Geodesics in CAT(0) Cubical Complexes

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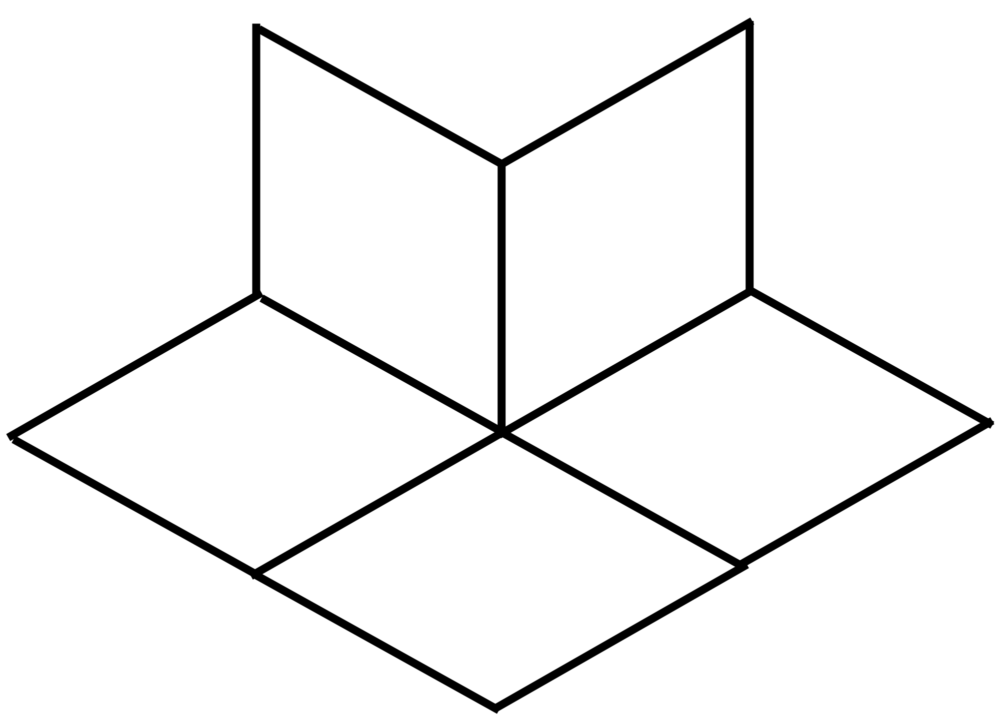
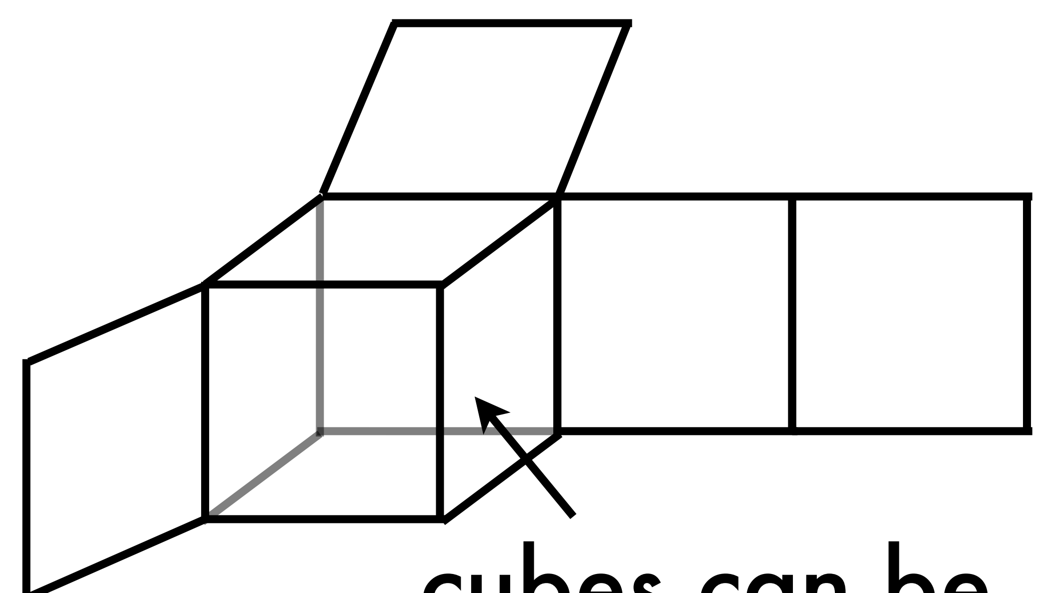
Cubical Complexes

- cubical complex = polyhedral complex of unit cubes + all attaching maps are injective
- metric on cubical complex induced by Euclidean L^2 metric on each cube



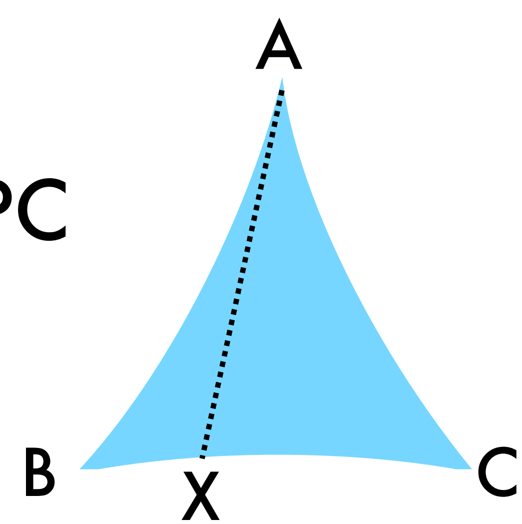


cubes can be different dimensions

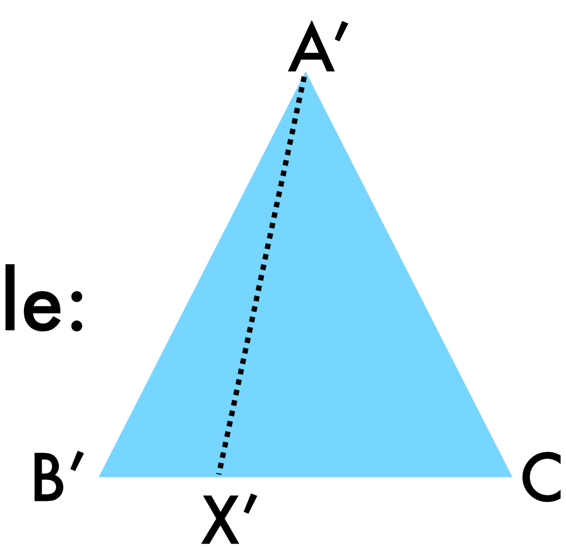
CAT(0)

- *non-positive curvature (NPC)* = triangles are at least as thin as in Euclidean space
- *global non-positive curvature* = all triangles are at least as thin as in Euclidean space = CAT(0)

triangle in a NPC space:



Euclidean comparison triangle:



$d(A,X) \leq d'(A',X')$

- CAT(0) $\Rightarrow$ unique shortest paths (*geodesics*)

CAT(0) Cubical Complexes

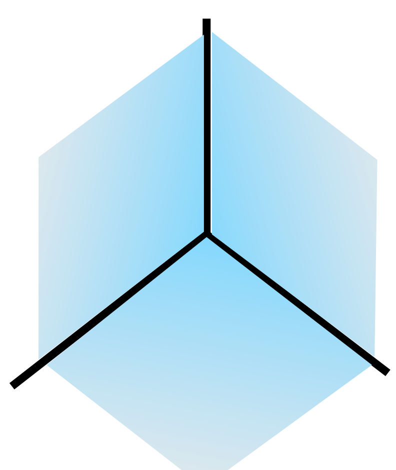
Theorem (Gromov, 1987):

A cubical complex is CAT(0)

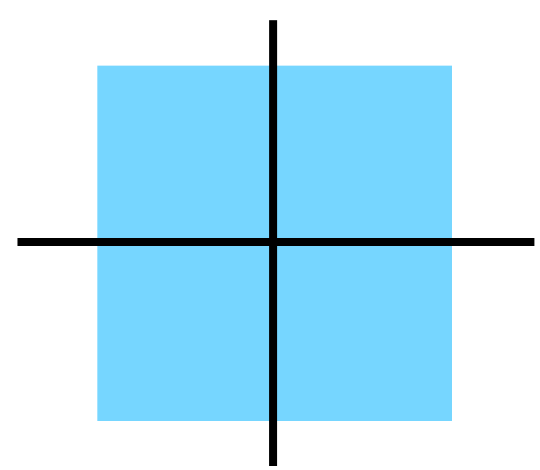
$\Leftrightarrow$ it is simply connected and the link of any vertex is a flag simplicial complex

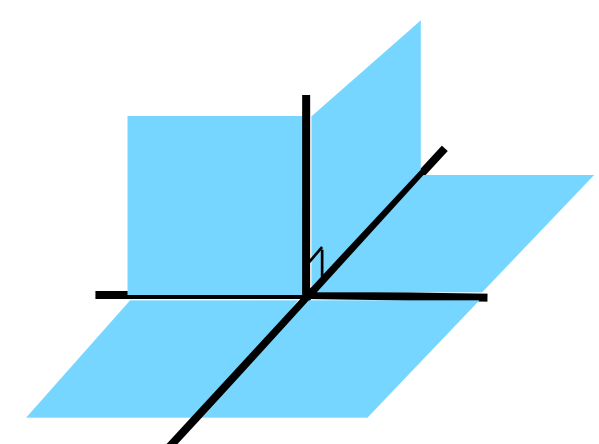
$\Leftrightarrow$ it is simply connected and if a vertex is incident to K edges, any pair of which specify a square, then these K edges also specify a K-dimensional cube.

not CAT(0):



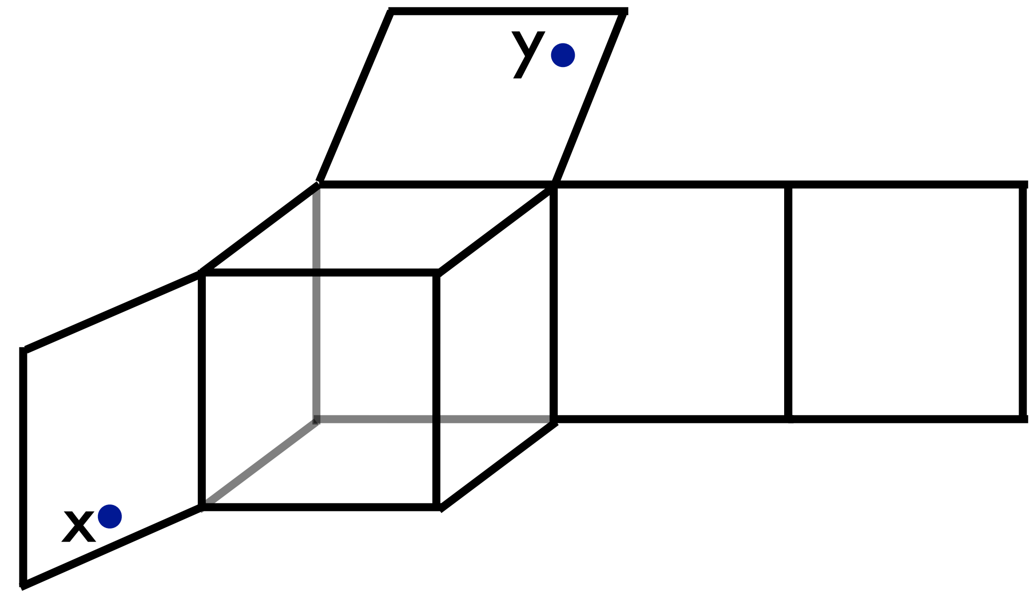
CAT(0):



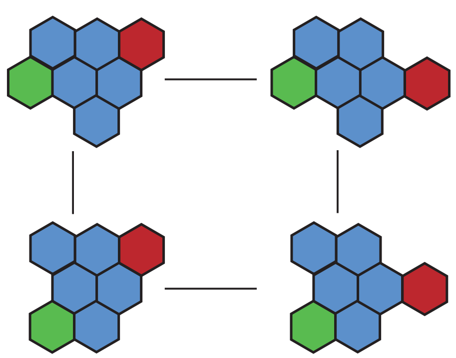


Problem

Given a CAT(0) cubical complex and two points x and y, find the geodesic from x to y.

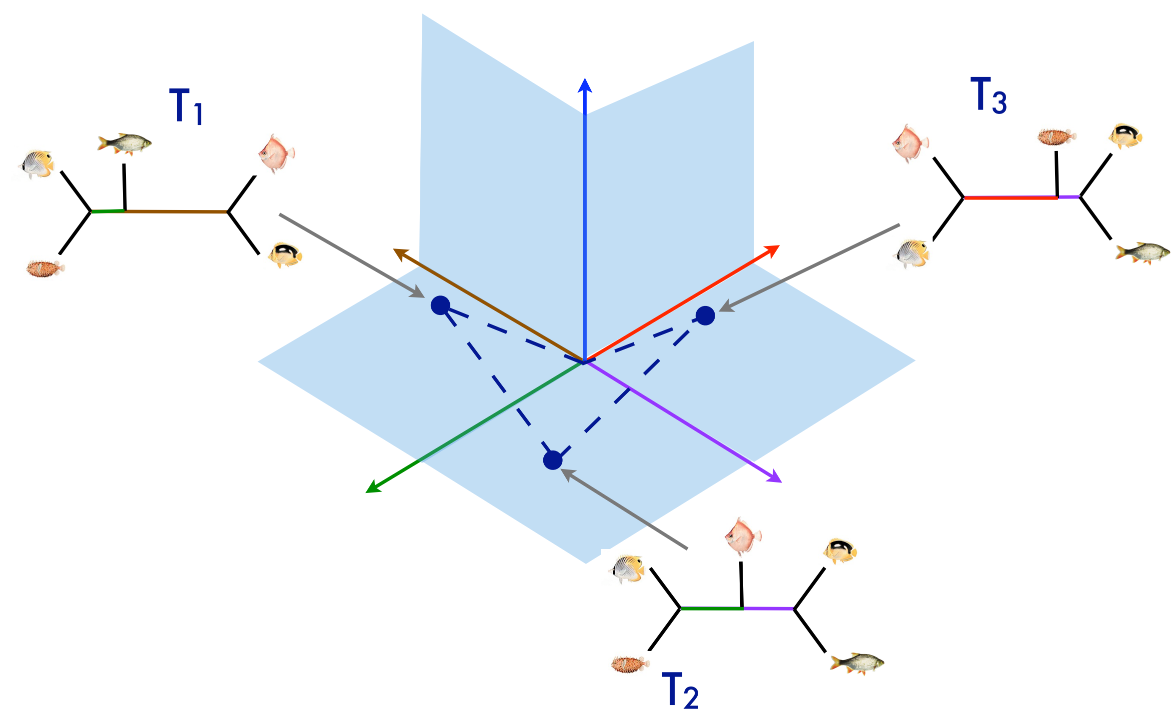


Applications:



(Ghrist and Peterson, 2007)

- reconfigurable systems



- space of phylogenetic trees

Solution

1. Coordinatize the CAT(0) complex: Establish a bijection with *posets with inconsistent pairs*. Coordinates = poset elements

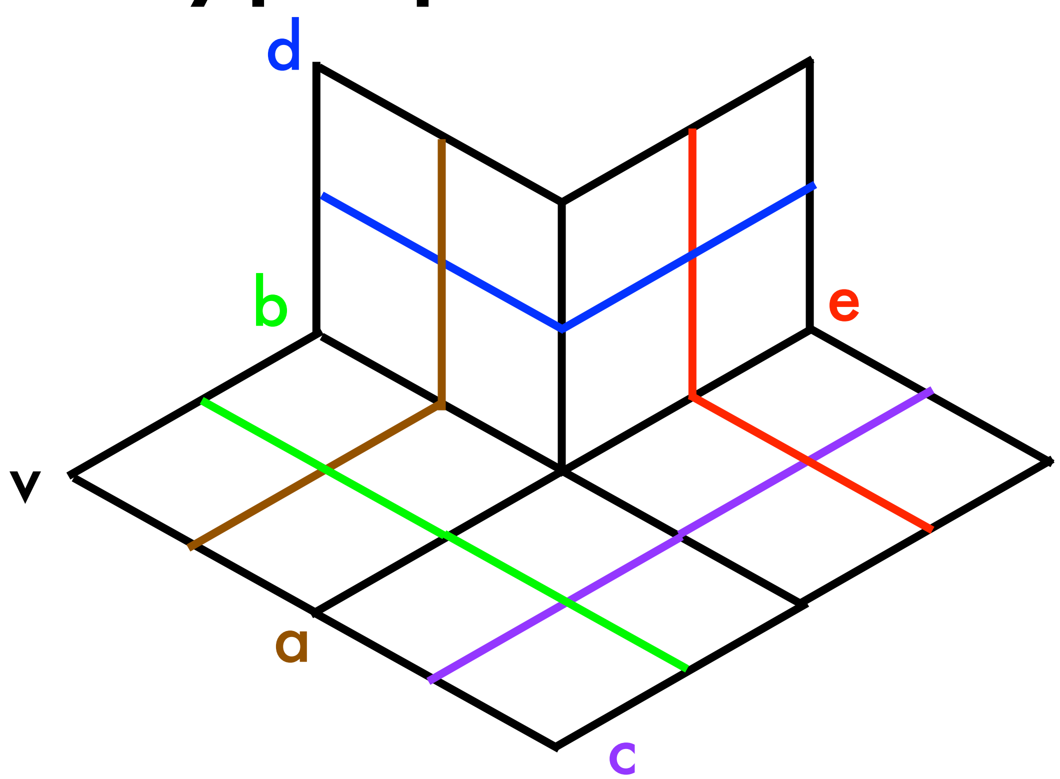
2. Reduce problem to subcomplex containing geodesic and find starting cube sequence.

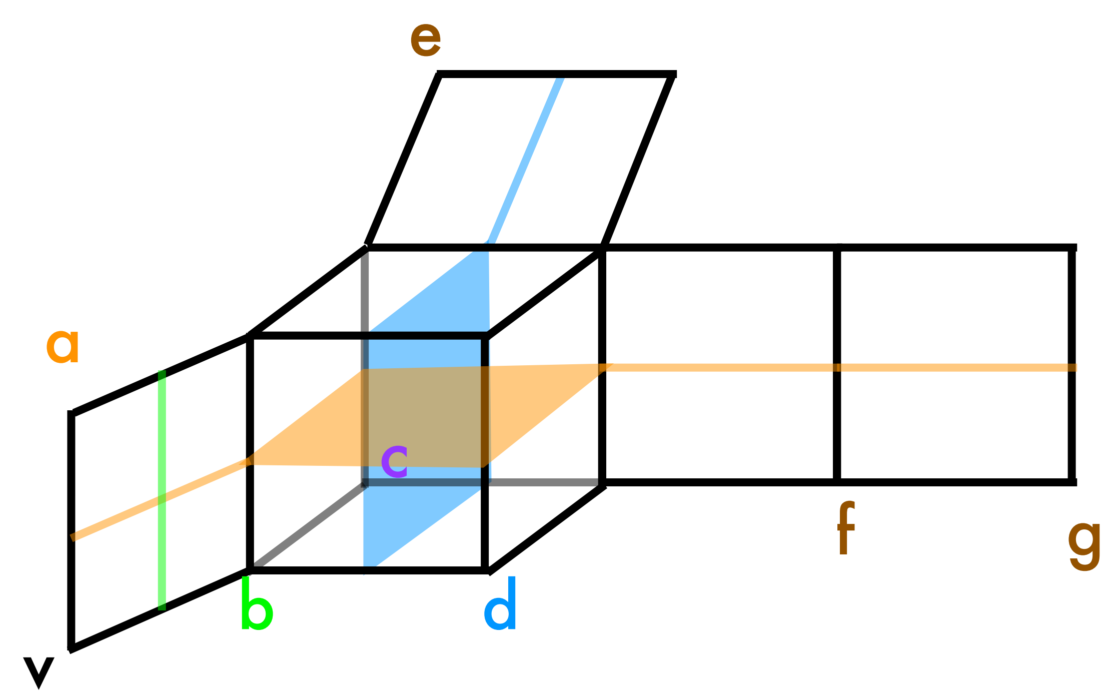
3. Find geodesic through this cube sequence.

4. If possible, improve cube sequence and repeat from 3.

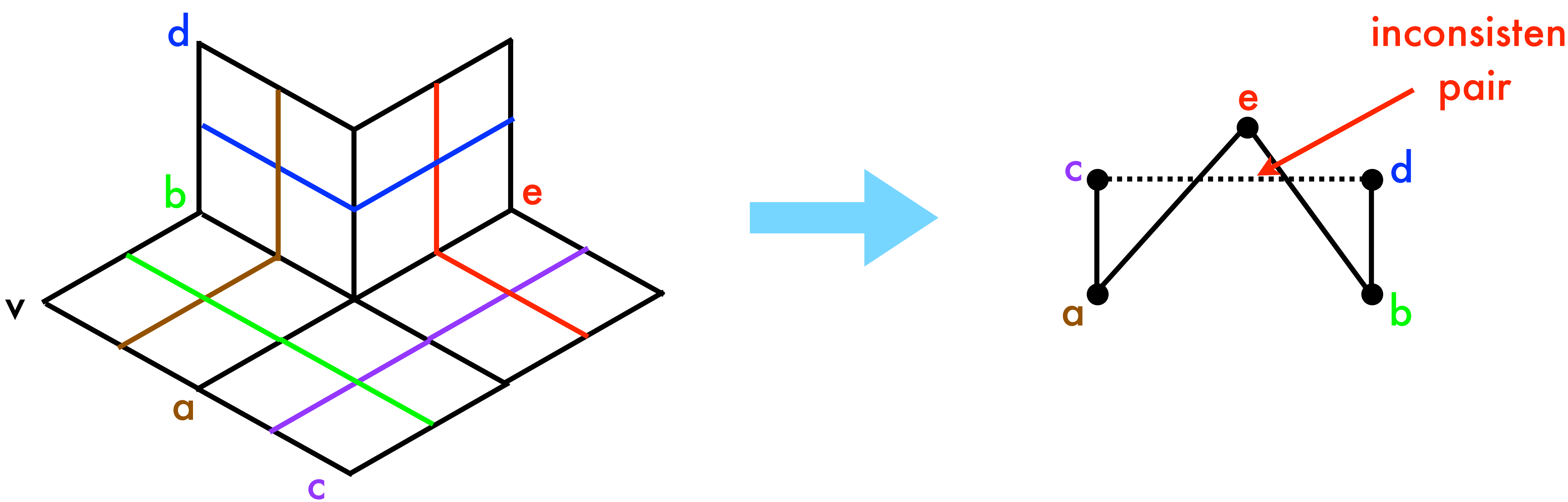
1. Poset Representation

- goal: represent cube complex as a poset to induce coordinate system
- associate each cube edge with the perpendicular “hyperplane” that bisects it
- hyperplanes act as coordinates





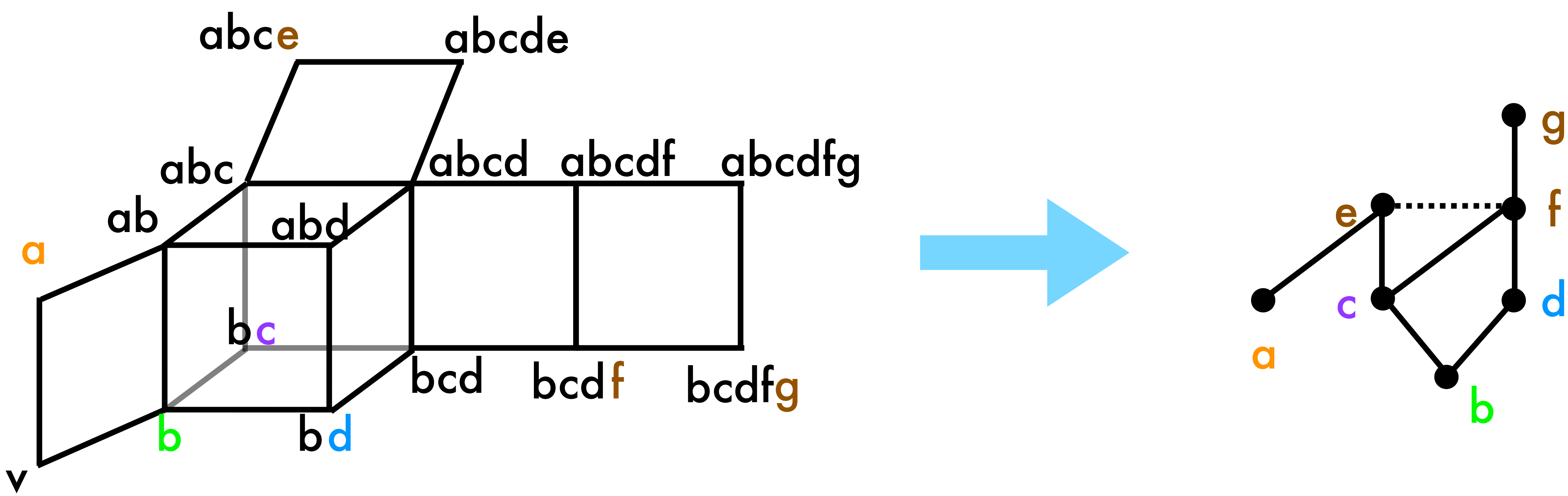
- fix a vertex v
- for each hyperplane, label the vertex closest to v on the opposite side of the hyperplane from v



- labeled vertices form *poset with inconsistent pairs*:
  - u < w ⇔ from v, must cross hyperplane u to reach vertex w
  - (p,q) is an *inconsistent pair* ⇔ no geodesic from v crosses both hyperplanes p and q

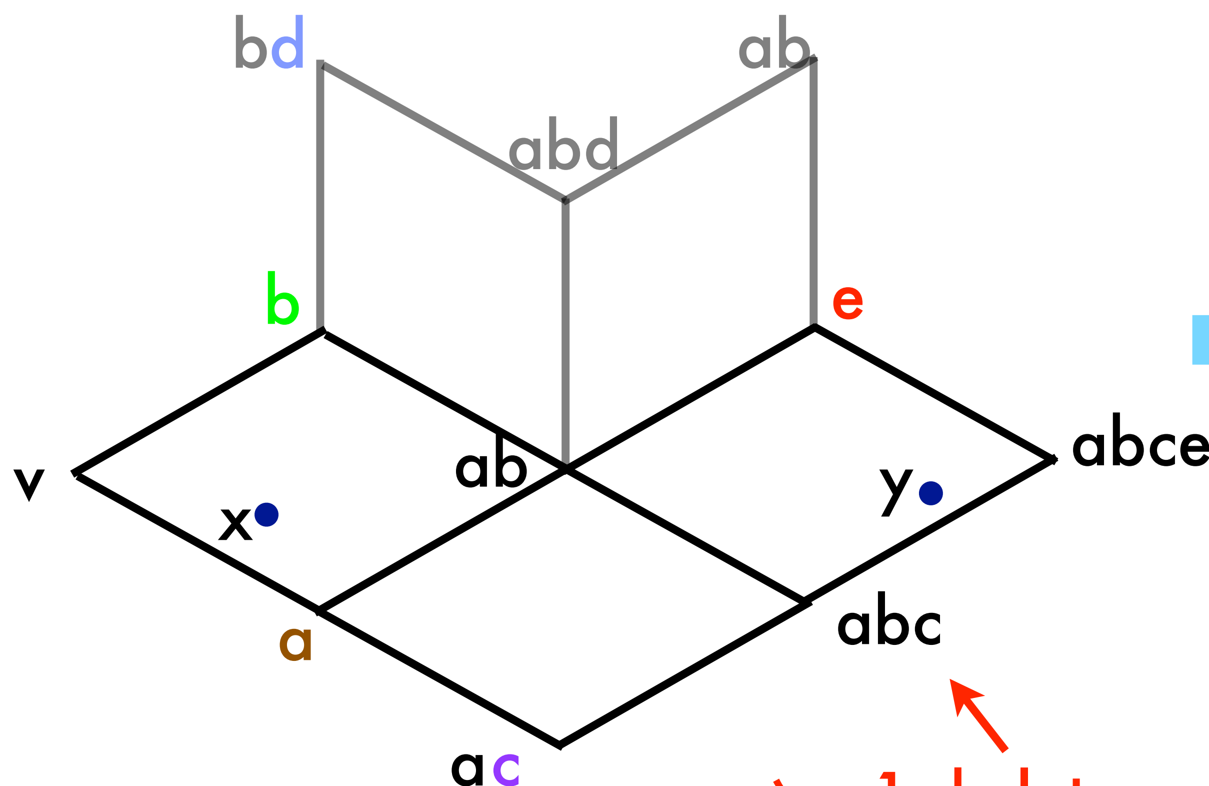
Theorem (Ardila, Owen, Sullivant):
Fixing a vertex, there is a bijection between CAT(0) cube complexes and posets with inconsistent pairs.

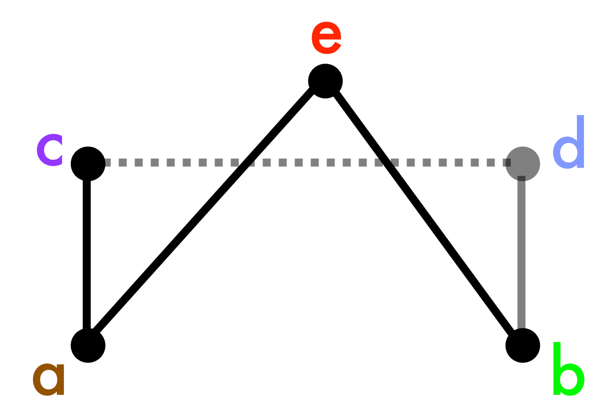
vertices in cube complex	$\longleftrightarrow$	order ideals with no inconsistent pairs in poset



2. Reduce Poset

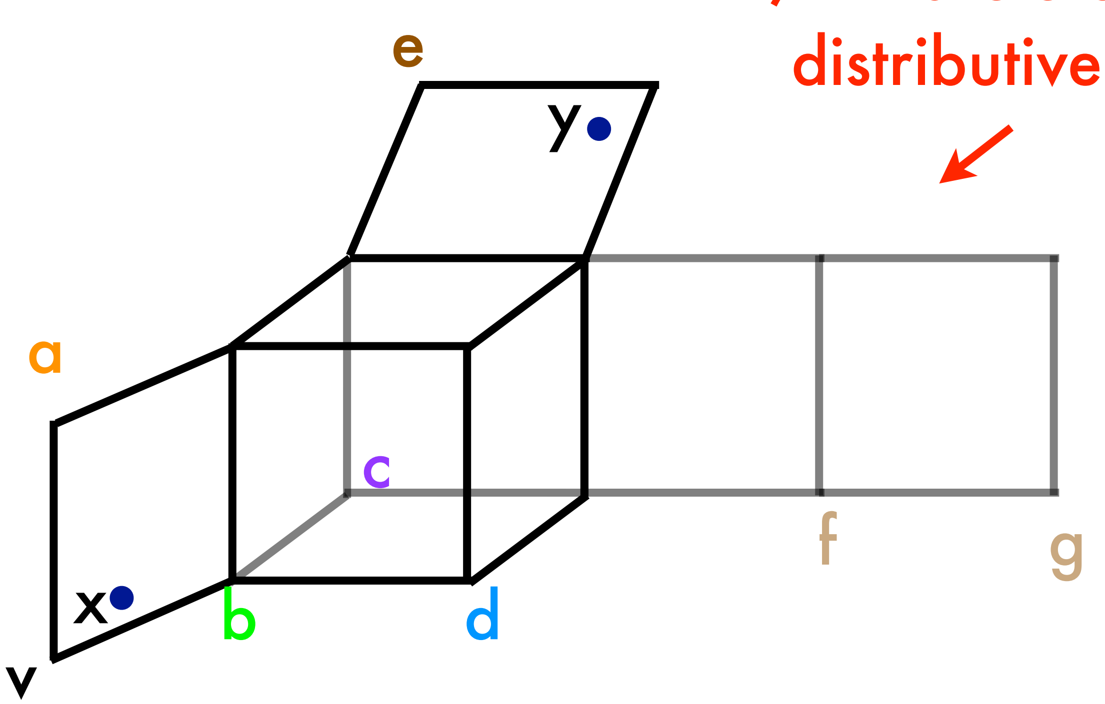
- delete cubes that could not contain the geodesic

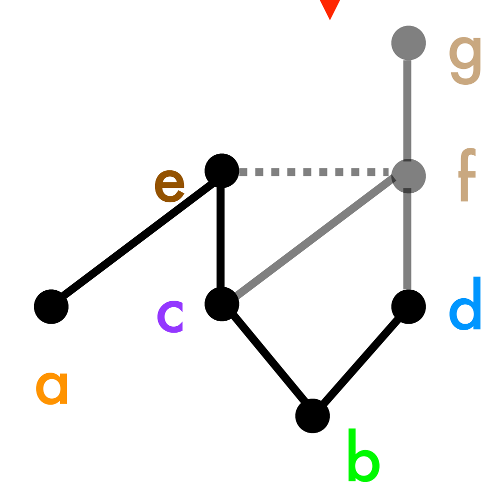




⇒ no inconsistent pairs

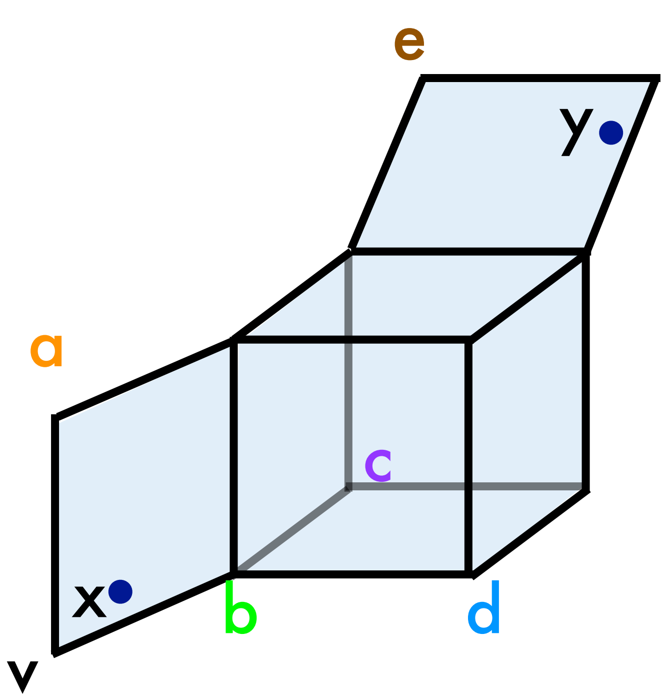
⇒ 1-skeleton = distributive lattice

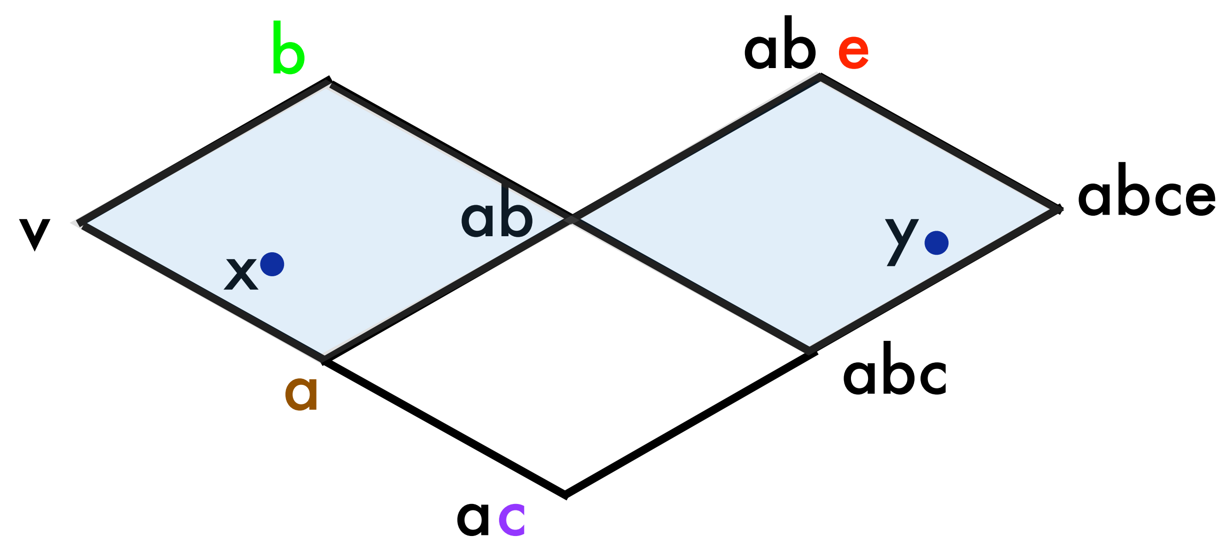




2. Starting Cube Sequence

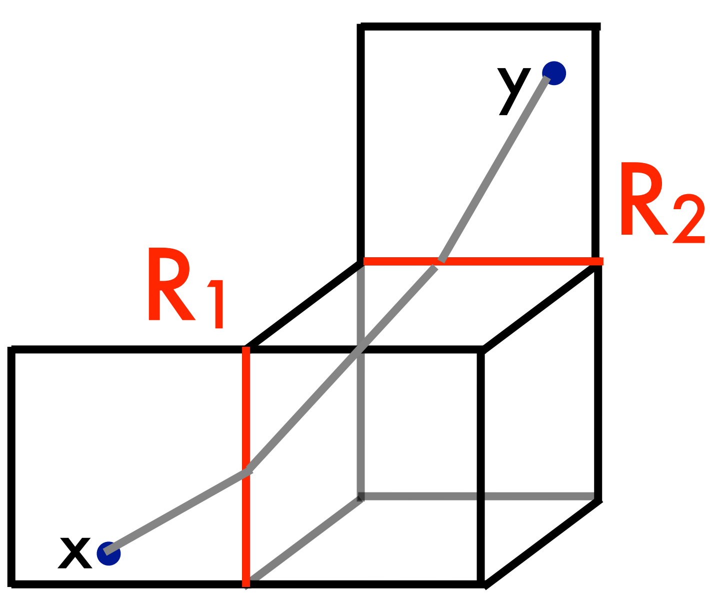
- choose a valid starting cube sequence based on x and y





3. Touring Problem

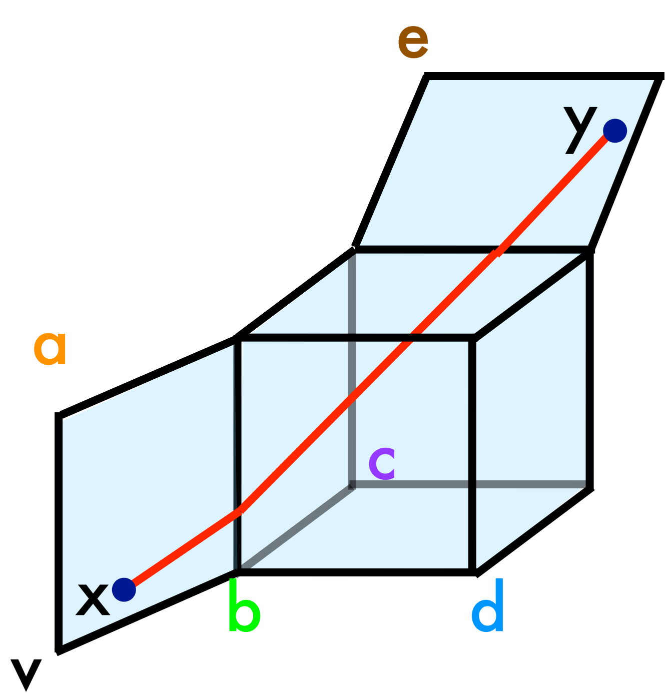
- rephrase finding the geodesic through the chosen cube sequence as a convex optimization problem (= *touring problem*)
- new problem solvable as a second order cone problem in polynomial time (Polishchuk and Mitchell, 2005)



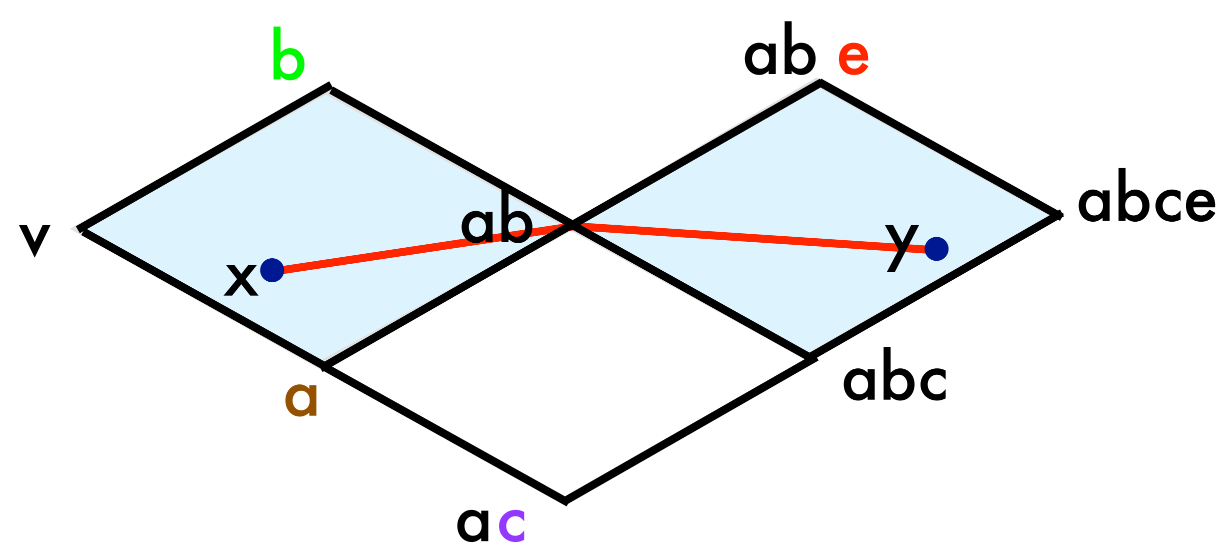
Find points in regions R_1 and R_2 that minimize length of path from x to y.

4. Improve the Path

4. Can geodesic through the current cube sequence be improved?
   If yes, get a new cube sequence; go to step 3. If no, then done.

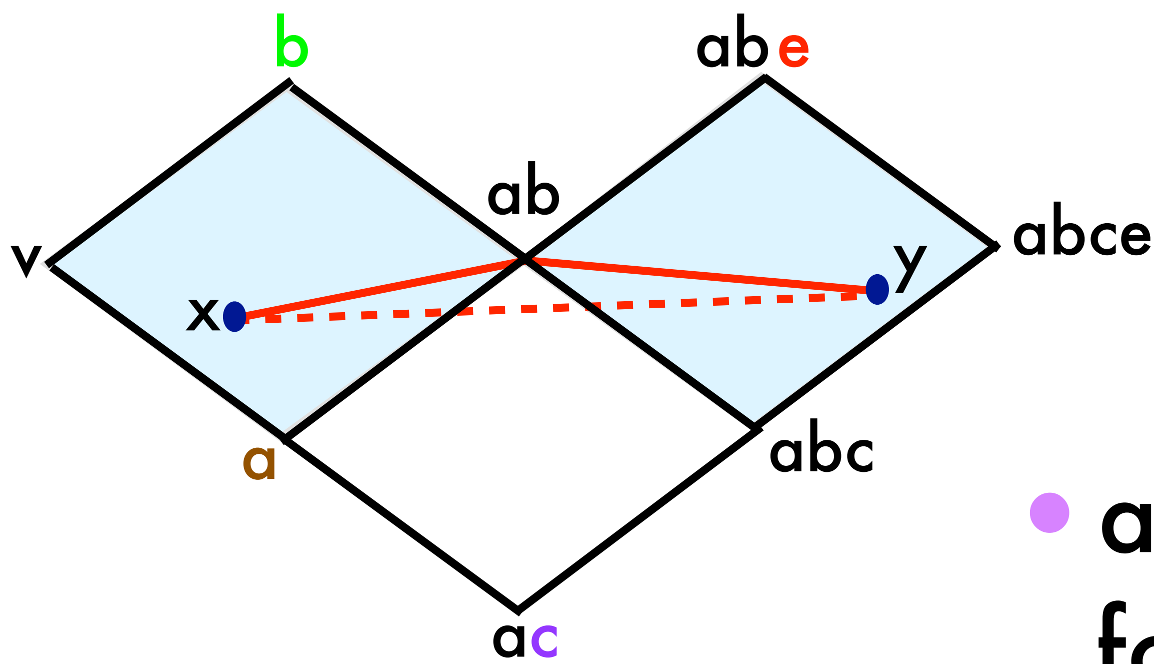


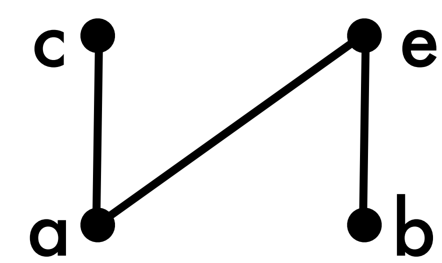
No improvement possible.



Path will be shorter if passes through white cube.

- can improve path iff there exists a cube giving a "short-cut" at any point where the path bends



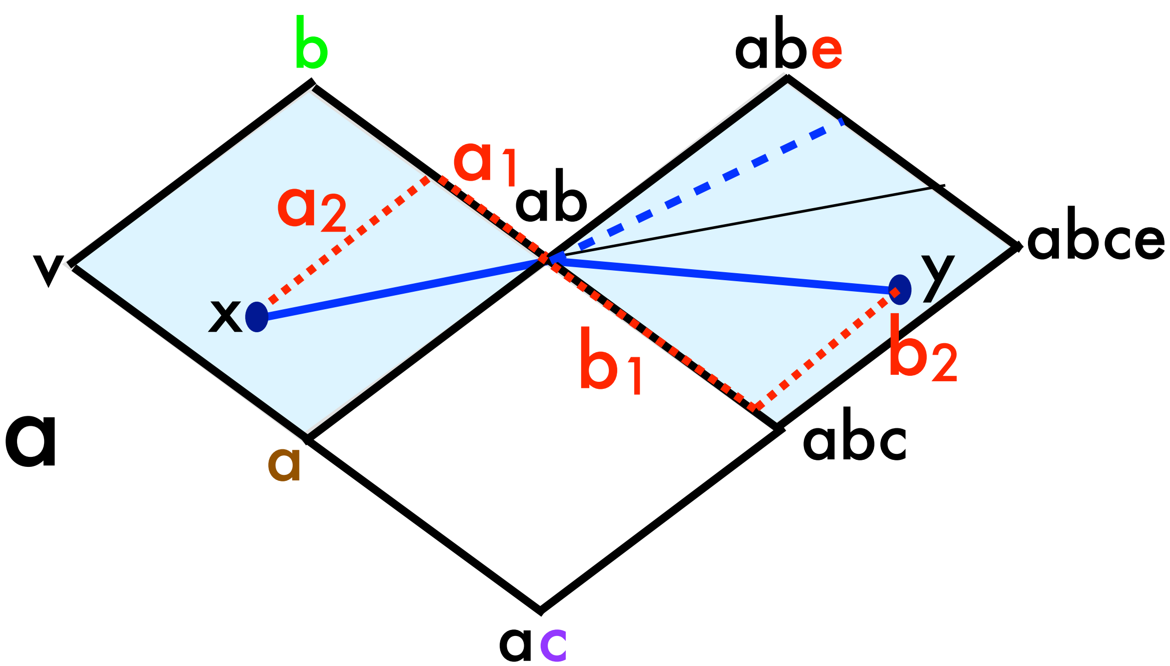


- a cube exists ⇔ its hyperplanes form an antichain in the cubical complex poset.

- also need $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$

- check for both by finding a min weight vertex cover.

(Owen and Provan, 2011)



Complexity

1. Coordinatize the CAT(0) complex: Establish a bijection with *posets with inconsistent pairs*. Coordinates = poset elements

polynomial

2. Reduce problem to subcomplex containing geodesic and find starting cube sequence.

polynomial

3. Find geodesic through this cube sequence.

polynomial

4. If possible, improve cube sequence and repeat from 3.

polynomial

unknown: # of iterations in general

References

1. F. Ardila, M. Owen and S. Sullivant. Geodesics in CAT(0) Cubical Complexes. Accepted *Advances in Applied Mathematics*, 2011. http://arxiv.org/abs/0907.3942

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