Promotion and Rowmotion

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1 Introduction

We present an equivariant bijection between two actions called rowmotion and promotion on order ideals in certain posets. This gives a uniform generalization of a result of Stanley concerning promotion on standard Young tableaux (SYT) and recent work of Armstrong, Stump, and Thomas on nonnesting and noncrossing partitions. We apply this perspective to the alternating sign matrix poset and show that rowmotion equals gyration.

2 Promotion

Promotion is an action on SYT or on linear extensions of posets or on order ideals. Let f be a linear extension of a poset $\mathcal{Q}$ and let $\rho_i(f)$ act on f by switching i and $i+1$ if they are not the labels of adjacent elements. Then f's *promotion* $\rho(f)$ is $\rho_{n-1}\rho_{n-2}\cdots\rho_1(f)$.

Stanley's 2009 survey paper also defines promotion using order ideals $J(\mathcal{Q})$. Linear extensions f have a natural interpretation as maximal chains $\emptyset = J_0 \subset J_1 \cdots \subset J_n = \mathcal{Q}$ in $J(\mathcal{Q})$ by taking $f(q) = i$ if $q \in J_{i+1} - J_i$. Then the promotion of f is $\tau_{n-1}\cdots\tau_1 f$, where τ_i acts on a chain by switching J_i to the other order ideal in $[J_{i-1}, J_{i+1}]$ if one exists.

A maximal chain in $J(\mathcal{Q})$ traces out an order ideal—defined by the boxes to the right of the maximal chain—in a related poset $J(\mathcal{Q})^o$.

Example 2.1. The two orbits of promotion on SYT of shape $(3,3)$ are:

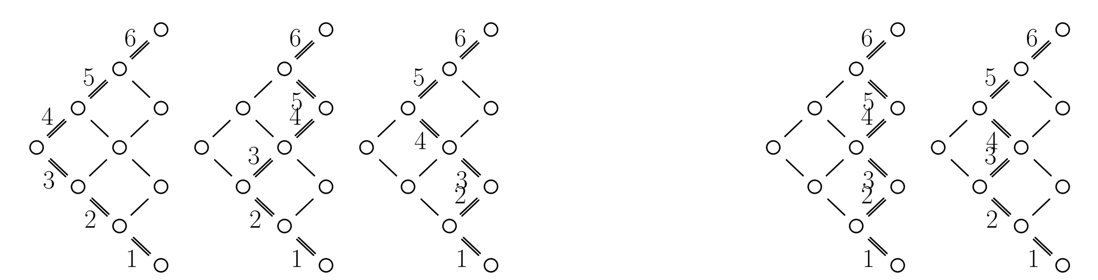
Orbit 1:

1	2	3	1	2	5	1	3	4
4	5	6	3	4	6	2	5	6

Orbit 2:

1	3	5	1	2	4
2	4	6	3	5	6

Using the maximal chain interpretation:



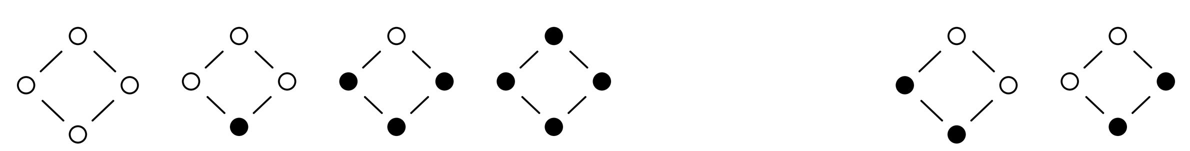
Using the order ideal interpretation:

3 Rowmotion

In 1973, Duchet defined an action on hypergraphs. It was generalized by Brouwer and Schrijver to an arbitrary poset, but remained unnamed. We will call this action *rowmotion*, denoted P.

Definition 3.1. Let $\mathcal{Q}$ be a poset, and let $J \in J(\mathcal{Q})$. We define J's *rowmotion* $P(J)$ to be the order ideal generated by the minimal elements of $\mathcal{Q}$ not in J.

Example 3.2. There are two orbits of $J([2]\times[2])$ under P (rowmotion).



4 New poset characterizations

Definition 4.1. For each $q \in \mathcal{Q}$, define $t_q : J(\mathcal{Q}) \to J(\mathcal{Q})$ to act by toggling q if possible. That is, if $J \in J(\mathcal{Q})$,

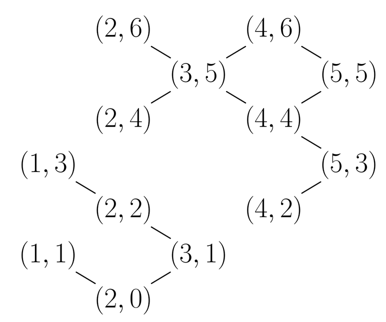
$$t_q(J) = \begin{cases} J \cup \{q\} & \text{if } q \notin J \text{ and if } q' < q \text{ then } q' \in J, \\ J - q & \text{if } q \in J \text{ and if } q' > q \text{ then } q' \notin J, \\ J & \text{otherwise.} \end{cases}$$

So $t_q^2 = 1$ and $(t_q t_{q'})^2 = 1$ if q and q' don't have a covering relation.

Definition 4.2. A *rowed-and-columned (rc) poset* $\mathcal{R}$ is a finite poset with vertices a subset of the positive integer span of $(2,0)$ and $(1,1)$, and covering relations a subset of $(i,j) > (i+1, j-1)$ and $(i,j) > (i-1,j-1)$.

Some examples of rc posets are the product of two chains $[n]\times[k]$ and the types A_n and B_n root posets.

Example 4.3.



This rc poset has $k = 5$ columns and $n = 7$ rows.

On rc posets rowmotion can be defined as an action on the *rows*, and promotion as an action on the *columns*.

Definition 4.4. Let $r_i(c_i) = \prod t_q$, where the product is over all elements in row (column) i. Also given $\omega \in S_n$ let $P_\omega = \prod_{i=1}^n r_{\omega(i)} = r_{\omega(1)} \cdot r_{\omega(2)} \cdots r_{\omega(n)}$ and $\rho_\omega = \prod_{i=1}^n c_{\omega(i)} = c_{\omega(1)} \cdot c_{\omega(2)} \cdots c_{\omega(n)}$.

Theorem 4.5. *On ranked posets, $P_{1..n}$ is P (rowmotion).*

Theorem 4.6. *If $J(\mathcal{Q})^o$ is an rc poset, $\rho_{k..1}$ is promotion on SYT of shape $\mathcal{Q}$.*

5 Main Theorem: Promotion = Rowmotion

We can do rowmotion on the rows of a ranked poset in any order without changing the orbit structure.

Theorem 5.1. *For any ranked poset $\mathcal{R}$ and any $\omega, \nu \in S_n$, there is an equivariant bijection between $\mathcal{R}$ under P_ω and $\mathcal{R}$ under P_ν.*

Applying this theorem to the alternating sign matrix poset we obtain an equivariant bijection between rowmotion and a known action called gyration.

Corollary 5.2. *Gyration equals $P_{135\ldots246\ldots}$ on the order ideals of the ASM poset, thus there is an equivariant bijection between rowmotion and gyration on ASMs.*

Similarly, we may permute the order in which promotion acts on the columns of any rc poset and not change the orbit structure.

Theorem 5.3. *For any rc poset $\mathcal{R}$ and any $\omega, \nu \in S_k$, there is an equivariant bijection between $\mathcal{R}$ under ρ_ω and $\mathcal{R}$ under ρ_ν.*

We now come to the main theorem which proves that in any rc poset, rowmotion and promotion have exactly the same orbit structure.

Theorem 5.4. *For any rc poset $\mathcal{R}$ and any $\omega \in S_n$ and $\nu \in S_k$, there is an equivariant bijection between $\mathcal{R}$ under P_ω and $\mathcal{R}$ under ρ_ν.*

Proof. By Theorems 5.1 and 5.3, we may restrict to considering only $P_{135\ldots246\ldots}$ and $\rho_{135\ldots246\ldots}$. But since every s_p with p in an odd (resp. even) column or row commute with one another, and since elements in an odd (resp. even) row are also necessarily in an odd (resp. even) column, we conclude that $P_{135\ldots246\ldots}$ is equal to $\rho_{135\ldots246\ldots}$. □

We may ask for an explicit equivariant bijection from rowmotion $P_{1\ldots n}$ to promotion $\rho_{k\ldots1}$. It is more convenient to go from $P^{-1} = P_{n\ldots1}$ to promotion $\rho_{k\ldots1}$.

Theorem 5.5. *An explicit equivariant bijection from rowmotion $P_{n\ldots1}$ to promotion $\rho_{k\ldots1}$ is given by acting on an order ideal by $D = \prod_{k=n}^{2}\prod_{i=k}^{n} d_i$ where $d_j = \prod t_q$ over all elements in diagonal j (from left to right) from smallest row to largest row.*

Example 5.6. *The orbits of the order ideals of the root poset $\Phi^+_{\mathcal{A}_2}$ under P_{12}^{-1}:*

Apply D:

The orbits of the order ideals of $J(\boxplus)^o$ under ρ_{321}:

6 Corollaries

Definition 6.1 (Reiner, Stanton, White)**.** Let X be a finite set, $X(q)$ a generating function for X, and C_X a cyclic group acting on X. Then the triple $(X, X(q), C_X)$ exhibits the *Cyclic Sieving Phenomenon (CSP)* if for $c \in C_X$,

$$X(\omega(c)) = |\{x \in X : c(x) = x\}|,$$

where $\omega : C_X \to \mathbb{C}$ is an isomorphism of C_X with the nth roots of unity.

As corollaries of Theorem 5.4 we obtain new proofs of the following.

Corollary 6.2 (Stanley, 2009)**.** *There is an equivariant bijection between SYT with two disjoint rows of sizes n and k under ρ and order ideals of $[n]\times[k]$ under P. Thus, $(J([n]\times[k]), \begin{bmatrix} n+k \\ k \end{bmatrix}_q, P)$ exhibits the CSP.*

Example 6.3. There are 2 orbits for SYT with 2 disjoint rows of length 2 under ρ.

Orbit 1: [1 2] [3 4], [1 4] [2 3], [3 4] [1 2], [2 3] [1 4]

Orbit 2: [1 3] [2 4], [2 4] [1 3]

Likewise, there are two orbits for $J([2]\times[2])$ under P (see Example 3.2).

Corollary 6.4 (Armstrong, Stump, Thomas, 2011)**.** *There is an equivariant bijection between non-nesting partitions of types A_n and B_n under P and the corresponding non-crossing partitions under Kreweras complementation. Thus, $(J(\Phi^+_{\mathcal{A}_n}), \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q, P)$ and $(J(\Phi^+_{\mathcal{B}_n}), \begin{bmatrix} 2n \\ n \end{bmatrix}_q, P)$ exhibit the CSP.*

Example 6.5. There are two orbits of $\mathcal{A}_2$ under P (rowmotion).

Likewise, there are 2 orbits of SYT of shape $(3,3)$ under ρ (promotion) (see Example 2.1).