Quantum Optimal Control Landscapes
— a “Simplicity” Theory —

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Outline

1. Motivation
2. Basic Concepts
3. Topological Analysis of Quantum Control Landscapes
4. Open questions
5. Concluding Remarks
Schemes for ultrafast laser control

- Frequency-domain approach: Two-pathway interference
- Time-domain approach: Pump-dump, STIRAP
- Optimal design approach: Optimal control theory, leaning control

Achievements

Optimization is supposed to be hard due to

- Limited bandwidth and severe noise in shaped pulses;
- A large number of control parameters.

What have been reported:

- > 1000 excellent simulation results (since 1985);
- ∼ 150 successful close-loop experiments (since 1998).

Observations:

- **dramatic enhancement** of the system yield;
- **robust** solutions to noises exist.
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Why is it easy to find a good quantum control?
Quantum Control Landscape: basic concepts

**Definition**: the graph of the mapping from the control variables to the cost functional.

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**Critical topology**: the topology of the set of critical points.

- Distribution of candidate solutions — algorithmic efficiency.
- Multiplicity of optimal solution set — robustness.
What we like...
What we dislike...
Control landscape for Observable Preparation

Schrödinger equation for an $N$-level closed quantum system:

$$\frac{\partial}{\partial t} \rho(t) = \frac{1}{i\hbar} \left[ H_0 - \epsilon(t)\mu, \rho(t) \right], \quad \rho(t_0) = \rho_0.$$ 

where $\epsilon(\cdot)$ is the control field. Consider the maximization of $\langle O \rangle$ at $t = T$:

$$J[\epsilon(\cdot)] = \text{Tr}\{\rho[T; \epsilon(\cdot)]O\}, \quad \epsilon(\cdot) \text{ admissible},$$

In principle, what does the landscape look like under unlimited control resources?
Control landscape at a coarse-grained scale
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Projection from the **dynamical control landscape**

\[ J[\epsilon(\cdot)] = \text{Tr}\{\rho[T;\epsilon(\cdot)]O\}, \quad \epsilon(\cdot) \text{ admissible} \]

onto the **kinematic control landscapes**:

\[ J(\rho) = \text{Tr}(\rho O), \quad \rho \text{ achievable}. \]
\[ J(U) = \text{Tr}(U\rho_0 U^\dagger O), \quad U \text{ achievable}. \]

where \( U \) is the propagator at \( t = T \).
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where \( U \) is the propagator at \( t = T \). In the case that the system is controllable

\[ J(U) = \text{Tr}(U \rho_0 U^\dagger O), \quad U \in \mathcal{U}(N). \]
Question

**Dynamical control landscape**
high-dimensional and highly nonlinear.

**Kinematic control landscape**
lower-dimensional and linear/quadratic.

What can be learned about the *dynamical landscape* from the kinematic one?
Landscape Reduction
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Suppose that $\epsilon(\cdot)$ is a critical point of $J(\epsilon(\cdot))$:

$$\delta J[\delta \epsilon(\cdot)] = \langle \nabla J(U(T)), \delta U(T) \rangle \equiv 0, \quad \forall \ \delta \epsilon(\cdot).$$
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- Moreover, $\epsilon(\cdot)$ is max. (min., saddle) .iff. $U(T)$ is max. (min., saddle).
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Moreover, $\epsilon(\cdot)$ is max. (min., saddle) iff $U(T)$ is max. (min., saddle).

**Conclusion**: critical topology preserved from the dynamical to the kinematic picture if all admissible controls are regular.
Conditions for kinematic landscape critical points

Take the parametrization $U \rightarrow U e^{i s A}$ in $\mathcal{U}(N)$ for any $A^\dagger = A$ and take the derivative of $J$:

$$\left. \frac{dJ}{ds} \right|_{s=0} = \text{Tr}(i A [U \rho_0 U^\dagger, O]) = 0, \quad \forall A^\dagger = A.$$

**Critical Condition:** $[U \rho_0 U^\dagger, O] = 0$.

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- Robustness of optimal controls on the “flat top” (maximum submanifold).
Control landscape for unitary gate fidelity

Fidelity defined as the distance from a desired quantum gate:

\[ J(U) = \|U - W\|^2 = 2N - 2\text{ReTr}(W^\dagger U), \quad U \in \mathcal{U}(N). \]

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How about open quantum systems?

In reality, environmental interactions are always present:

\[ H = H_S \otimes \mathbb{I}_\lambda + \mathbb{I}_N \otimes H_E + H_{SE} \]
Kinematic Control Landscape for Open Quantum Systems

Definition

\[ J(\{K_j\}) = \sum_j \text{Tr}(K_j \rho_0 K_j^\dagger O), \quad \sum_{j=1}^{\lambda} K_j^\dagger K_j = I_N. \]
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**Assumptions**

- all Kraus maps are achievable;
- all admissible controls are regular.
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The equation $\sum_j K_j^\dagger K_j = \mathbb{I}_N$ implies that the following $K$ is the first $N$ columns of some enlarged unitary matrix:

$$K = \begin{pmatrix} K_1 \\ \vdots \\ K_\lambda \end{pmatrix} = U \begin{pmatrix} I_N \\ \vdots \\ 0_N \end{pmatrix}, \quad U = \begin{pmatrix} K_1 \cdots * \\ \vdots \vdots \vdots * \\ K_\lambda \cdots * \end{pmatrix}$$
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\end{pmatrix}
\]

\( J(K) = \text{Tr}\{U (\rho_0 \otimes |1\rangle\langle1|) U^\dagger (O \otimes \mathbb{I}_\lambda)\} \triangleq J(U) \)
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Auxiliary control landscape for “system” + “environment”.

Landscape Mapping
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\[ \epsilon(\cdot) \xrightarrow{S-equation} J[\epsilon(\cdot)] \xrightarrow{dynamical} \]
\[ K \xrightarrow{kinematic} J(K) \xrightarrow{J(\cdot)} \]
\[ U \xrightarrow{kinematic(lifted)} \]
\[ K = U(I_N \otimes |1\rangle\langle 1|) \]
Landscape Mapping

\[ \varepsilon(\cdot) \rightarrow S\text{–equation} \rightarrow K \rightarrow U \]

\[ \text{dynamical} \rightarrow \text{controllable/regular} \rightarrow \text{kinematic} \rightarrow \text{kinematic(lifted)} \]

\[ J[\varepsilon(\cdot)] \rightarrow J(K) \rightarrow J(U) \]

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Landscape Mapping

\[ \epsilon(\cdot) \xrightarrow{\text{controllable/regular}} K \xleftarrow{\text{surjective}} \]

\[ \text{dynamical} \quad \rightarrow \quad \text{kinematic} \quad \leftrightarrow \quad \text{kinematic(lifted)} \]

\[ S - \text{equation} \]

\[ K = U(\mathbb{1}_N \otimes |1\rangle \langle 1|) \]

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Controllable/regular

Topologically equivalent
Landscape Topology for open quantum systems
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Open question: the role of controllability?

Almost all quantum systems are controllable (C. Altafini, J. Math. Phys. 43, 2051 (2002).) BUT...

Gate fidelity landscape $J = \|\text{Tr}(W^\dagger U)\|^2$, $U \in SU(2) \subset U(8)$. 
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The role of Controllability beyond Yes-or-No
not only the existence of “wanted” controls but also nonexistence of “unwanted” controls

Open question: the role of singularity?
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Look at the critical condition for $\epsilon(\cdot)$:

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where $\delta U(T)$ is dependent on $\delta \epsilon(\cdot)$. 
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The mapping $\delta \epsilon(\cdot) \mapsto \delta U(T)$ can be singular.
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Invisible critical points in the kinematic picture!
Open question: the role of singularity?

- Singular controls may become traps, e.g. zero field;
  P. Fouquieres, S. Schirmer, arXiv:1004.3492
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**Important in time optimal control (Lapert et al, PRL 2010) !**
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Search efforts scaling with the system dimension and objectives, e.g., $N$, $\rho$, $O$ or $W$?
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Concluding Remarks

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- A strong support for evident laboratory successes;
- Open up perspectives in developing more efficient algorithms (e.g., gradient and evolutionary-strategy algorithms are going on in Princeton laboratory).
THANK YOU!