Kostant partition functions and flow polytopes of signed graphs

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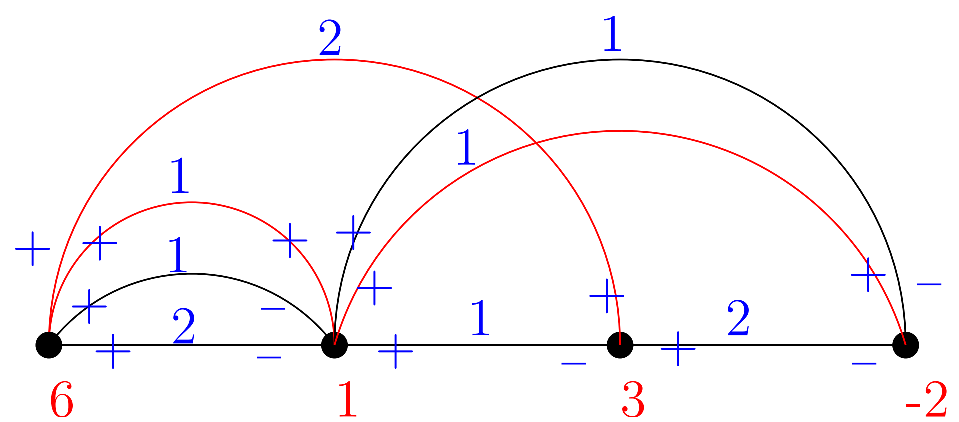
Flow polytopes of signed graphs

Example

A nonnegative integer flow with excess flow vector $\mathbf{a} = (6, 1, 3, -2)$

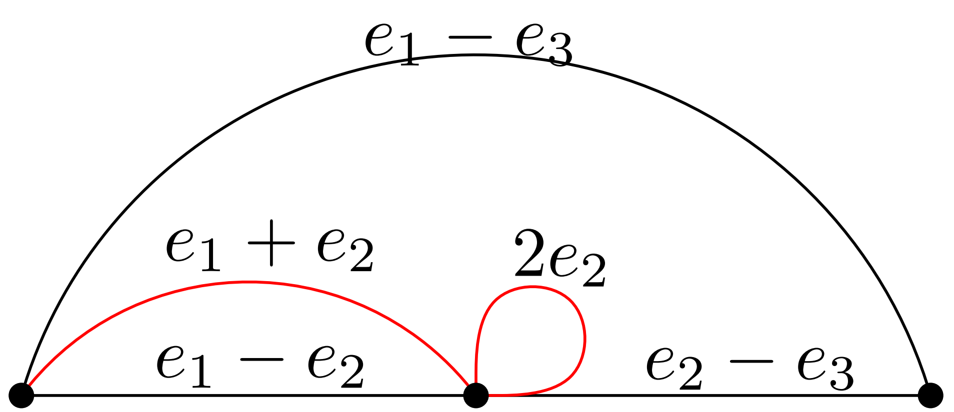


The flow polytope $\mathcal{F}_G(\mathbf{a})$ associated to the signed graph G and excess flow vector $\mathbf{a}$ is the set of all $\mathbf{a}$-flows $f : E \to \mathbb{R}_{\geq 0}$.

The Kostant partition function of a signed graph G

$K_G(\mathbf{v})$ is the number of ways to write the vector $\mathbf{v}$ as a nonnegative integer linear combination of the positive type C_n roots corresponding to the edges of G, without regard to order.

Example



$K_G(e_1 + 3e_2) = 2$, since

$e_1 + 3e_2 = (e_1 + e_2) + (2e_2) = (e_1 - e_2) + 2(2e_2)$.

Kostant partition functions and flow polytopes

The number of vertices of $\mathcal{F}_G(\mathbf{a})$ equals the Kostant partition function $K_G(\mathbf{a})$, for the special vectors $\mathbf{a} \in \{(2,0,\ldots,0)$, $(1,1,0,\ldots,0)$, $(1,0,\ldots,0,1,0,\ldots,0), (1,0,\ldots,0,-1)$, $(1,0,\ldots,0,-1,0,\ldots,0)\}$

Ehrhart polynomial: $L_{\mathcal{F}_G(\mathbf{a})}(t) = K_G(t\mathbf{a})$

The volume of $\mathcal{F}_G(\mathbf{a})$ is also expressed in terms of Kostant partition functions. However, regardless of whether the graph is signed or not, the volume is expressed in terms of type A_n Kostant partition functions!

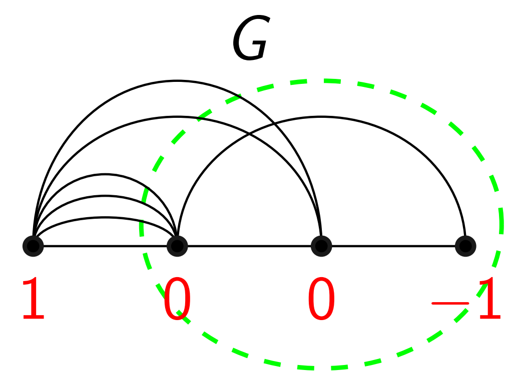
Volumes of flow polytopes: only negative edges

Theorem 1 (Postnikov-Stanley)

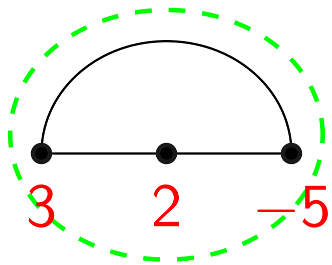
G graph with **negative edges**,
$\mathrm{vol}(\mathcal{F}_G(e_1 - e_{n+1}) = K_G(0, d_2, \ldots, d_n, -\sum_{i=2}^{n} d_i)$, *where* $d_i = indeg_G(i) - 1$ *for* $i \in \{2, \ldots, n\}$.

Example

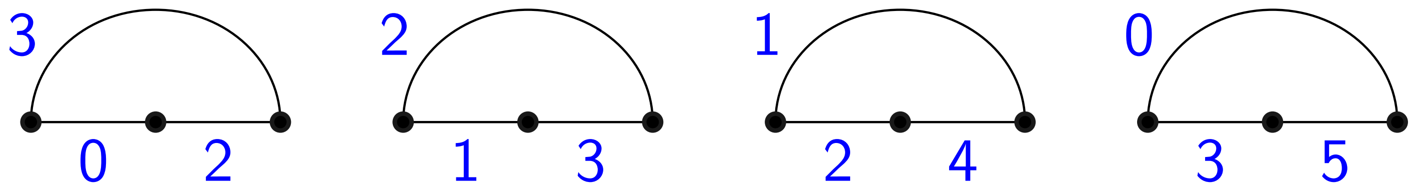
Volume flow polytope $\mathcal{F}_G(1, 0, 0, -1)$ for



$= \#$ of flows on



$= 4$:

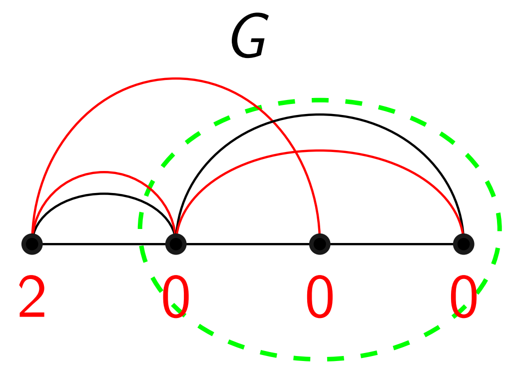


Theorem 2

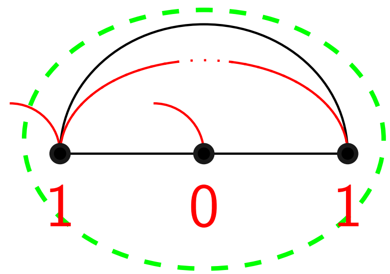
*G **signed** graph,* $\mathrm{vol}(\mathcal{F}_G(2e_1)) = K_G^{dynamic}(0, d_2, \ldots, d_n, -\sum_{i=2}^n d_i)$, *where $d_i = indeg_G(i) - 1$ for $i \in \{2, \ldots, n\}$.*

Example

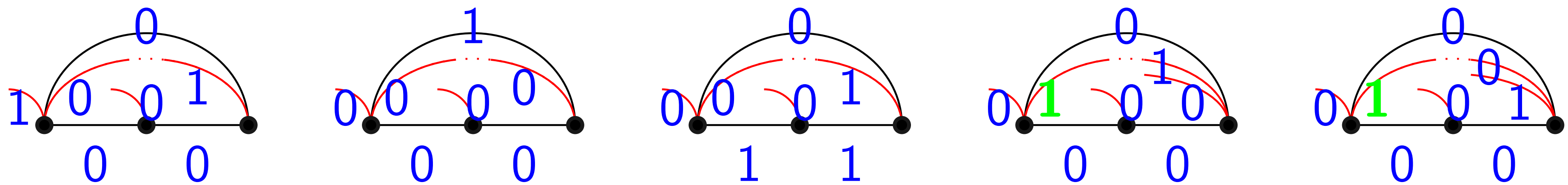
Volume flow polytope $\mathcal{F}_G(2,0,0,0)$ for



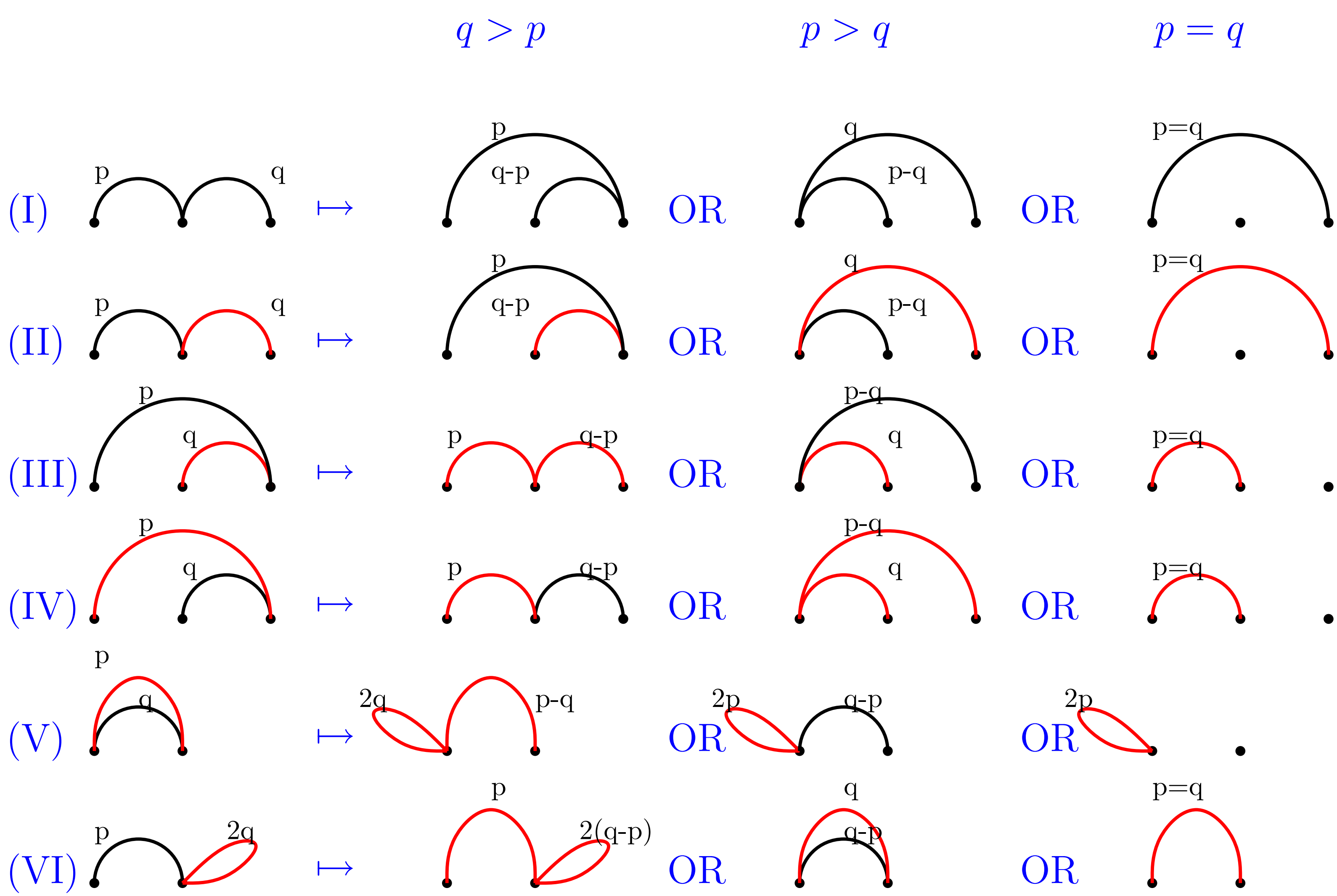
$= \#$ of *dynamic* flows on



$= 5$:

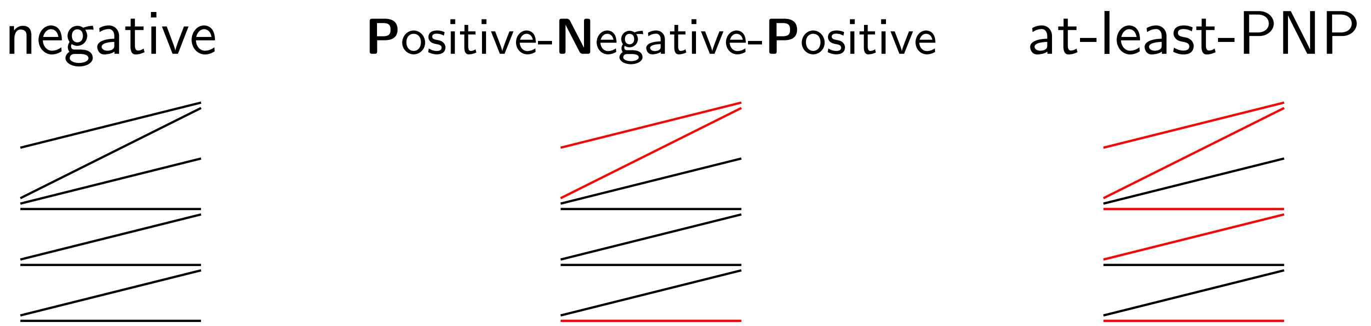


Reduction rules



Subdivison of Flow polytopes I/III: noncrossing trees

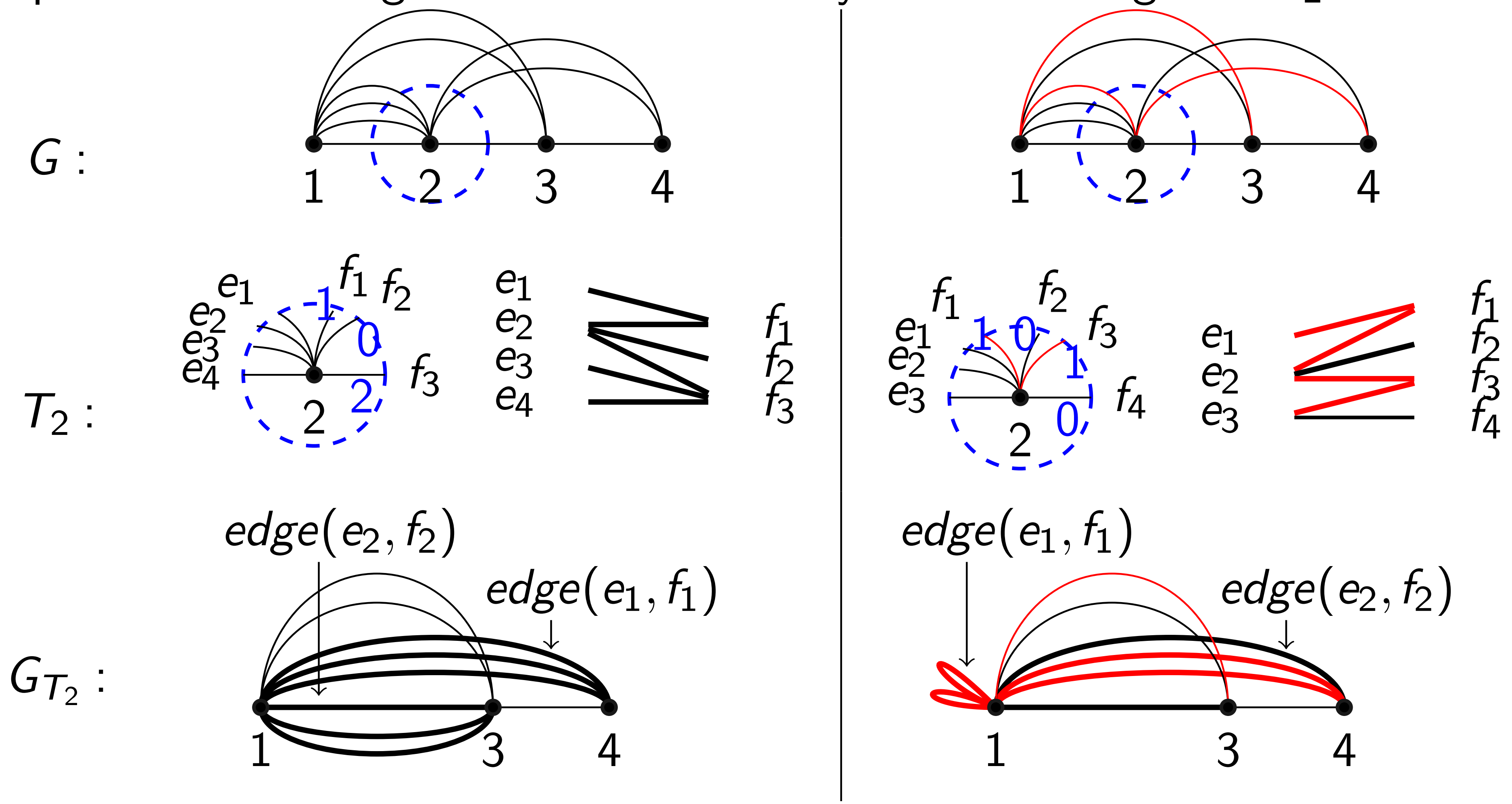
We use the **reduction rules** for signed graphs to subdivide flow polytopes. Subdivisions are indexed by signed bipartite non-crossing trees (*i.e.* signed compositions):



$T_{(1^-,0^-,1^-,1^-,0^-)}$ $T_{(1^+,0^-,1^-,1^-,0^+)}$ $T_{(1^+,0^-,1^+,1^-,0^+)}$

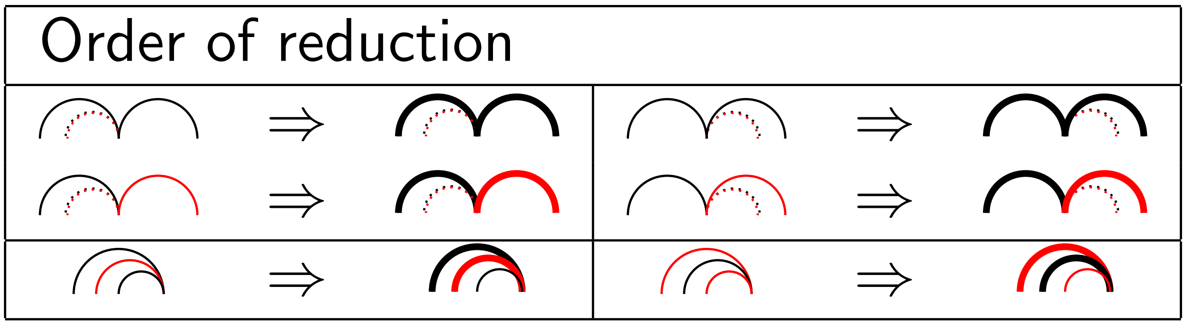
Subdivison of Flow polytopes II/III: Removing vertex from signed graph G

Replace incident edges of vertex 2 in G by a noncrossing tree T_2



Subdivision of Flow polytopes III/III: Descending order and Main Subdivision Lemma

We use the following order for subdivision: selected edges are **bold**

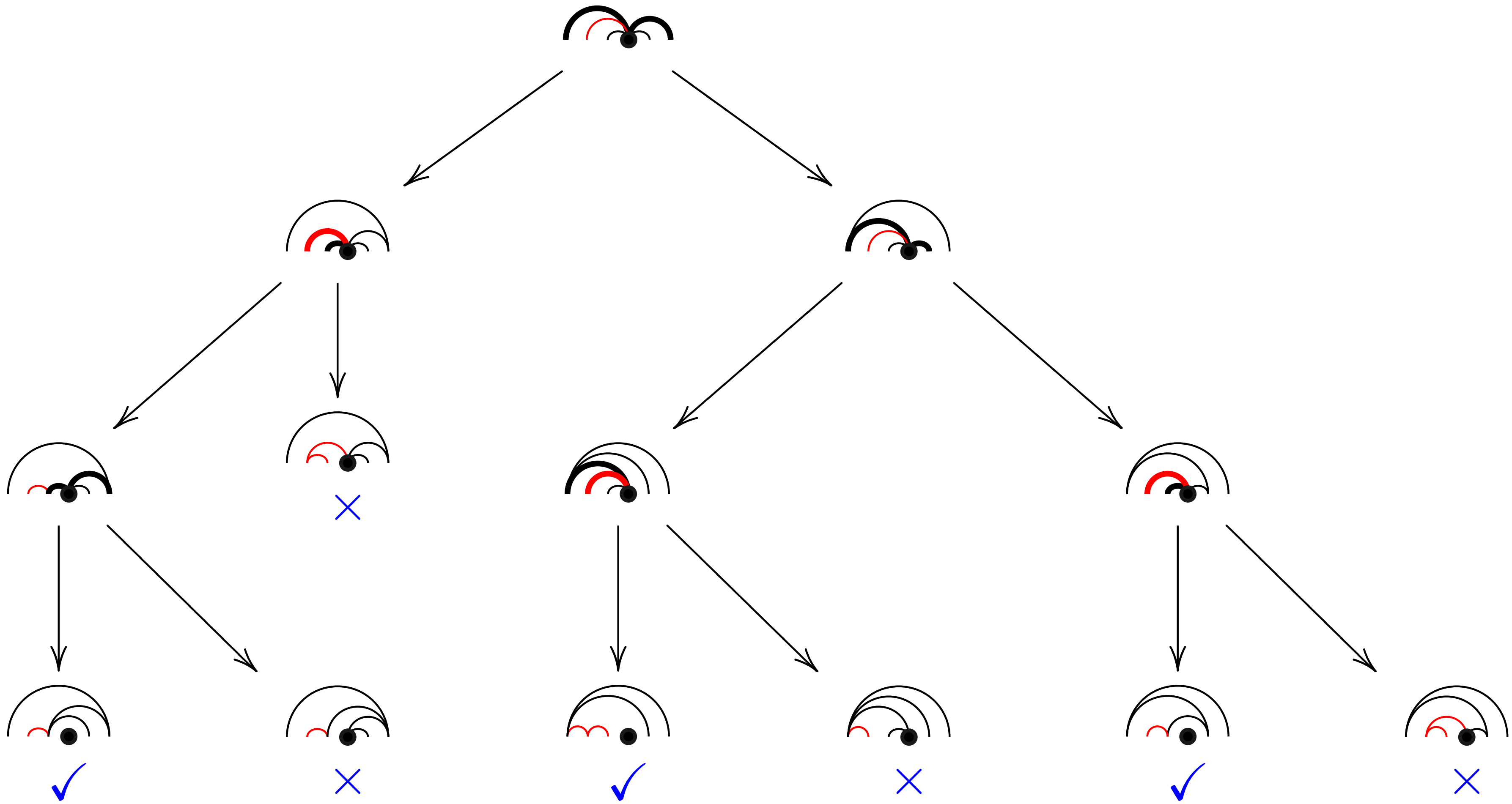


Lemma

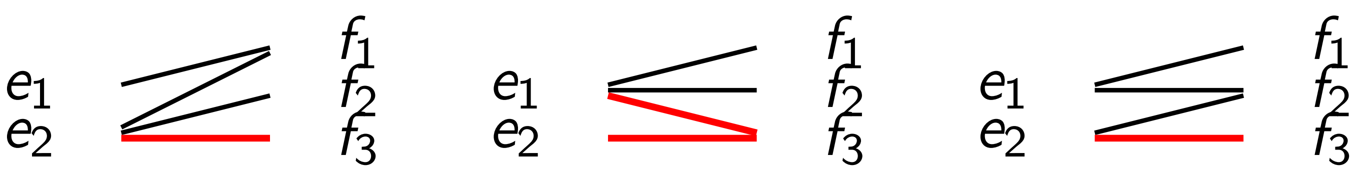
Let G be a signed graph, vertex set $[n+1]$, $\mathcal{F}_G(\mathbf{a})$ be its flow polytope. If $a_i = 0$, using **reduction rules** *to edges incident to i in the order* **above**, *the polytope* **decomposes** *as:*

$$\mathcal{F}_G(\mathbf{a}) = \bigcup_{T \text{ at-least-PNP trees}} \mathcal{F}_{G_T^{(i)}}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, 2y - \sum a_i),$$

Example of the Lemma

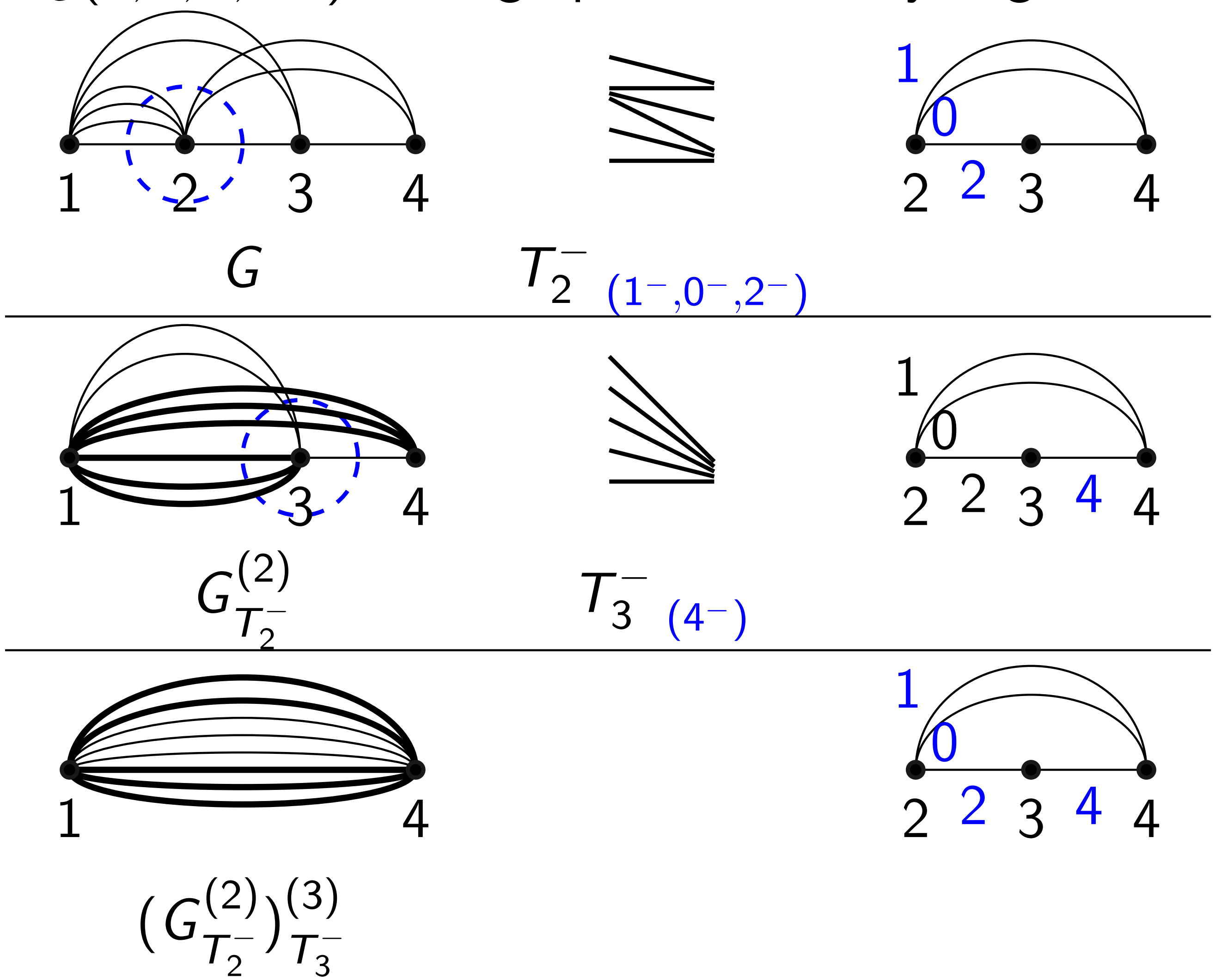


Three outcomes ✓ indexed by bipartite trees:



Using the Lemma to prove Theorem 1

Example of the subdivision to compute the volume of $\mathcal{F}_G(1,0,0,-1)$ for a graph G with only negative edges.



Using the Lemma to prove Theorem 2

Example of the subdivision to compute the volume of $\mathcal{F}_G(2,0,0,0)$ for a signed graph G.