I. Build a ring R_P from a poset P
II. Geometric/topological motivation
III. Open problems

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I. Ring R_P : We build a ring R_P from a labeled poset P

SET-UP • P is a poset with vertices V and (directed) edges E

- Each edge e is labeled by a linear form α_e (= homogeneous deg-1 poly)
- Additional conditions: we won't worry about them (see Guillemin-Zara, or open qstns below).
- n a positive integer

RING: $R_P \subseteq \bigoplus_{v \in V} \mathbb{C}[t_1, \ldots, t_n]$ is defined by

$$R_P = \left\{ f : V \longrightarrow \mathbb{C}[t_1, \ldots, t_n] \text{ such that } \begin{array}{l} \text{for each edge } e = vw \\ f(v) - f(w) \text{ is divisible by } \alpha_e \end{array} \right\}$$

- Each $f(v)$ is a polynomial. Can equivalently write $f(v) - f(w) \in \underbrace{(\alpha_e)}_{\text{ideal}}$
- Can represent elements f pictorially by drawing poset & labeling vertex v by $f(v)$
- Then R_P is polynomial-labeled graphs so that polynomials on either side of edge have difference divisible by label on edge

EG: $n = 1$ $\quad P = \begin{array}{c} w \\ |\, t \\ v \end{array}$ $\quad$ Possible classes $\quad \begin{array}{l} ②, ③, ④ \text{ ask} \\ | \\ ① \text{ ask} \end{array}$ $\quad \rightsquigarrow \quad \begin{array}{l} p(t) + t\,q(t) \\ | \\ p(t) \end{array}$ $\quad$ for any $p(t), q(t)$ in $\mathbb{C}[t]$

$\mathbb{C}[t]$ is ambient polynomial ring

Ring structure: add/multiply vertex-wise. $\mathbb{C}[t_1, \ldots, t_n]$-module structure: ditto.

$$\begin{array}{l} p(t) + t\,q(t) \\ | \\ p(t) \end{array} = \begin{array}{l} p(t) \\ | \\ p(t) \end{array} + \begin{array}{l} t\,q(t) \\ | \\ 0 \end{array} = p(t) \begin{array}{l} 1 \\ | \\ 1 \end{array} + q(t) \begin{array}{l} t \\ | \\ 0 \end{array}$$

ie $\begin{array}{l} 1 \\ | \\ 1 \end{array}, \begin{array}{l} t \\ | \\ 0 \end{array}$ are free (!) $\mathbb{C}[t]$-module basis for R_P

II. Why bother? Geometry/topology

Given a "good" algebraic variety X with action of torus T, can build poset P_X

GKM: $R_{P_X} \cong H_T^*(X)$ is the equivariant cohomology ring of X

PS: $\underbrace{H^*(X) \cong}_{\text{ring } \cong} \dfrac{H_T^*(X)}{\langle t_1, \ldots, t_n \rangle H_T^*(X)}$ $\quad$ "Can obtain ordinary cohomology straightforwardly from equivariant cohomology"