Cores, Shi Arrangements, and Catalan Numbers

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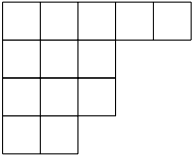
Partitions

A *partition* is a weakly decreasing sequence of positive integers of finite length.

The *Young diagram* of the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a diagram with a left-justified array of λ_1 boxes in row 1, λ_2 boxes in row 2, etc.

$\lambda = (5, 3, 3, 2)$ has Young diagram 

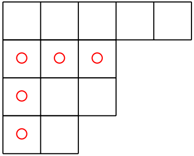
$|\lambda| = \#$ of boxes $= \sum_i \lambda_i = 13$.

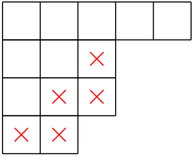
Representation theory of $\mathfrak{S}_d$ where $d = |\lambda|$.

Partitions index the irreps over $\mathbb{Q}$. You can use them to construct the irreps—they encode a wealth of information.

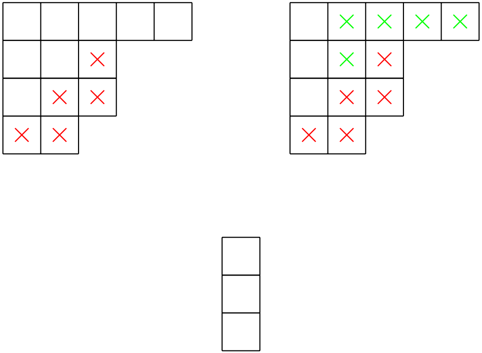
Hooks

8	7	5	2	1
5	4	2		
4	3	1		
2	1			



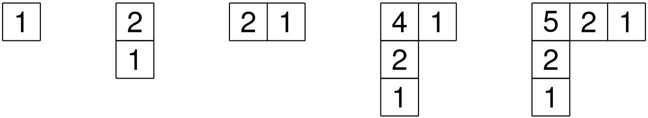


Rim-hooks



n-cores

An *n*-core is an integer partition λ such that $n \nmid h_{ij}$ for all boxes (i, j) in λ.



Some 3-cores. Boxes contain their hook numbers.

If you successively remove all n-rim-hooks, you are left with an n-core. $d = wn + |\text{core}|$
Independent of order removed.

If p is prime and λ is a p-core, the irrep corresponding to λ is still irreducible and projective over $\mathbb{F}_p$.
(If n is not prime, use Hecke algebra at an nth root of unity.)

p-cores are the matrix algebras when you decompose the group algebra into “blocks:"
$\mathbb{F}_p \mathfrak{S}_d = \prod_i B_i$. $B_i = \mathbb{M}_f(\mathbb{F}_p)$.

Otherwise, many irreps can belong to the same block.
The blocks are indexed by p-cores.

The affine symmetric group $\widehat{\mathfrak{S}}_n$ acts on $\{n\text{-cores}\}$.

In fact, $\widehat{\mathfrak{S}}_n$ acts on all partitions and the orbit $\widehat{\mathfrak{S}}_n \cdot \emptyset = \{n\text{-cores}\}$.

All their corresponding blocks are matrix algebras over $\mathbb{F}_p$, and so Morita equivalent. This is part of a larger story of Chuang-Rouquier who show blocks in the same orbit are derived equivalent.

This is also part of the larger story whereby the $\{n\text{-cores}\}$ are the extremal vectors in a highest weight crystal for $\widehat{\mathfrak{sl}}_n$.

$\widehat{\mathfrak{S}}_n$ acts on *n*-cores

The box in row i, column j has residue $j - i \mod n$.

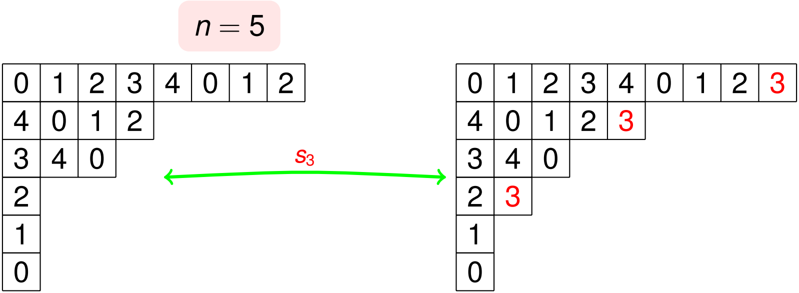
0	1	2	3	0	1
3	0	1			

$n = 4$

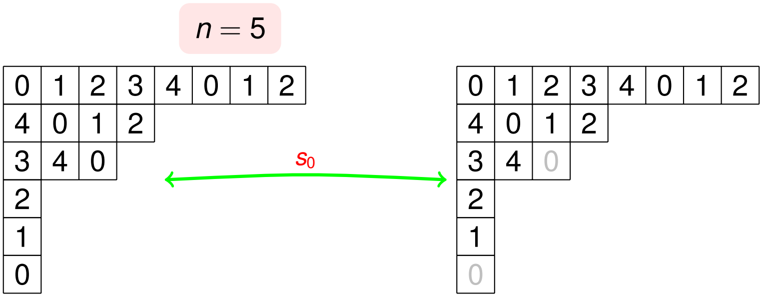
s_k acts on the n-core λ by removing/adding all boxes with residue k

The residues encode information about the central character and more specifically how a large commutative subalgebra acts.

$\widehat{\mathfrak{S}}_n$ acts on n-cores

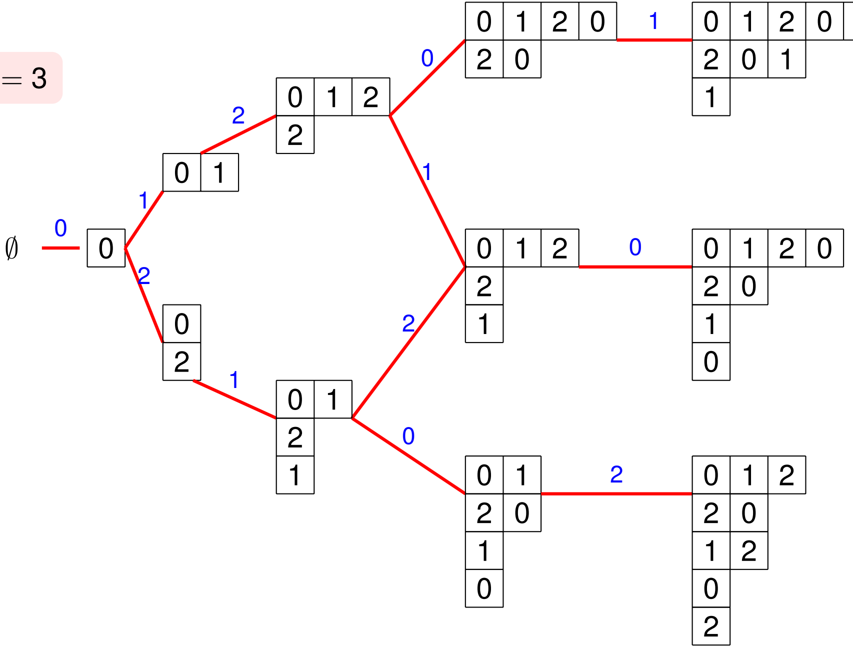


$\widehat{\mathfrak{S}}_n$ acts on n-cores



$\widehat{\mathfrak{S}}_n$ acts on cores

$n = 3$



The affine symmetric group

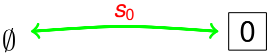
The affine symmetric group, denoted $\widehat{\mathfrak{S}}_n$, is defined as

$$\widehat{\mathfrak{S}}_n = \langle s_1, \ldots, s_{n-1}, s_0 \mid s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ if } i \not\equiv j \pm 1 \mod n, \\ s_i s_j s_i = s_j s_i s_j \text{ if } i \equiv j \pm 1 \mod n \rangle$$

for $n > 2$, and $\widehat{\mathfrak{S}}_2 = \langle s_1, s_0 \mid s_i^2 = 1 \rangle$.

The affine symmetric group contains the symmetric group $\mathfrak{S}_n$ as a subgroup. $\mathfrak{S}_n$ is the subgroup generated by the s_i, $0 < i < n$.

$$w \in \mathfrak{S}_n \quad \Longleftrightarrow \quad w \cdot \emptyset = \emptyset$$



$$\widehat{\mathfrak{S}}_n \cdot \emptyset = \{n\text{-cores}\} \simeq \widehat{\mathfrak{S}}_n / \mathfrak{S}_n.$$

$\widehat{\mathfrak{S}}_n$ acts by affine transformations

s_i = reflection over hyperplane $\{x_i = x_{i+1}\} =: H_{\alpha_i,0}$.
$V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\} \subseteq \mathbb{R}^n$

s_0 = affine reflection over hyperplane $\{x_1 - x_n = 1\} =: H_{\theta,1}$.

$$w \in \mathfrak{S}_n \quad \Longleftrightarrow \quad w \cdot (0, 0, \ldots, 0) = (0, 0, \ldots, 0)$$

orbit $\widehat{\mathfrak{S}}_n \cdot (0, 0, \ldots, 0) \simeq \widehat{\mathfrak{S}}_n / \mathfrak{S}_n$.

$\widehat{\mathfrak{S}}_n \cdot (0, 0, \ldots, 0)$ = root lattice $= Q = \bigoplus_i \mathbb{Z}\alpha_i$, where
$\alpha_i = (0, \ldots, \underbrace{1, -1}_{i \quad i+1}, \ldots, 0)$ are the simple roots.

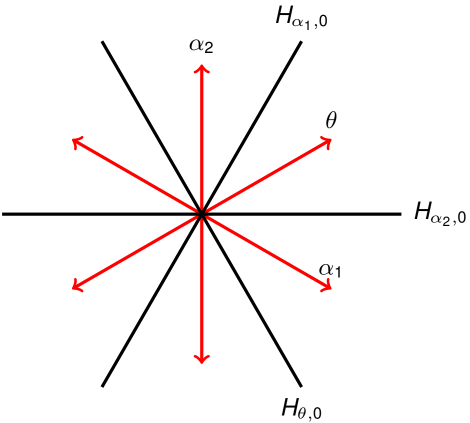
Notation

$\alpha_{ij} = \alpha_1 + \cdots \alpha_{j-1} \in V$, where $1 \leq i \leq j \leq n$ are the positive roots.

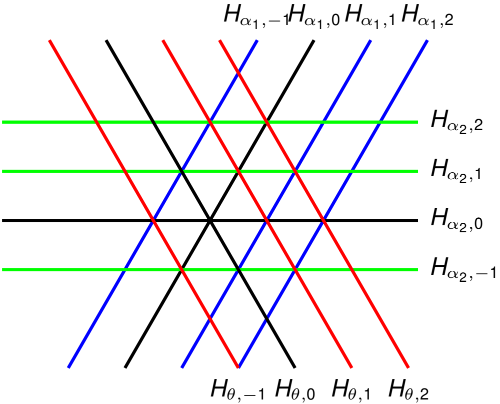
$\theta = \alpha_1 + \cdots + \alpha_{n-1} = (1, 0, \ldots, 0, -1)$ is the highest root

$H_{\alpha,k} = \{x \in V | \langle x \mid \alpha \rangle = k\}$, $H^+_{\alpha,k} = \{x \in V | \langle x \mid \alpha \rangle \geq k\}$

Roots and hyperplanes $n = 3$

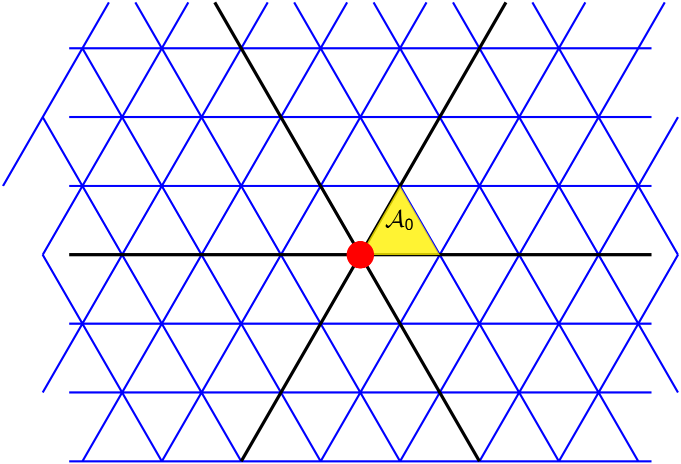


The roots α_1, α_2, and θ and their reflecting hyperplanes.


$H_{\alpha_1,-1}$ $H_{\alpha_1,0}$ $H_{\alpha_1,1}$ $H_{\alpha_1,2}$
$H_{\alpha_2,2}$
$H_{\alpha_2,1}$
$H_{\alpha_2,0}$
$H_{\alpha_2,-1}$
$H_{\theta,-1}$ $H_{\theta,0}$ $H_{\theta,1}$ $H_{\theta,2}$

Alcoves

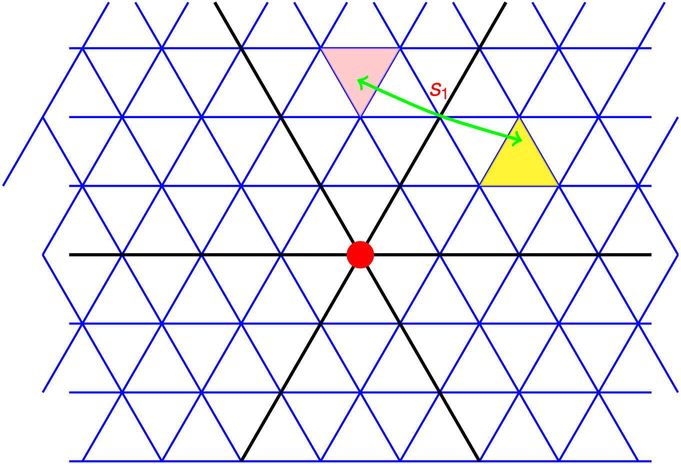
Each connected component of $V \setminus \bigcup_{\substack{\alpha_{ij} 1 \le i \le j \le n-1 \\ k \in \mathbb{Z}}} H_{\alpha_{ij},k}$ is called an alcove.



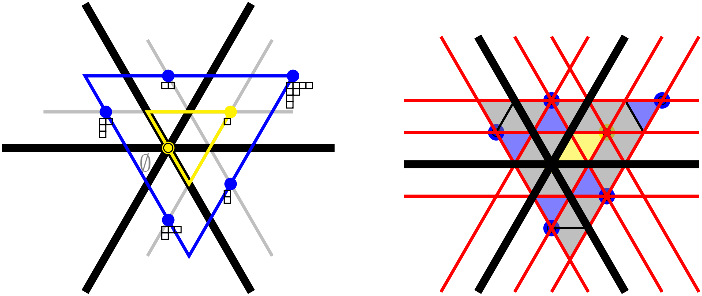
The fundamental alcove $\mathcal{A}_0$ is yellow.

$\widehat{\mathfrak{S}}_n$ acts on alcoves

s_i reflects over $H_{\alpha_i,0}$ for $1 \leq i \leq 0$ and s_0 reflects over $H_{\theta,1}$.



Bijection n-cores to alcoves

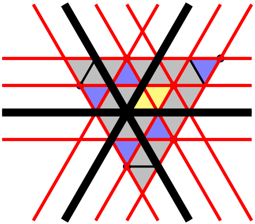


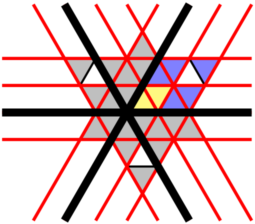
Certain statistics on partitions $\lambda = w \cdot \emptyset$ correspond to linear equations or inequalities satisfied by lattice points $w \cdot (0, \ldots, 0)$ or more precisely alcoves $w \cdot \mathcal{A}_0$.

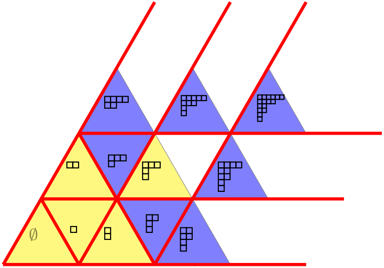
Bijection alcoves to alcoves

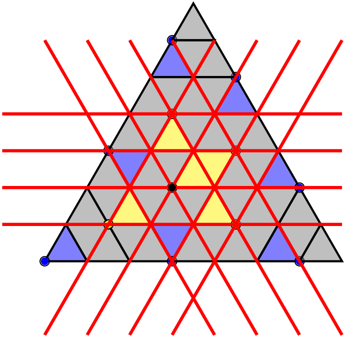
$$w \cdot \mathcal{A}_0 \leftrightarrow w^{-1}\mathcal{A}_0$$

The orbit of $\mathcal{A}_0$ under minimal length right representatives $w \in \mathfrak{S}_n \backslash \widehat{\mathfrak{S}}_n$ is the dominant chamber.

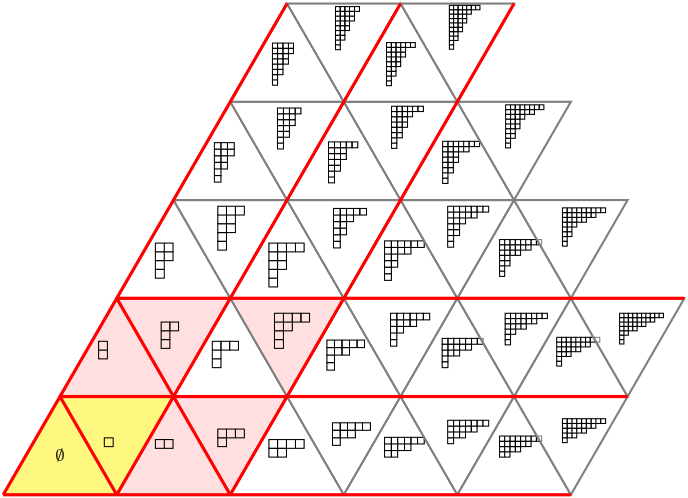






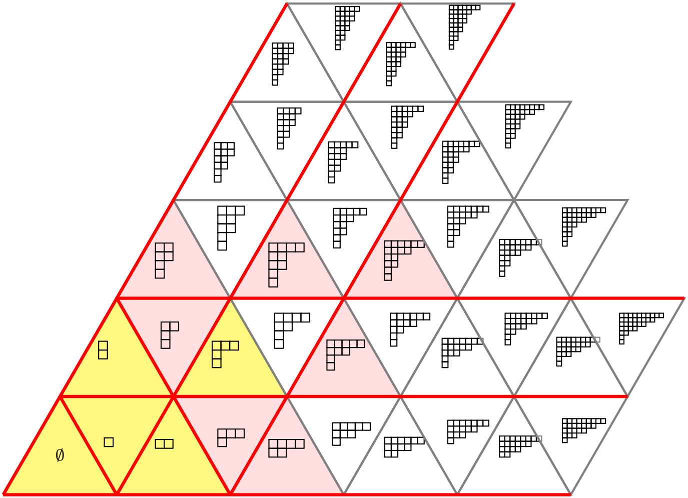


n-cores to dominant alcoves



All n-cores which are also t-cores

Above shows $n = 3$, $t = 5 = mn - 1$.
Below shows $n = 3$, $t = 7 = mn + 1$.



In 2002[1], Jaclyn Anderson showed that there are $\frac{1}{n+t}\binom{n+t}{n}$ partitions which are both n-cores and t-cores when n and t are relatively prime.

There are extended Catalan number $= C_{nm}$ partitions which are simultaneously n-cores and $(nm+1)$-cores, the same as the number of dominant Shi regions.

Take the "minimal" alcove in each region.

The partitions which are simultaneously n-cores and $(nm-1)$-cores are in bijection with the bounded dominant Shi regions.
Take the "maximal" alcove in each region.

[1] "Partitions which are simultaneously t_1- and t_2-core", Discrete Mathematics

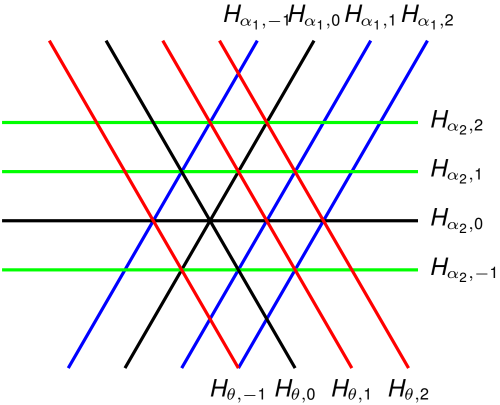
Extended Shi arrangement

For any positive integers n and m, the extended Shi arrangement is

$$\{H_{\alpha_{ij},k} | k \in \mathbb{Z}, -m < k \leq m \text{ and } 1 \leq i \leq j \leq n\}.$$

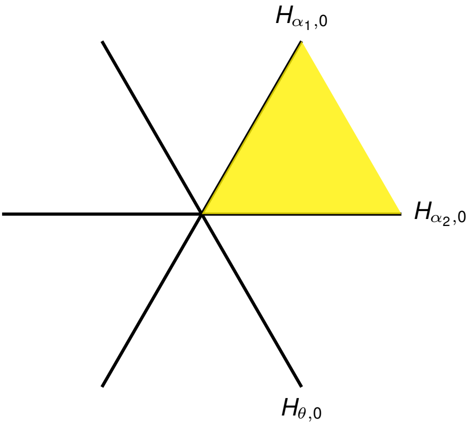
We also call it the m-Shi arrangement.

Shi arrangement for $n = 3$ and $m = 2$



Dominant/fundamental chamber

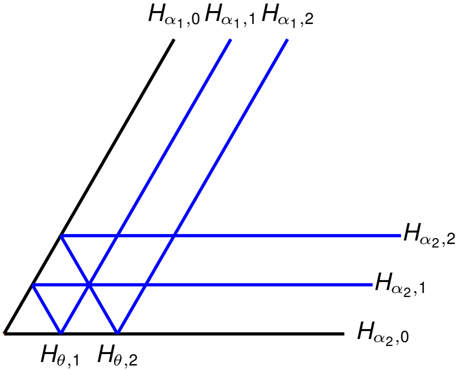
The fundamental or dominant chamber is $\cap_{\alpha_{ij}} H^+_{\alpha_{ij},0}$.



Regions

The regions of an arrangement are the connected components of the complement of the arrangement. Regions in the dominant chamber are called dominant regions.

Dominant regions



Dominant Shi regions for $n = 3$ and $m = 2$.

Number of regions in the dominant chamber

When $m = 1$, there are the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+n+1}\binom{n+n+1}{n}$$

regions in the dominant chamber.
When $m > 1$, there are the extended Catalan number

$$C_{nm} = \frac{1}{nm+1}\binom{n(m+1)}{n} = \frac{1}{n+nm+1}\binom{n+nm+1}{n}$$

regions in the dominant chamber. $C_n = C_{n1}$.

Bounded regions

There are

$$\frac{1}{n+nm-1}\binom{n+nm-1}{n}$$

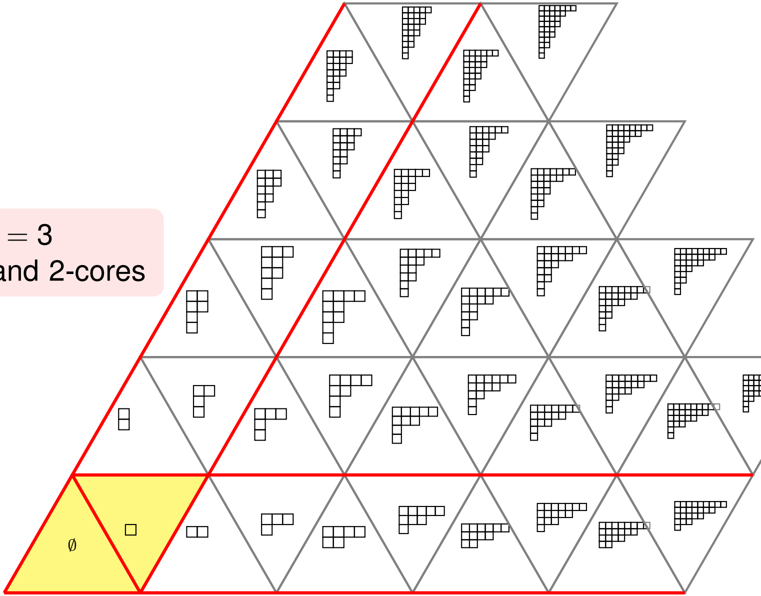
partitions which are both n-cores and $(nm-1)$-cores and there are

$$\frac{1}{n+nm-1}\binom{n+nm-1}{n}$$

bounded regions in the m-Shi arrangements.

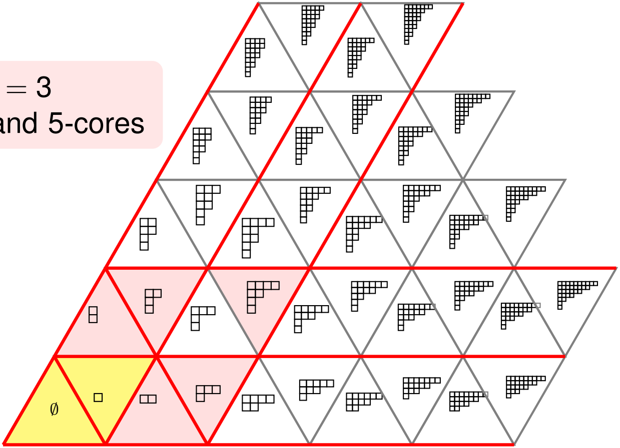
Alcoves $\Longleftrightarrow$ n-cores

$m = 1$, $n = 3$
3-cores and 2-cores



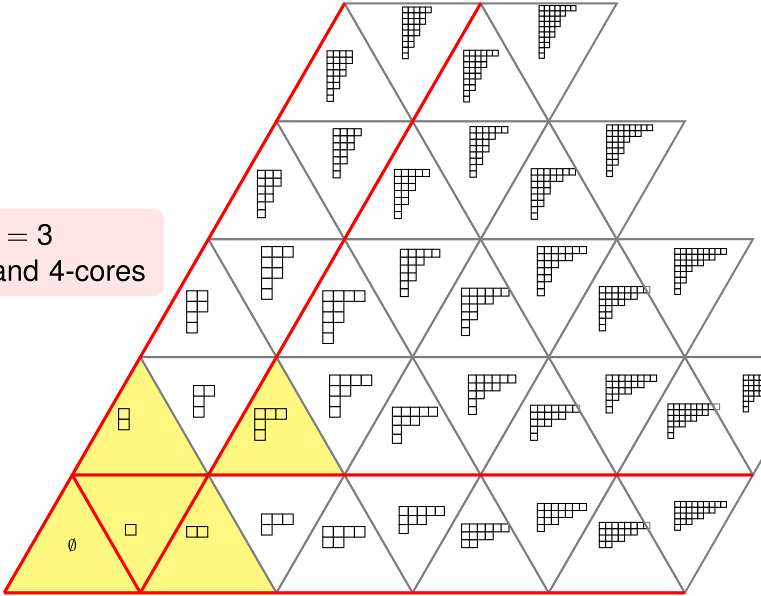
Alcoves $\Longleftrightarrow$ n-cores

$m = 2$, $n = 3$
3-cores and 5-cores



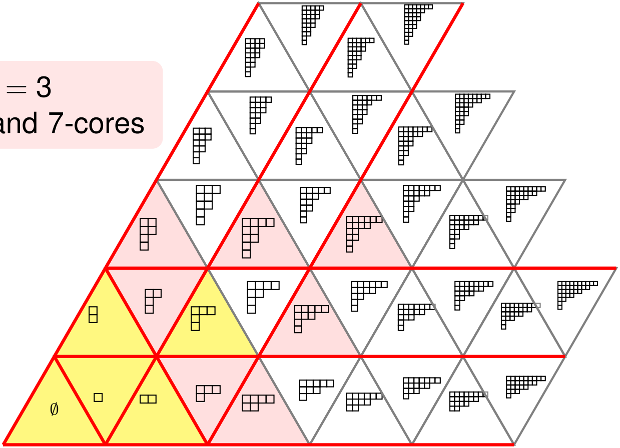
Alcoves $\Longleftrightarrow$ n-cores

$m = 1$, $n = 3$
3-cores and 4-cores



Alcoves $\Longleftrightarrow$ n-cores

$m = 2$, $n = 3$
3-cores and 7-cores



m-minimal alcoves

An alcove is *m*-minimal if it is the alcove in its *m*-Shi region separated from $\mathcal{A}_0$ by the least number of hyperplanes in the *m*-Shi arrangement.

We show the *m*-minimal alcoves have the same characterization as the *n*-cores which are also $(nm+1)$-cores.

Addable and removable boxes

$$w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha_1,3}}^{-} \cap {H_{\theta,4}}^{+} \cap {H_{\alpha_2,2}}^{-},$$

$\lambda = (5, 3, 2, 2, 1, 1) = w\emptyset$

0	1	2	0	1
2	0	1		
1	2			
0	1			
2				
1				

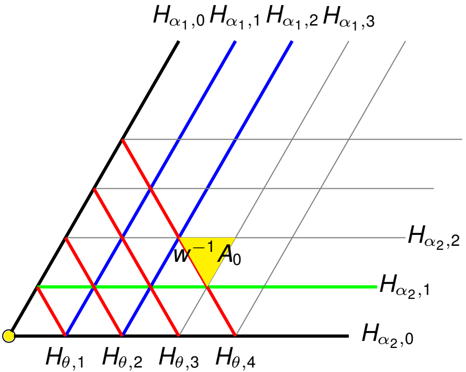
α_i	$w^{-1}(\alpha_i)$	$\frac{1}{2}$-space wall	i-boxes	$\langle w(\vec{0}) \mid \alpha_i \rangle$
α_0	$-\alpha_1 + 3\delta$	${H_{\alpha_1,3}}^{-}$	3 addable 0-boxes	$-3+1$
α_1	$\theta - 4\delta$	${H_{\theta,4}}^{+}$	4 removable 1-boxes	-4
α_2	$-\alpha_2 + 2\delta$	${H_{\alpha_2,2}}^{-}$	2 addable 2-boxes	2

Addable and removable boxes

0	1	2	0	1	2
2	0	1	2		
1	2				
0	1				
2					
1					

0	1	2	0	1
2	0	1		
1	2	0		
0	1			
2	0			
1				
0				

Addable and removable boxes



α_i	$w^{-1}(\alpha_i)$	(w)	$w^{-1}\mathcal{A}_0 \subseteq H_{\alpha,k}{}^+$
α_0	$-\alpha_1 + 3\delta$	$\{-\alpha_1 + \delta, -\alpha_1 + 2\delta,$	$H_{\alpha_1,1}{}^+, H_{\alpha_1,2}{}^+,$
α_1	$\theta - 4\delta$	$-\theta + \delta, -\theta + 2\delta,$	$H_{\theta,1}{}^+, H_{\theta,2}{}^+,$
		$-\theta + 3\delta, -\theta + 4\delta,$	$H_{\theta,3}{}^+, H_{\theta,4}{}^+$
α_2	$-\alpha_2 + 2\delta$	$-\alpha_2 + \delta\}$	$H_{\alpha_2,1}{}^+$