Robustness of quantum information processing to control noise

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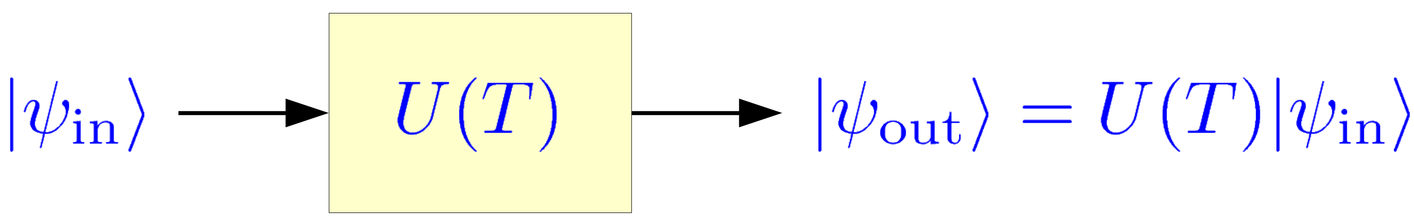
Basic definitions: Unitary quantum gates

A unitary quantum gate is the basic functioning element of a quantum circuit. Some basic notation:

n	number of qubits in the quantum gate system
$N = 2^n$	dimension of the system's Hilbert space
$W \in \mathrm{U}(N)$	the target unitary transformation
$U(T) \in \mathrm{U}(N)$	the actual evolution operator of the system at the final time T

The same unitary transformation is applied to any input state:



Controlled quantum gate

An external classical control $c(t)$ is necessary to operate the quantum gate. The Hamiltonian and evolution operator are functionals of the control:

$$H = H_0 + H_\mathrm{c}[c(t)], \quad U = U[c(t)], \quad t \in [0, T]$$

Gate fidelity is a measure of how well the target transformation was performed:

$$F = 1 - \|W - U(T)\|$$

It is convenient to use a normalized fidelity:

$$F = \frac{1}{N}\mathrm{Re}\mathrm{Tr}\left(W^\dagger U\right) \quad \text{or} \quad F = \frac{1}{N}\left|\mathrm{Tr}\left(W^\dagger U\right)\right|$$

Gate fidelity is also a functional of the control: $$F = F[c(t)]$$

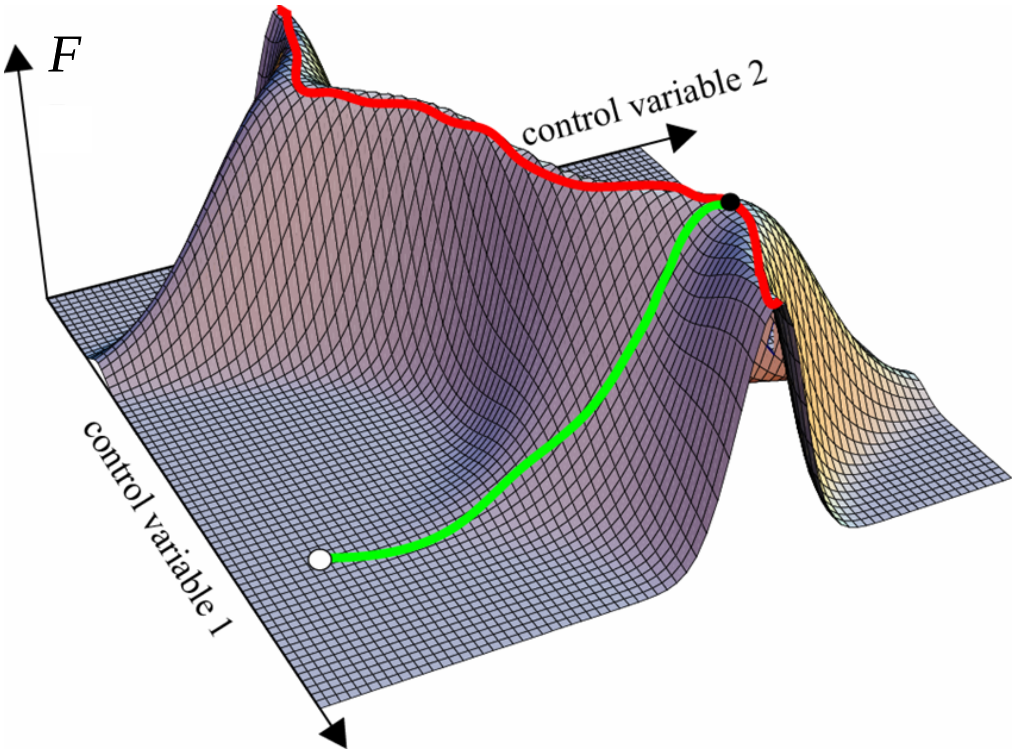
Quantum control landscape and optimality

The functional dependence $F = F[c(t)]$ is called the **control landscape**.

The **critical points** of the control landscape satisfy:

$$\frac{\delta F}{\delta c(t)} = 0,$$

$$\forall t \in [0, T]$$



A sufficient condition for **optimality** of a critical point is negative semidefiniteness of the Hessian matrix:

$$\mathcal{H}(t, t') = \frac{\delta^2 F}{\delta c(t') \delta c(t)}$$

For a recent review, see
C. Brif, R. Chakrabarti, and H. Rabitz,
New J. Phys. **12**, 075008 (2010)

Optimally controlled quantum gate

The analysis of regular critical points on the control landscape reveals that:

- There is one **maximum** manifold: $F = 1$
- There is one **minimum** manifold: $F = 0$
- All other critical manifolds are **saddles** (can be avoided by a smart optimization algorithm)

An **optimal control solution** $c_0(t)$ is perfect in ideal conditions

(no environment, no noise, no uncertainties):

$$U[c_0] = W \Rightarrow F[c_0] = 1$$

The Hessian at any optimal control solution has only non-positive eigenvalues. The “flatness” of the control landscape in the vicinity of an optimal control solution depends on the number of zero Hessian eigenvalues and magnitude of negative Hessian eigenvalues.

Optimal quantum gate with noisy control

All real controls are noisy! Consider a unitary quantum gate operating in the vicinity of an optimal control:

$$c(t) = c_0(t) + z(t)$$

Expanding for small noise:

$$F[c] \approx 1 + \frac{1}{2}\int_0^T \int_0^T \mathcal{H}_0(t,t')z(t)z(t')dtdt'$$

In the case of random noise, the control error $z(t)$ is a stochastic variable, with an auto-correlation function:

$$R_z(t,t') = \mathbb{E}\{z(t)z(t')\}$$

Statistical expectation value of the quantum gate fidelity:

$$\mathbb{E}\{F[c]\} \approx 1 + \frac{1}{2}\int_0^T \int_0^T \mathcal{H}_0(t,t')R_z(t,t')dtdt'$$

Robustness to white control noise

For **white noise** with any zero-mean distribution:

$$R_z(t,t') = \sigma^2 \delta(t-t')$$

This is a good model for thermal noise, which is the dominant source of control errors for solid-state qubits controlled by time-dependent voltages.

The statistical expectation value of the quantum gate fidelity:

$$\mathbb{E}\{F[c]\} \approx 1 - \frac{1}{2}\sigma^2 \left|\mathrm{Tr}(\mathcal{H}_0)\right|$$

The expected fidelity decrease is determined by the trace of the Hessian:

$$\mathrm{Tr}(\mathcal{H}_0) = \int_0^T \mathcal{H}_0(t,t)dt = -\sum_m |h_m|$$

Robustness to white control noise

For control through a dipole coupling:

$$H_c(t) = -c(t)\mu$$

the Hessian (at the maximum) for unitary gate control is given by

$$\mathcal{H}_0(t,t') = -\frac{1}{N}\mathrm{Tr}\left[\mu(t)\mu(t')\right]$$

The trace of the Hessian is then independent of the details of the applied control and depends only on the norm of the dipole operator and the total control time:

$$\mathrm{Tr}(\mathcal{H}_0) = -\frac{1}{N}\|\mu\|_F^2 T$$

$$\|\mu\|_F^2 = \mathrm{Tr}(\mu^2)$$

Strategies for enhancing robustness

- For **white noise**, the expected fidelity decrease is determined by the trace of the Hessian. For unitary gate control, this yields:

$$\mathbb{E}\{F[c]\} \approx 1 - \frac{1}{2N}\sigma^2 \mathrm{Tr}(\mu^2) T$$

  - Explore minimum control time that preserves controllability, given the system Hamiltonian (including the free Hamiltonian and the dipole operator)
  - Explore scaling of the fidelity decrease with the number of gate qubits.

- For **non-white noise**, the expected fidelity decrease is determined by the overlap of the Hessian and the noise autocorrelation function:

$$\mathbb{E}\{F[c]\} \approx 1 + \frac{1}{2}\int_0^T \int_0^T \mathcal{H}_0(t,t') R_z(t,t') dt dt'$$

  - Minimize the gate error by searching for optimal controls with the Hessian “orthogonal” to the control noise (use the null space of the Hessian).

Multiple objectives: High fidelity and robustness

We would like to maintain a high value of the nominal fidelity while at the same time decreasing sensitivity to the control noise (i.e., enhancing the robustness). This a **multiobjective optimization** problem.

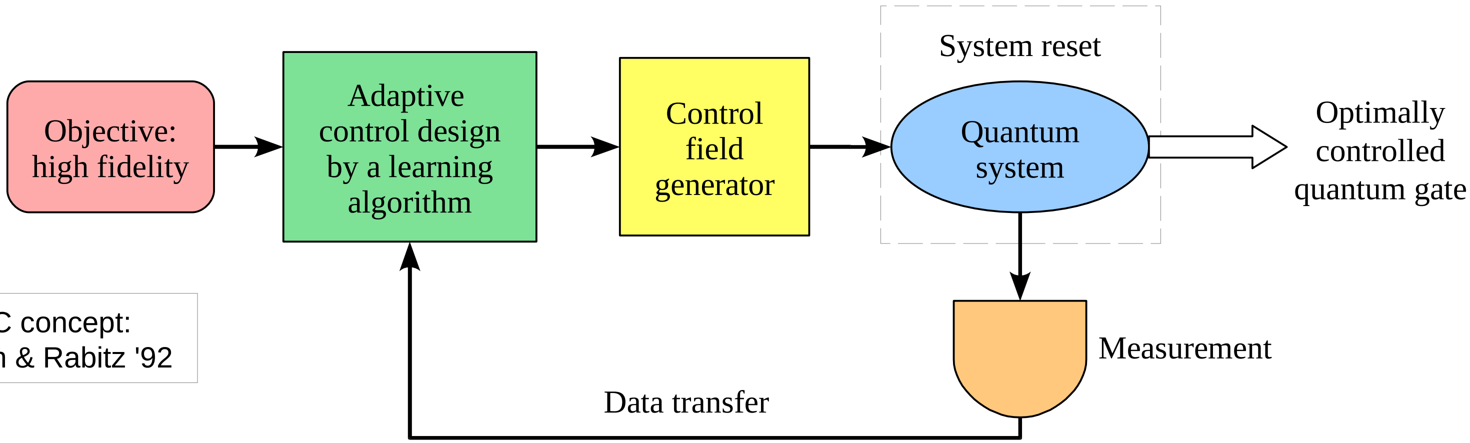
An important feature of multiobjective optimization is the **Pareto front** – the set of all controls such that no further improvement in one objective can be achieved without a detrimental effect on another.

- Exploring the Pareto front: interesting, but computationally expensive.
- Staying on the top of the landscape (the maximum of the nominal fidelity) and searching for optimal controls which minimize the absolute value of the overlap between the Hessian and the noise autocorrelation function.
  - A local (gradient-based) search, e.g., a second-order D-MORPH algorithm using the functional derivative of the Hessian with respect to the field (the third derivative of the objective with respect to the field).
  - A global (stochastic) serach, e.g., a genetic algorithm: for each new individual in the population, climb to the top of the landscape to evaluate the robustness there.

Adaptive optimization of quantum gate fidelity

We seek **improved robustness** – i.e., want to minimize the decrease in fidelity for a given control noise.

A laboratory-oriented approach – closed-loop optimization using **adaptive feedback control** (AFC) in the laboratory (or numerical simulation)



AFC concept:
Judson & Rabitz '92

Advantages of laboratory AFC:

- Optimization is for actual system with actual noise, not a simplified model;
- Each trial is very fast (~ps for system evolution, ~ms for control generation).

Drawback of laboratory AFC: Fidelity estimation requires process tomography (very expensive in number of experiments for multi-qubit systems)