An Inverse Approach to the Littlewood-Richardson Rule for the K-theoretic Coproducts

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The K-theory of Grassmannians

The *stable Grothendieck polynomial* G_λ is

$$G_\lambda = \sum_\gamma \mathbb{K}_{\lambda\gamma} m_\gamma \,,$$

where $\mathbb{K}_{\lambda\gamma} = (-1)^{|\gamma|-|\lambda|}$ times the number of *set-valued tableaux* of shape λ and type that rearranges to γ.

$T =$

45	57	
3	4	457
12	2	2

is of shape $(3,3,2)$ and type $(1,3,1,3,3,0,2)$ and $w(T) = 4531257424572$.

Let $\{g_\lambda\}$ denote the basis dual to $\{G_\lambda\}$. By duality, $h_\gamma = \sum_\lambda \mathbb{K}_{\lambda\gamma} g_\lambda$. So

$$g_\lambda = \sum_\gamma \mathbb{K}^{-1}_{\gamma\lambda} h_\gamma.$$

The LR Rule for K-theoretic Coproduct

(♂) $\Delta G_\nu = \sum_{\lambda,\mu} d^\nu_{\lambda\mu} G_\lambda \otimes G_\mu \,,$ $\quad d^\nu_{\lambda\mu} = \sum_{\substack{\sigma\subset\lambda \\ \lambda/\sigma \text{ rook strip}}} (-1)^{|\lambda|+|\mu|-|\nu|} \alpha^\lambda_{\nu/\sigma,\mu}$

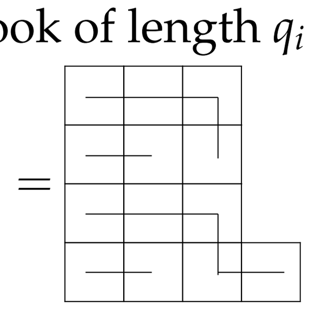
where $\alpha^\lambda_{\nu/\sigma,\mu}$ is the number of set-valued tableaux of shape ν/σ weight μ whose reading word is Yamanouchi.

Our Result

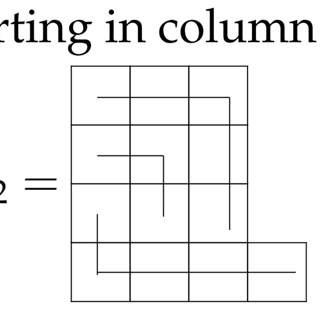
We give a combinatorial proof of (♂) using the Pieri Rule for g_λ and an involution on set-valued tableaux.

Tabloids and Elegant Fillings

A *special rim hook tabloid* (s.r.h.) T of shape μ and type $(q_1, q_2, \ldots, q_k)$ is a filling of μ with a rim hook of length q_i starting in column 1 and row i, for all i.

$T_1 =$ 

$\text{sign}(T_1) = 1 = (-1)^{(1-1)+(2-1)+(1-1)+(2-1)}$

$T_2 =$ 

$\text{sign}(T_2) = 1 = (-1)^{(2-1)+(2-1)+(3-1)}$

special rim hook tabloids of shape (4,3,3,3) and type $(2,5,2,4)$ and $(0,5,3,5)$, respectively.

$$\text{sign}(T) = \prod_H (-1)^{\text{ht}(H)-1},$$

where the sum is over all special rim hooks H in T.

$$m_\gamma = \sum_\beta K^{-1}_{\gamma\beta} s_\beta \,,$$

where $K^{-1}_{\gamma\beta} = \sum_T \text{sign}(T)$ over s.r.h. tabloids T of shape β and any type that rearranges to γ.

An *elegant filling* of shape μ/β is a SSYT with entries in the ith row restricted to $1, 2, \ldots, i-1$.

$E =$

2	3	
1	2	2
		1

is of shape $(3,3,3,2)/(3,2)$.

$$s_\beta = \sum_{\mu\supset\beta} E_{\beta\mu} G_\mu \,,$$

where $E_{\beta\mu}$ is the number of all elegant fillings E of shape μ/β.

Algebraic Aspect

$$\Delta G_\nu = \sum_{\lambda,\mu} d^\nu_{\lambda\mu} G_\lambda \otimes G_\mu \qquad (♂)$$

↓ duality

$$g_\lambda g_\mu = \sum_\nu d^\nu_{\lambda\mu} g_\nu \qquad (♀)$$

$(*)$ ↙

$$g_\lambda \sum_\gamma \sum_{\beta\subset\mu} K^{-1}_{\gamma\beta} E_{\beta\mu} h_\gamma$$

↘

$$\sum_{\substack{\sigma\subset\lambda \\ \lambda/\sigma \text{ rook strip}}} (-1)^{|\lambda|+|\mu|-|\nu|} \alpha^\lambda_{\nu/\sigma,\mu}$$

Pieri ↙

$$\sum_\nu \sum_{\substack{\sigma\subset\lambda \\ \lambda/\sigma \text{ rook strip}}} \sum_\gamma \sum_{\beta\subset\mu} (-1)^{|\lambda|+|\gamma|-|\nu|} \mathbb{K}^\lambda_{\nu/\sigma,\gamma} K^{-1}_{\gamma\beta} E_{\beta\mu}\, g_\nu$$

$$\sum_\gamma \sum_{\beta\subset\mu} (-1)^{|\gamma|} \mathbb{K}^\lambda_{\nu/\sigma,\gamma} K^{-1}_{\gamma\beta} E_{\beta\mu} = (-1)^{|\mu|} \alpha^\lambda_{\nu/\sigma,\mu}$$

$$m_\gamma = \sum_\mu \mathbb{K}^{-1}_{\gamma\mu} G_\mu \qquad m_\gamma = \sum_\beta K^{-1}_{\gamma\beta} s_\beta \qquad s_\beta = \sum_{\mu\supset\beta} E_{\beta\mu} G_\mu$$

↓ duality

$$g_\mu = \sum_\gamma \mathbb{K}^{-1}_{\gamma\mu} h_\gamma \qquad \mathbb{K}^{-1}_{\gamma\mu} = \sum_{\beta\subset\mu} K^{-1}_{\gamma\beta} E_{\beta\mu}$$

$$(*) \qquad g_\mu = \sum_\gamma \sum_{\beta\subset\mu} K^{-1}_{\gamma\beta} E_{\beta\mu} h_\gamma$$

Pieri Rule

$$g_\lambda h_\gamma = \sum_\nu \sum_{\substack{\sigma\subset\lambda \\ \lambda/\sigma \text{ rook strip}}} (-1)^{|\lambda|+|\gamma|-|\nu|} \mathbb{K}^\lambda_{\nu/\sigma,\gamma} g_\nu$$

where $\mathbb{K}^\lambda_{\nu/\sigma,\gamma}$ is the number of set-valued tableaux of shape ν/σ weight γ.

Idea of Combinatorial Proof

We find a sign-reversing involution

$$\iota : (S, T, E) \longmapsto (\hat{S}, \hat{T}, E)$$

for given $\lambda, \mu, \nu, \sigma$ with $\sigma \subset \lambda$ and λ/σ a rook strip:

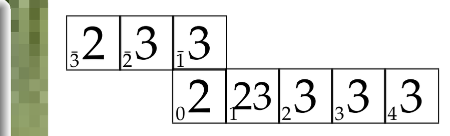
T: a special rim hook tabloid of shape β and type $(q_1, q_2, \ldots, q_k)$ that rearranges to γ, where $\beta \subset \mu$;

S: a set-valued tableau of shape ν/σ and type $(q_1, q_2, \ldots, q_k)$;

E: an elegant filling of shape μ/β;

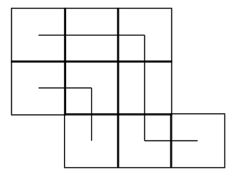
- $\text{sign}(\hat{S}, \hat{T}, E) = -\,\text{sign}(S, T, E)$, where $\text{sign}(S, T, E) = \text{sign}(T)$;
- fixed points of ι are $(Y, \widetilde{T}, E)$ where Y is a Yamanouchi set-valued tableau of shape ν/σ type μ and $\widetilde{T}$ is the tabloid shape μ type μ.

Example

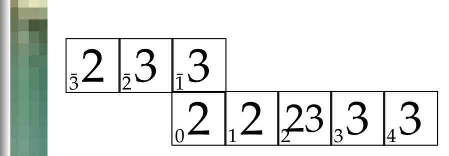


$C_2 = (-3, 0, 1)$

$C_3 = (-2, -1, 1, 2, 3, 4)$



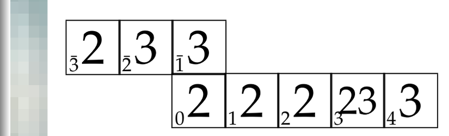
↓



$C_2 = (-3, 0, 1, 2)$

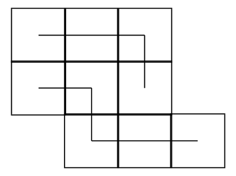
$C_3 = (-2, -1, 2, 3, 4)$

↓



$C_2 = (-3, 0, 1, 2, 3)$

$C_3 = (-2, -1, 3, 4)$



References

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