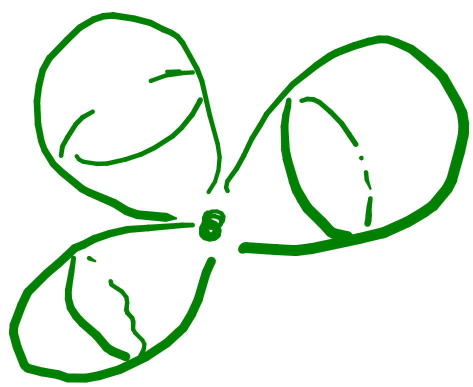


$\{3,5\} \leftrightsquigarrow \{1,3,5\}$

Faces in $\overline{F}_j \setminus \bigcup_{i<j} \overline{F}_i$ $\longleftrightarrow$ subsets of ranks $\{1,2,\ldots,5\}$ hitting $\{1,2,3\}$ & $\{3,4\}$ & $\{5\}$

Remarks • lexicographic shellability of Björner & Wachs is special case with intervals size one

• useful even when $\Delta(P)$ not homotopy equivalent to wedge of spheres



• implies lexicographic order facet attachments each change at most one betti number

Philosophy:

Use "natural" labelings where you can get a good handle on types of intervals arising in interval systems

e.g. weak order: intervals are homotopy equiv to balls or spheres (not top dim'l)

$$
\begin{tikzcd}
 & 321 \arrow[dl, dash, "s_2"'] \arrow[dr, dash, "s_1"] & \\
312 \arrow[d, dash, "s_1"'] & & 231 \arrow[d, dash, "s_2"] \\
132 \arrow[dr, dash, "s_2"'] & & 213 \arrow[dl, dash, "s_1"] \\
 & 123 &
\end{tikzcd}
$$

Rk: Tends to work well for posets arising in algebra

Some Applications

- $GL_n(\mathbb{F}_q)$-analogue of partition lattice is Cohen-Macaulay
  (with Hanlon & Shareshian)
- Multigraded betti numbers bound for affine semigroup ring in terms of Gröbner basis degree
  (with Welker)
- Topology of Multiset Partition Posets
  (with Bobson, partial results..)
- more recently, work of others...

- Sagan-Vatter, The Möbius function of a Composition Poset
  - use BH construction to calculate Möbius functions by producing CW complexes w/ easier-to-calculate Euler characteristic

P = any finite poset

P^* = finite words whose letters are in P

$u \leq_{P^*} w \iff w$ has subsequence $w(i_1) \ldots w(i_{|u|})$ with $u(j) \leq_P w(i_j)$ for all j

- Hetyei-Krattenthaler, The Poset of Bipartitions
  - use BH construction to prove $\Delta(P) \simeq S^{n-2}$ where P has rank $3n-2$

P = poset of bipartitional relations

Defn: A relation $U \subseteq X \times X$ for a finite set X is a bipartitional relation if both U and $X \times X \setminus U$ are transitive

e.g. $X = \{1, 2, 3\}$ $U^c = \{(3,3), (3,1), (3,2)\}$

$U = \{(1,1), (1,2), (2,2), (2,1), (1,3), (2,3)\}$

Papers (on Discrete Morse Theory for Posets)

- Babson - H., Discrete Morse functions from lexicographic orders, Trans. Amer. Math. Soc., 2005
- On optimizing discrete Morse fns, Advances Applied Math, 2005

Written later with better explanations, notation, etc

- H.-Welker, Gröbner basis degree bounds on $\mathrm{Tor}_{\cdot}^{k[\Lambda]}(k,k)$ and discrete Morse theory for posets, Contemp. Math, 2005