q-Eulerian polynomials

Michelle Wachs
University of Miami

Permutation Statistics

MacMahon at the beginning of the 20th century studied 4 basic permutations statistics

- descent number
- excedance number
- inversion number
- major index

Eulerian Permutation Statistics

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

Eulerian Permutation Statistics

For $\sigma \in \mathfrak{S}_n$,

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1, 3, 4\}$$

Define $\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|$. So

$$\mathrm{des}(32541) = 3$$

Excedance set: $\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i)\}$

$$\sigma = 32541 \qquad \mathrm{EXC}(\sigma) = \{1, 3\}$$

Define $\mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|$. So

$$\mathrm{exc}(32541) = 2$$

Eulerian Permutation Statistics

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

Eulerian Permutation Statistics

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} = \sum_{j=0}^{n-1} a_{n,j} t^j$$

$$A_3(t) = 1 + 4t + t^2$$

Eulerian Permutation Statistics

Euler's exponential generating function formula:

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z}-t}$$

Symmetry and Unimodality

Eulerian numbers $a_{n,j}$

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

Symmetry and Unimodality

Eulerian numbers $a_{n,j}$

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

A stronger property: γ-positivity

$$A_n(t) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,i} t^i (1+t)^{n-1-2i}, \qquad \gamma_{n,i} \in \mathbb{N}$$

Foata & Shützenberger (1970):

$$\gamma_{n,i} = |\{\sigma \in \mathfrak{S}_n \mid \sigma 0 \text{ has no double descents } \& \operatorname{des}(\sigma) = i\}|$$

Geometric interpretation

$(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ is the h-vector of the type A Coxeter complex.

Stanley (1980): The *h*-vector of every simplicial polytope is unimodal (and symmetric).

The γ vector of a d-dimensional simplicial polytope Δ is defined by

$$\sum_{i=0}^{d} h_i(\Delta)t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i(\Delta)t^i(1+t)^{n-1-2i}$$

Geometric interpretation

$(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ is the h-vector of the type A Coxeter complex.

Stanley (1980): The h-vector of every simplicial polytope is unimodal (and symmetric).

The γ vector of a d-dimensional simplicial polytope Δ is defined by

$$\sum_{i=0}^{d} h_i(\Delta) t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i(\Delta) t^i (1+t)^{n-1-2i}$$

Gal's Conjecture (2005): The γ vector of every flag homology sphere is nonnegative, i.e. $\gamma_i(\Delta) \geq 0$ for all i.

Geometric interpretation

$(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ is the h-vector of the type A Coxeter complex.

Stanley (1980): The *h*-vector of every simplicial polytope is unimodal (and symmetric).

The γ vector of a *d*-dimensional simplicial polytope Δ is defined by

$$\sum_{i=0}^{d} h_i(\Delta) t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i(\Delta) t^i (1+t)^{n-1-2i}$$

Gal's Conjecture (2005): The γ vector of every flag homology sphere is nonnegative, i.e. $\gamma_i(\Delta) \geq 0$ for all i.

Karu: true for all barycentric subdivisions
Peterson, Stembridge: true for all Coxeter complexes
Postnikov, Reiner and Williams: true for all chordal nestohedra

Mahonian Permutation Statistics

Inversion Number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\mathrm{inv}(32541) = 6$$

Mahonian Permutation Statistics

Inversion Number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n, \quad \sigma(i) > \sigma(j)\}|.$$

$$\mathrm{inv}(32541) = 6$$

Major Index:

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{maj}(32541) = \mathrm{maj}(3.25.4.1) = 1 + 3 + 4 = 8$$

Mahonian Permutation Statistics

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Mahonian Permutation Statistics

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

Mahonian Permutation Statistics

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

q-analog

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

q-Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

q-Eulerian polynomials

Theorem (Stanley 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

q-Eulerian polynomials

Theorem (Stanley 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv,des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}.$$

q-Eulerian polynomials

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)}$$

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,tq^{-1})\frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{\exp_q(zt) - t\exp_q(z)}$$

Let

$$A_n(q,t) := A_n^{\mathrm{maj,exc}}(q,tq^{-1}) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}t^{\mathrm{exc}(\sigma)}$$

Symmetry and Unimodality of $A_n(q,t)$

$n\backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\ldots$	$6+6q+11q^2+\ldots$	$4+3q+5q^2+\ldots$	1

Symmetry and Unimodality of $A_n(q, t)$

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\ldots$	$6+6q+11q^2+\ldots$	$4+3q+5q^2+\ldots$	1

$$A_n(q,t) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,i}(q) t^i (1+t)^{n-1-2i}$$

Symmetry and Unimodality of $A_n(q, t)$

$n\backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\dots$	$6+6q+11q^2+\dots$	$4+3q+5q^2+\dots$	1

$$A_n(q, t) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,i}(q) t^i (1+t)^{n-1-2i}$$

Shareshian and MW:

$$\gamma_{n,i}(q) = \sum_{\sigma \in ND_{n,i}} q^{\mathrm{inv}(\sigma)} \in \mathbb{N}[q]$$

where

$$ND_{n,i} := \{\sigma \in \mathfrak{S}_n \mid \sigma 0 \text{ has no double descents } \& \ \mathrm{des}(\sigma) = i\}$$

Eulerian quasisymmetric functions

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each excedance.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

Define

$$\mathrm{DEX}(\sigma) := \mathrm{DES}(\bar{\sigma})$$

$$\mathrm{DEX}(531462) = \mathrm{DES}(\bar{5}.\bar{3}14.\bar{6}2) = \{1, 4\}$$

Eulerian quasisymmetric functions

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each excedance.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

Define

$$\mathrm{DEX}(\sigma) := \mathrm{DES}(\bar{\sigma})$$

$$\mathrm{DEX}(531462) = \mathrm{DES}(\bar{5}.\bar{3}14.\bar{6}2) = \{1, 4\}$$

$$\sum_{i \in \mathrm{DEX}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$$

Eulerian quasisymmetric functions

For $T \subseteq [n-1]$, Gessel's fundamental quasisymmetric function

$$F_T(x_1, x_2, \dots) := \sum_{\substack{s_1 \geq \cdots \geq s_n \\ i \in T \Rightarrow s_i > s_{i+1}}} x_{s_1} \dots x_{s_n}$$

For $j \in [n-1]$, define the Eulerian quasisymmetric function

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}$$

Symmetric Function Generalization

Stable principal specialization:

$$\mathbf{ps}(f(x_1, x_2, \dots)) := f(1, q, q^2, \dots)$$

From Gessel's theory of quasisymmetric functions we have

$$\mathbf{p}s(F_T) = (q;q)_n^{-1} q^{\Sigma T}$$

where $(p;q)_n := (1-p)(1-pq)\dots(1-pq^{n-1})$

Symmetric Function Generalization

Stable principal specialization:

$$\mathbf{ps}(f(x_1, x_2, \dots)) := f(1, q, q^2, \dots)$$

From Gessel's theory of quasisymmetric functions we have

$$\mathbf{p}s(F_T) = (q;q)_n^{-1} q^{\Sigma T}$$

where $(p;q)_n := (1-p)(1-pq)\dots(1-pq^{n-1})$

Hence

$$\mathbf{p}s(F_{\mathrm{DEX}(\sigma)}) = (q;q)_n^{-1} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}$$

which implies

$$\mathbf{p}s(Q_{n,j}) := (q;q)_n^{-1} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}$$

Symmetric function analog of Euler's formula

Shareshian and MW (2006):

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(tz)-tH(z)},$$

where $H(z)=\sum_{n\geq 0} h_n z^n$.

Symmetric function analog of Euler's formula

Shareshian and MW (2006):

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(tz)-tH(z)},$$

where $H(z) = \sum_{n\geq 0} h_n z^n$.

$x_i \mapsto q^{i-1}$ and $z \mapsto z(1-q) \implies$

$$\sum_{n\geq 0} A_n(q,t)\frac{z^n}{[n]_q!} \quad = \quad \frac{(1-t)\exp_q(z)}{\exp_q(tz)-t\exp_q(z)}$$

Another occurrence of this symmetric function

Gessel:

$$1 + \sum_{n\geq 1} z^n \sum_{w \in ND_{n,i}(\mathbb{P})} x_w t^i (1+t)^{n-1-2i} = \frac{(1-t)H(z)}{H(tz) - tH(z)}$$

where $x_w := x_{w_1} x_{w_2} \ldots x_{w_n}$ and

$$ND_{n,i}(\mathbb{P}) := \{w \in \mathbb{P}^n \mid w0 \text{ has no double descents } \& \operatorname{des}(w) = i\}$$

$$779.1558.25 \ \in \ ND_{9,2}(\mathbb{P})$$

Another occurrence of this symmetric function

Gessel:

$$1 + \sum_{n \geq 1} z^n \sum_{w \in ND_{n,i}(\mathbb{P})} x_w t^i (1+t)^{n-1-2i} = \frac{(1-t)H(z)}{H(tz) - tH(z)}$$

where $x_w := x_{w_1} x_{w_2} \ldots x_{w_n}$ and

$$ND_{n,i}(\mathbb{P}) := \{ w \in \mathbb{P}^n \mid w0 \text{ has no double descents } \& \operatorname{des}(w) = i \}$$

$$779.1558.25 \ \in \ ND_{9,2}(\mathbb{P})$$

$$\begin{array}{ccccccc} & & & & & \boxed{2} & \boxed{5} \\ & & \boxed{1} & \boxed{5} & \boxed{5} & \boxed{8} & \\ \boxed{7} & \boxed{7} & \boxed{9} & & & & \end{array}$$

Symmetric function analog of Foata-Shützenberger

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \Gamma_{n,i} \, t^i (1+t)^{n-1-2i}$$

where

$$\Gamma_{n,i} := \sum_{\mu \in SH_{n,i}} s_\mu$$

and $SH_{n,i}$ is the set of skew hooks of size n where

- all columns have size at most 2
- last column has size 1
- i columns have size 2

Thus $\Gamma_{n,i}$ is Schur-positive.

Symmetric function analog of Foata-Shützenberger

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \Gamma_{n,i} \; t^i (1+t)^{n-1-2i}$$

where

$$\Gamma_{n,i} := \sum_{\mu \in SH_{n,i}} s_\mu$$

and $SH_{n,i}$ is the set of skew hooks of size n where

- all columns have size at most 2
- last column has size 1
- i columns have size 2

Thus $\Gamma_{n,i}$ is Schur-positive.

$\implies \sum_{j=0}^{n-1} Q_{n,j} t^j$ is Schur-unimodal (i.e., $Q_{n,j} - Q_{n,j-1}$ is Schur-positive for all $j < \frac{n-1}{2}$)

q-analog of Foata-Shützenberger

$$\mathbf{p}s(\Gamma_{n,i}) = (q;q)_n^{-1} \sum_{\sigma \in ND_{n,i}} q^{\mathrm{inv}(\sigma)}$$

$$\Downarrow$$

$$A_n(q,t) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{\sigma \in ND_{n,i}} q^{\mathrm{inv}(\sigma)} \right) t^i (1+t)^{n-1-2i}$$

Geometric Interpretation

Geometric Interpretation

$X_n :=$ the toric variety associated with the type A Coxeter complex.

Symmetric group $\mathfrak{S}_n$ acts naturally on X_n and this induces a representation of $\mathfrak{S}_n$ on each cohomology $H^{2j}(X_n)$.

Geometric Interpretation

$X_n :=$ the toric variety associated with the type A Coxeter complex.

Symmetric group $\mathfrak{S}_n$ acts naturally on X_n and this induces a representation of $\mathfrak{S}_n$ on each cohomology $H^{2j}(X_n)$.

Procesi and Stanley (1985)

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(tz)-tH(z)}$$

We now have the geometric interpretation:

$$Q_{n,j} = \mathrm{ch} H^{2j}(X_n)$$

Thus Schur-unimodality (and symmetry) of $(Q_{n,j})_{j=0,\ldots,n}$ follows from the Hard Lefschetz Theorem and Schur's Lemma.

Geometric Interpretation

$X_n :=$ the toric variety associated with the type A Coxeter complex.

Symmetric group $\mathfrak{S}_n$ acts naturally on X_n and this induces a representation of $\mathfrak{S}_n$ on each cohomology $H^{2j}(X_n)$.

Procesi and Stanley (1985)

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(tz)-tH(z)}$$

We now have the geometric interpretation:

$$Q_{n,j} = \mathrm{ch} H^{2j}(X_n)$$

Thus Schur-unimodality (and symmetry) of $(Q_{n,j})_{j=0,\dots,n}$ follows from the Hard Lefschetz Theorem and Schur's Lemma.
Equivariant version of Gal's Conjecture?

Cycle type refinements

For $\lambda \vdash n$, let $\mathfrak{S}_\lambda := \{\sigma \in \mathfrak{S}_n \mid \lambda(\sigma) = \lambda\}$,

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_\lambda \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}$$

$$a_{\lambda,j}(q) := \sum_{\substack{\sigma \in \mathfrak{S}_\lambda \\ \mathrm{exc}(\sigma) = j}} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)}$$

$$A_\lambda(q,t) := \sum_{j=0}^{n-1} a_{\lambda,j}(q) t^j = \sum_{\sigma \in \mathfrak{S}_\lambda} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

Cycle type refinements

For $\lambda \vdash n$, let $\mathfrak{S}_\lambda := \{\sigma \in \mathfrak{S}_n \mid \lambda(\sigma) = \lambda\}$,

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_\lambda \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{DEX}(\sigma)}$$

$$a_{\lambda,j}(q) := \sum_{\substack{\sigma \in \mathfrak{S}_\lambda \\ \mathrm{exc}(\sigma) = j}} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)}$$

$$A_\lambda(q,t) := \sum_{j=0}^{n-1} a_{\lambda,j}(q) t^j = \sum_{\sigma \in \mathfrak{S}_\lambda} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

We have

$$a_{\lambda,j}(q) = (q;q)_n \, \mathbf{ps}(Q_{\lambda,j})$$

Unimodality of cycle type refinements

Brenti (1993): $A_\lambda(1, t)$ is symmetric and unimodal with center of symmetry $c = \frac{n - m_1(\lambda)}{2}$.

Unimodality of cycle type refinements

Brenti (1993): $A_\lambda(1,t)$ is symmetric and unimodal with center of symmetry $c = \frac{n-m_1(\lambda)}{2}$.

Henderson and MW (2010)

(1) $\sum_{j\geq 0} Q_{\lambda,j} t^j$ is a symmetric and Schur unimodal polynomial in t with center of symmetry c.

(2) $A_\lambda(q,t)$ is a symmetric and unimodal polynomial in t with center of symmetry c.

Unimodality of cycle type refinements

Brenti (1993): $A_\lambda(1,t)$ is symmetric and unimodal with center of symmetry $c = \frac{n-m_1(\lambda)}{2}$.

Henderson and MW (2010)

(1) $\sum_{j\geq 0} Q_{\lambda,j} t^j$ is a symmetric and Schur unimodal polynomial in t with center of symmetry c.

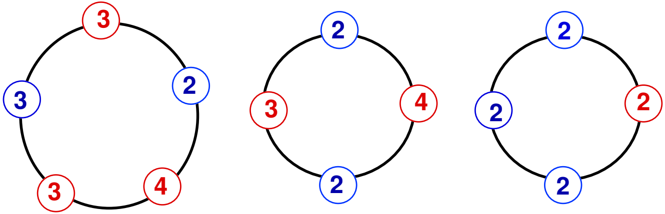
(2) $A_\lambda(q,t)$ is a symmetric and unimodal polynomial in t with center of symmetry c.

(1) $\Longrightarrow$ (2). Use stable principal specialization.

To prove (1) we use an alternative characterization of $Q_{\lambda,j}$, which was used in the proof of the symmetric function version of Euler's formula.

Alternative characterization of $Q_{\lambda,j}$ - Shareshian and MW

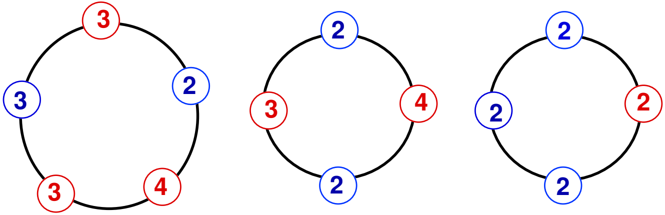
An ornament of type λ is a multiset of bicolored necklaces whose necklace sizes form partition λ



type $= (5, 4, 4)$ weight $= x_2^7 x_3^4 x_4^2$

Alternative characterization of $Q_{\lambda,j}$ - Shareshian and MW

An ornament of type λ is a multiset of bicolored necklaces whose necklace sizes form partition λ



type $= (5,4,4)$ weight $= x_2^7 x_3^4 x_4^2$

Shareshian and MW (2006) Let $\mathcal{R}_{\lambda,j}$ = set of ornaments of type λ with j red letters. Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} wt(R)$$

Analogous to a result of Gessel and Reutenauer (1993).

Plethystic identity - Shareshian and MW

For $\lambda = 1^{m_1} 2^{m_2} \cdots k^{m_k}$,

$$\sum_{j=0}^{n-1} Q_{\lambda,j} t^j = \prod_{i=1}^{k} h_{m_i} \left[\sum_{j=0}^{i-1} Q_{(i),j} t^j \right].$$

Plethystic identity - Shareshian and MW

For $\lambda = 1^{m_1} 2^{m_2} \cdots k^{m_k}$,

$$\sum_{j=0}^{n-1} Q_{\lambda,j} t^j = \prod_{i=1}^{k} h_{m_i} \left[\sum_{j=0}^{i-1} Q_{(i),j} t^j \right].$$

Summing over all partitions λ yields,

$$\sum_{n,j \geq 0} Q_{n,j} t^j = \sum_{m \geq 0} h_m \left[\sum_{i,j \geq 0} Q_{(i),j} t^j \right].$$

Plethystic identity - Shareshian and MW

For $\lambda = 1^{m_1}2^{m_2}\cdots k^{m_k}$,

$$\sum_{j=0}^{n-1} Q_{\lambda,j}t^j = \prod_{i=1}^{k} h_{m_i}\left[\sum_{j=0}^{i-1} Q_{(i),j}t^j\right].$$

Summing over all partitions λ yields,

$$\sum_{n,j\geq 0} Q_{n,j}t^j = \sum_{m\geq 0} h_m\left[\sum_{i,j\geq 0} Q_{(i),j}t^j\right].$$

The plethystic inverse of $\sum_{m\geq 0} h_m$ is,

$$L := \sum_{n\geq 0}(-1)^n \; lie_n,$$

where lie_n is the Frobenius characteristic of the Lie representation. Hence

$$\sum_{n,j\geq 0} Q_{(n),j}t^j = L\left[\sum_{i,j\geq 0} Q_{i,j}t^j\right].$$

A new formula for $\sum_{n,j\geq 0} Q_{\lambda,j} t^j z^n$ - Henderson and MW

From the symmetric function version of Euler's formula and the plethystic identity we derive

$$\sum_{n,j\geq 0} Q_{(n),j} t^j z^n = h_1 + \sum_{m\geq 1} lie_m \left[\sum_{i\geq 2} t[i-1]_t h_i z^i \right].$$

A new formula for $\sum_{n,j\geq 0} Q_{\lambda,j} t^j z^n$ - Henderson and MW

From the symmetric function version of Euler's formula and the plethystic identity we derive

$$\sum_{n,j\geq 0} Q_{(n),j} t^j z^n = h_1 + \sum_{m\geq 1} lie_m \left[\sum_{i\geq 2} t[i-1]_t h_i z^i \right].$$

Consequences: For all $\lambda \vdash n$,

(1) $Q_{\lambda,j}$ is Schur-positive: immediate.

(2) $\sum_{j=0}^{n-1} Q_{\lambda,j} t^j$ is Schur-unimodal.

A new formula for $\sum_{n,j\geq 0} Q_{\lambda,j} t^j z^n$ - Henderson and MW

From the symmetric function version of Euler's formula and the plethystic identity we derive

$$\sum_{n,j\geq 0} Q_{(n),j} t^j z^n = h_1 + \sum_{m\geq 1} lie_m \left[\sum_{i\geq 2} t[i-1]_t h_i z^i \right].$$

Consequences: For all $\lambda \vdash n$,

(1) $Q_{\lambda,j}$ is Schur-positive: immediate.

(2) $\sum_{j=0}^{n-1} Q_{\lambda,j} t^j$ is Schur-unimodal.

Proof of (2). We construct

- an $\mathfrak{S}_n$-module $V_{\lambda,j}$ whose Frobenius characteristic is $Q_{\lambda,j}$ by using the plethystic formula
- an injection $V_{\lambda,j-1} \to V_{\lambda,j}$ (for $1 \leq j \leq n/2$) that commutes with $\mathfrak{S}_n$ action.

(q,p)-Eulerian polynomials

$$A_\lambda(p,q,t) := \sum_{j=0}^{n-1} a_{\lambda,j}(p,q)t^j = \sum_{\sigma\in\mathfrak{S}_\lambda} p^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

(q,p)-Eulerian polynomials

$$A_\lambda(p,q,t) := \sum_{j=0}^{n-1} a_{\lambda,j}(p,q)t^j = \sum_{\sigma\in\mathfrak{S}_\lambda} p^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

Shareshian and Wachs (2007): If λ has the form $(\mu, 1^k)$, where μ is a partition of $n-k$ with no parts equal to 1, then

$$a_{\lambda,j}(p,q) = (p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im}\, \mathbf{ps}_m(Q_{\mu,j} h_{k-i}),$$

where $\mathbf{ps}_m$ is the (nonstable) principal specialization of order m.

Unimodality of $A_\lambda(p, q, t)$ - Henderson and MW

Follows that

$$a_{\lambda,j}(p, q) - a_{\lambda,j-1}(p, q)$$
$$= (p; q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im}\, \mathbf{ps}_m((Q_{\mu,j} - Q_{\mu,j-1})h_{k-i}).$$

For $j \leq \frac{n-k}{2}$, $Q_{\mu,j} - Q_{\mu,j-1}$ is Schur-positive. So it is F-positive.

Unimodality of $A_\lambda(p, q, t)$ - Henderson and MW

Follows that

$$a_{\lambda,j}(p,q) - a_{\lambda,j-1}(p,q)$$
$$= (p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im}\, \mathbf{ps}_m((Q_{\mu,j} - Q_{\mu,j-1})h_{k-i}).$$

For $j \leq \frac{n-k}{2}$, $Q_{\mu,j} - Q_{\mu,j-1}$ is Schur-positive. So it is F-positive.

Lemma: For any subset S of $[n-k-1]$,

$$(p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im} \mathbf{ps}_m(F_{S,n-k} h_{k-i}) \in \mathbb{N}[p,q].$$

Some other related work

- MacMahon (1915), Askey-Ismail (1976): enumeration of multiset derangements with j excedances
- Carlitz-Scoville-Vaughan (1976), Dollhopf-Goulden-Greene (2006), Stanley (1995, 2006): enumeration of words with no adjacent repeats, with j descents
- Shareshian-MW (2005): representation of symmetric group on top homology of a Rees product poset.
- Foata-Han (2007): (p, q)-Eulerian numbers,
- Foata-Han (2008): q-Euler numbers, type B analog of (p, q)-Eulerian polynomials
- Hyatt (2009): wreath product analog of Eulerian quasisymmetric functions
- Sagan-Shareshian-MW (2009): cyclic sieving
- Shareshian-MW (2009): chromatic quasisymmetric functions and generalized Eulerian numbers.