Geodesics in CAT(0) Cubical Complexes

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Cubical Complexes

• cubical complex = polyhedral complex of unit cubes + all attaching maps are injective

• metric on cubical complex induced by Euclidean $L^2$ metric on each cube

Cubes can be different dimensions
CAT(0)

- **non-positive curvature (NPC)** = triangles are at least as thin as in Euclidean space
- **global non-positive curvature** = all triangles are at least as thin as in Euclidean space = CAT(0)

\[ d(A, X) \leq d'(A', X') \]

- CAT(0) \(\Rightarrow\) unique shortest paths (geodesics)
**CAT(0) Cubical Complexes**

**Theorem (Gromov, 1987):**
A cubical complex is CAT(0) if and only if it is simply connected and the link of any vertex is a flag simplicial complex. It is also simply connected and if a vertex is incident to $K$ edges, any pair of which specify a square, then these $K$ edges also specify a $K$-dimensional cube.

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not CAT(0):  

CAT(0):
Problem

Given a CAT(0) cubical complex and two points $x$ and $y$, find the geodesic from $x$ to $y$.

Applications:

- reconfigurable systems
- space of phylogenetic trees

(Ghrist and Peterson, 2007)
Solution

1. Coordinatize the CAT(0) complex: Establish a bijection with posets with inconsistent pairs. Coordinates = poset elements

2. Reduce problem to subcomplex containing geodesic and find starting cube sequence.

3. Find geodesic through this cube sequence.

4. If possible, improve cube sequence and repeat from 3.
1. Poset Representation

- **goal**: represent cube complex as a poset to induce coordinate system

- associate each cube edge with the perpendicular "hyperplane" that bisects it

- hyperplanes act as coordinates
• fix a vertex \( v \)
• for each hyperplane, label the vertex closest to \( v \) on the opposite side of the hyperplane from \( v \)

labeled vertices form \textit{poset with inconsistent pairs}:

• \( u < w \) \iff from \( v \), must cross hyperplane \( u \) to reach vertex \( w \)
• \((p,q)\) is an \textit{inconsistent pair} \iff no geodesic from \( v \) crosses both hyperplanes \( p \) and \( q \)
Theorem (Ardila, Owen, Sullivant): Fixing a vertex, there is a bijection between CAT(0) cube complexes and posets with inconsistent pairs.

vertices in cube complex $\leftrightarrow$ order ideals with no inconsistent pairs in poset
2. Reduce Poset

- delete cubes that could not contain the geodesic

$\Rightarrow$ 1-skeleton = distributive lattice

$\Rightarrow$ no inconsistent pairs
2. Starting Cube Sequence

- choose a valid starting cube sequence based on x and y
3. Touring Problem

- rephrase finding the geodesic through the chosen cube sequence as a convex optimization problem ( = touring problem)

- new problem solvable as a second order cone problem in polynomial time (Polishchuk and Mitchell, 2005)

Find points in regions $R_1$ and $R_2$ that minimize length of path from $x$ to $y$. 
4. Improve the Path

4. Can geodesic through the current cube sequence be improved? If yes, get a new cube sequence; go to step 3. If no, then done.

No improvement possible.

Path will be shorter if passes through white cube.
• can improve path iff there exists a cube giving a “short-cut” at any point where the path bends

• a cube exists $\Leftrightarrow$ its hyperplanes form an antichain in the cubical complex poset.

• also need $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$

• check for both by finding a min weight vertex cover. (Owen and Provan, 2011)
Complexity

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unknown: # of iterations in general
References


