Protecting Quantum Information with Optimal Control

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Quantum Information & Decoherence

Realistic physical systems are (noisy) open systems → they interact with the surrounding environment.

\[ C(t) + \delta C(t) \]
Model Open System:
Interacting Quantum Spins

\[ H(t) = H_s(t) + H_{\text{int}} + H_e \]

Schrödinger equation: \[ \dot{U}(t) = -iH(t)U(t) \]

Objective: Generate target system \textit{time evolutions}
Model Open System: Interacting Quantum Spins

\[ H = H_S + H_{\text{int}} + H_e \]

\[ H = H_0 - \vec{\sigma} \cdot \vec{C}(t) - \vec{\sigma} \cdot \vec{\Gamma} + H_e \]

Control pulse area: \[ \theta(t) = \int_0^t \vec{C}(\tau) \, d\tau \]
Qubit Dynamics & the Bloch Sphere

“Pure” spin states have the form:

\[ |\psi\rangle = c_0 |S_0\rangle + c_1 |S_1\rangle, \]

where \( |c_0|^2 + |c_1|^2 = 1 \).

\[ |\psi\rangle = \cos(\theta/2) |S_0\rangle + \exp(i\phi) \sin(\theta/2) |S_1\rangle. \]
Qubit Dynamics & the Bloch Sphere: Memory Channels

Consider a decoherence process for states in the $xy$-plane, i.e.,

$$|\psi\rangle = \frac{|S_0\rangle + \exp(i\phi)|S_1\rangle}{\sqrt{2}}.$$
Qubit Dynamics & the Bloch Sphere: Memory Channels

With the “right” set of physical rotations, this error can be corrected → “Hahn-echo”

Hahn-echo pulse sequence:

\[ \tau \rightarrow Y_\pi \rightarrow \tau \rightarrow Y_\pi \]

- Free evolution
- \(\pi\)-rotation about the \(y\)-axis

\(\pi\)-pulses \(\Rightarrow \theta(t_f) = \pi\)
Quantum Memory Channels: Dynamical-Decoupling Pulse Sequences

\[ \prod_{i}^{N} U_i \approx \mathcal{I}, \quad \text{where } U_i \text{ represent } \pi \text{ and } \pi/2 \text{ rotations and free evolutions.} \]

This is an approximation to \( \mathcal{I} \) because

- \( \{U_i\} \) and \( N \) are finite
- Non-unitary evolution is corrected with unitary “time-reversal” operations
Dynamical-Decoupling Pulses

Aside from satisfying geometric and pulse area constraints, e.g.,

$$\theta(t_f) = \int_0^{t_f} \overline{C}(\tau) d\tau,$$

What other control-field conditions can improve gate fidelity?
Dynamical-Decoupling Pulses

To remove 1st- and 2nd-order errors in $\pi$ and $\pi/2$ $z$-axis rotations when $\vec{\Gamma} = \Gamma_x \Rightarrow \vec{\eta} = 0$:

$$\eta_1 = \int_0^{t_f} \sin[\theta(t)] dt, \quad \eta_2 = \int_0^{t_f} \cos[\theta(t)] dt,$$

$$\eta_3 = \int_0^{t_f} t \sin[\theta(t)] dt, \quad \eta_4 = \int_0^{t_f} t \cos[\theta(t)] dt,$$

$$\eta_5 = \int_0^{t_f} \int_0^{t_f} \sin[\theta(t_1) - \theta(t_2)] \text{sign}(t_1 - t_2) dt_1 dt_2,$$

Double Quantum Dot: Effective One-Qubit Model


\[ H = H_0 - \vec{\sigma} \cdot \vec{\mathcal{D}}(t) - \vec{\sigma} \cdot \vec{\Gamma} + H_e \]

\[ \rightarrow H = C_z(t) \sigma_z + \epsilon \sigma_x \]
Dynamical-Decoupling Pulses

Feasible controls satisfying $\vec{n} = 0$

Gate distance:

$$\Delta \left( Z_\theta, U_{t_f} \right) = \sqrt{1 - \frac{1}{2} \left| \text{Tr} \left( Z_\theta^+ U_{t_f} \right) \right|}$$

Dynamical Decoupling + Optimal Control Pulses

By searching the space of controls satisfying

1. \( \vec{\eta} \approx 0 \)

2. \[
\Delta (Z_\pi, U_{t_f}) = \sqrt{1 - \frac{1}{2} \left| \text{Tr} \left( Z_\pi U_{t_f}^\dagger \right) \right|} \approx 0,
\]

and incorporating parameter estimates for \( \epsilon \), we improve control fidelities for \( Z_\pi \).

Systematic searching \( \rightarrow \) Optimal control theory
Quantum Optimal Control Theory

- Define an objective:
  \[ \Delta (Z_\theta, U_{t_f}) = \sqrt{1 - \frac{1}{2} \left| \text{Tr} \left( Z_\theta^+ U_{t_f} \right) \right|} \]

- Incorporate constraints:
  - Schrödinger’s equation
  - Experimental limitations of the control field

\[ J = \Delta \circ U_{t_f} + \frac{\alpha}{2} \int_0^{t_f} \left\| \tilde{\mathcal{O}}(t) \right\|^2 \, dt \]

- Optimize iteratively
  - Evolutionary algorithms
  - Gradient-based methods \[ \nabla_c J = 0 \]
DD+OC Optimization Procedure

After calculating $\nabla_c \mathcal{J}$, all gradient directions $\nabla_c \eta_i$ are removed:

$$\nabla_c \mathcal{J} \rightarrow \nabla_c \mathcal{J} - \sum_i p_i \nabla_c \eta_i,$$

so $\langle \nabla_c \mathcal{J}, \nabla_c \eta_i \rangle = 0$,

where $\vec{p}$ is constructed from

$$G_{ij} = \langle \nabla_c \eta_i, \nabla_c \eta_j \rangle \quad \text{and} \quad \langle \nabla_c \mathcal{J}, \nabla_c \eta_i \rangle.$$
DD+OC Pulses: Gate Distances

Distance, $\Delta[Z_{\pi}] \sim U(t; C)$

- $C_p(t), Z_{\pi}$
- $\text{DD+OC, } \varepsilon_0 = 0$
- $\text{DD+OC, } \varepsilon_0 = 1$
- $\text{DD+OC, } \varepsilon_0 = 2$
- $\text{DD+OC, } \varepsilon_0 = 3$
- $\text{DD+OC, } \varepsilon_0 = 4$
- $\text{DD+OC, } \varepsilon_0 = 5$

$x$-axis: $\varepsilon$

$y$-axis: $10^{-7}$ to $10^{-1}$
Gate Distances from OC and DD+OC

Distance, $\Delta [Z_{\pi}(t, \mathcal{U})]$
DD+OC Pulses: Improved Robustness 1
DD+OC Pulses: Improved Robustness 2

![Graph showing distance, δZ_{\text{opt}}, U(t_p; C)]
DD+OC Pulses

Control field, $C(t)$

Time, $t$

$C_p(t)$

$\epsilon_0 = 0$

$\epsilon_0 = 1$

$\epsilon_0 = 2$

$\epsilon_0 = 3$

$\epsilon_0 = 4$

$\epsilon_0 = 5$
Conclusions and Current Work

- Demonstrated dynamical decoupling + optimal control for improved gate fidelity and robustness

- Extend formalism to arbitrary rotation axes and perturbative expansions about arbitrary $\epsilon$.

- Explore robustness to control field variations
Calculating the Distance Measure with the Hilbert-Schmidt Norm

\[ \Delta(U, V) = \lambda \min_{\Phi_e} \left\{ \| U - (1_s \otimes \Phi_e)V \|_{HS} \right\} \]

After some relatively straightforward linear algebra . . .

\[ = \left[ 1 - \frac{1}{n} \| \text{Tr}_s (UV^\dagger) \|_{\text{Tr}} \right]^{1/2} \]

Nice analytical result; solved numerically in practice.

Quantum Control Results: One-Qubit Operations

Multiparticle environment: Hadamard gate
Gate Robustness to System Variations

Optimal Hadamard gate with a four-particle environment:

\[ \gamma = 0.02, \quad \gamma' = 0.0175, \quad \text{and} \quad F \approx 0.9934. \]

This control is applied to an ensemble of systems with random variations in \( \gamma \) and \( \gamma' \) given by \( \Delta \gamma / \gamma = 1/8 \).