On the maximal measure of sections of the n-cube

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defines a probability measure on the n-cube B_∞^n. For $a \in S^{n-1}$ let

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be the $(n-1)$-dimensional measure of the central section orthogonal to a.
For $k \in \{1, \cdots, n\}$, let $f_k := \frac{1}{\sqrt{k}}(\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0) \in S^{n-1}$.

K. Ball: For Lebesgue measure ($h = 1$), $A(a,1) \leq A(f_2, 1)$.
A. Zvavitch: False in general for Gaussian measure, $h_\lambda(s) = \exp(-\lambda s^2)$:
For $n > 3$ and large $\lambda > 0$, $A(f_n, h_\lambda) > A(f_2, h_\lambda)$.

The main result

Theorem 1

Let $h : [-1,1] \to \mathbb{R}_{>0}$ *be even and in* C^3 *with* $h' \le 0$, $h'' \le 0$, $h''' \ge 0$ *on* $[0,1]$ *and* $h(0) \le \frac{3}{2} h(1)$. *Suppose further that*

$$\pi \left(\int_0^1 r^2 h(r) dr \right) \left(\int_0^1 h(r)^2 dr \right)^2 \ge \left(\int_0^1 h(r) dr \right)^5 . \qquad (1)$$

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Remark: The conditions $h''' \ge 0,\ h(0) \le \frac{3}{2}h(1)$ are technical and can be weakened. Condition (1) is essential, without it, in general, for large n $A(f_n,h) > A(f_2,h)$.

Corollary. For $\lambda > 0$ consider the Gaussian measure with $h(r) = \exp(-\lambda r^2)$

$$d\mu(s) = \exp\left(-\lambda \|s\|_2^2\right) ds / \left(\int_{-1}^{1} \exp(-\lambda r^2) dr\right)^n , \quad s \in B_\infty^n.$$

Then for $\lambda \leq 0.196262$ and $a_1 \leq 1/\sqrt{2}$

$$A(a, h) \leq A(f_2, h)$$

while for $\lambda > 0.196263$ and large n

$$A(f_n, h) > A(f_2, h).$$

Condition (1) is satisfied for $h(r) = \exp(-\lambda r^2) \Leftrightarrow \lambda \leq 0.1962627...$

Formula for the measure of the cube section

Proposition 1

Let $h : [-1,1] \to \mathbb{R}_{>0}$ *be even and in* C^1. *Let*

$$f(t) := \int_0^1 \cos(tr)h(r)dr \,/ \int_0^1 h(r)dr, \quad t \in \mathbb{R}. \tag{2}$$

Then the measure μ_h *of the section of* B_∞^n *orthogonal to* $a = (a_j)_{j=1}^n \in S^{n-1}$ *is given by*

$$A(a,h) = \frac{1}{\pi} \int_0^\infty \prod_{j=1}^n f(a_j r)\, dr$$

Analogue of K. Ball's main inequality

Proposition 2

Assume $h \in C^3[0,1], \quad h > 0, \quad h' \leq 0, \quad h'' \leq 0, \quad h''' \geq 0, \quad h(0) \leq \frac{3}{2} h(1)$ *and (1). Let*

$$f(t) := \int_0^1 \cos(tr) h(r) dr \ / \ \int_0^1 h(r) dr,$$

$$H(p) := \sqrt{p} \int_0^\infty |f(t)|^p dt, \quad p \geq 0.$$

Then for all $p \geq 2, \quad H(p) \leq H(2)$.

Proof of the Theorem using Proposition 2:
Let $a = (a_j)_{j=1}^n \in S^{n-1}$, $\quad 0 \leq a_1 \leq 1/\sqrt{2}$. Then
$p_j := a_j^{-2} \geq 2$, $\quad \sum_{j=1}^n \frac{1}{p_j} = 1$ and Hölder's inequality yields

$$\begin{aligned} A(a,h) &= \frac{1}{\pi} \int_0^\infty \prod_{j=1}^n f(a_j r) dr \\ &\leq \frac{1}{\pi} \prod_{j=1}^n \left(\int_0^\infty |f(a_j r)|^{p_j} dr \right)^{1/p_j}, \quad a_j r = t \\ &= \frac{1}{\pi} \prod_{j=1}^n \left(\sqrt{p_j} \int_0^\infty |f(t)|^{p_j} dt \right)^{1/p_j} \end{aligned}$$

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$$A(a,h) = \frac{1}{\pi} \int_0^\infty \prod_{j=1}^n f(a_j r) dr$$

$$\le \frac{1}{\pi} \prod_{j=1}^n \left(\int_0^\infty |f(a_j r)|^{p_j} dr \right)^{1/p_j}, \quad a_j r = t$$

$$= \frac{1}{\pi} \prod_{j=1}^n \left(\sqrt{p_j} \int_0^\infty |f(t)|^{p_j} dt \right)^{1/p_j}$$

$$= \frac{1}{\pi} \prod_{j=1}^n H(p_j)^{1/p_j} \le \frac{1}{\pi} H(2)$$

$$= \frac{1}{\pi} \int_0^\infty f(r/\sqrt{2})^2 dr = A(f_2, h)$$

Nazarov-Podkorytov's Lemma on distribution functions

Lemma 1

(X,λ) *measure space,* $f,g : X \to \mathbb{R}$ *in* $L_p(X,\lambda)$ *for any* $p>0$,

$$F(x) = \lambda\{t \in X \mid |f(t)| > x\}, \quad G(x) = \lambda\{t \in X \mid |g(t)| > x\}, \quad x \in \mathbb{R}_{\geq 0}.$$

Assume there is $x_0 > 0$ *such that*

$$G \leq F \quad \text{on} \quad (0,x_0) \quad \text{and} \quad G \geq F \quad \text{on} \quad (x_0,\infty). \tag{3}$$

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Then $\varphi(p) = \frac{1}{p x_0^p} \int_X (|g(t)|^p - |f(t)|^p) dt$ *is increasing in* $p \in \mathbb{R}_{>0}$.

Therefore $\int_X |f(t)|^{p_0} dt \leq \int_X |g(t)|^{p_0} dt$ *for some* $p_0 > 0$
implies $\int_X |f(t)|^p dt \leq \int_X |g(t)|^p dt$ *for all* $p \geq p_0$.

The L_2-norm of f

We apply Lemma 1 with $p_0 = 2$, f and a suitable exponential function g. For this we need

Lemma 2

Let h be as in Proposition 1 and for $t > 0$

$$f(t) = \int_0^1 \cos(tr)h(r)dr \;/\; \int_0^1 h(r)dr,$$

$$g(t) = \exp(-dt^2), \quad d := \frac{1}{2\pi}\left(\int_0^1 h(r)dr\right)^4 \;/\; \left(\int_0^1 h(r)^2 dr\right)^2.$$

Then $\int_0^\infty |f(t)|^2 dt = \int_0^\infty |g(t)|^2 dt.$

Zeros of f

Under the conditions imposed on h, $f(t)$ resembles $\dfrac{\sin t}{t}$:

Proposition 3

(Polya-Szegö)
Assume $h \in C^2[0,1]$ with $h > 0$, and $[h' \le 0,\ h'' < 0]$ or $[h' \ge 0]$ and let

$$f(t) = \int_0^1 \cos(tr)h(r)dr \ / \int_0^1 h(r)dr.$$

Then f has infinitely many zeros, all of which are real.
For all $n \in \mathbb{N}$, there is exactly one zero between $n\pi$ and $(n+1)\pi$ (and between $-(n+1)\pi$ and $-n\pi$).

Comparison of f and g near 0

To prove Proposition 2 using Lemma 1, take f, g as above and let F, G denote their distribution functions. We have to show (3): there is $x_0 > 0$ with

$$G \leq F \quad \text{on} \quad (0, x_0), \quad G \geq F \quad \text{on} \quad (x_0, 1].$$

Note $|f|,\ g \leq 1$. For $G \geq F$ on $(x_0, 1]$ we need $|f| \leq g$ near 0, where $|f|, g$ are maximal. Let

$$d' := \frac{1}{2} \int_0^1 r^2 h(r) dr \ / \int_0^1 h(r) dr.$$

Near $t = 0$: $\cos(tr) \simeq 1 - \frac{1}{2} t^2 r^2, \quad f(t) \simeq 1 - d' t^2 \simeq \exp(-d' t^2).$

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Thus $|f(t)| \leq g(t) = \exp(-dt^2)$ near $t = 0$ requires $d' \geq d$ or

$$\left(\int_0^1 h(r) dr \right)^5 \leq \pi \left(\int_0^1 r^2 h(r) dr \right) \left(\int_0^1 h(r)^2 dr \right)^2 . \qquad (1)$$

Lemma 3

$h \in C^2$, $h > 0$, $h' \le 0$, $h'' < 0$, $\bar{t_1}$ = *smallest positive zero of* f,

$$f(t) = \int_0^1 \cos(tr)h(r)dr \;/\; \int_0^1 h(r)dr.$$

Then for all $\quad 0 \le t \le \bar{t_1}, \quad |f(t)| \le \exp(-d' t^2)$,

$$d' = \frac{1}{2} \int_0^1 r^2 h(r)dr \;/\; \int_0^1 h(r)dr.$$

Proof. Denote the zeros of f by
$\bar{t}_n \in (n\pi, (n+1)\pi$ and $-\bar{t}_n$, $f(0) = 1$, $f'(0) = 0$:

$$f(t) = \prod_{n \in \mathbb{N}} \left(1 - \frac{t^2}{\bar{t}_n^2}\right), \quad \ln f(\sqrt{x}) = \sum_{n \in \mathbb{N}} \ln\left(1 - \frac{x}{\bar{t}_n^2}\right)$$

$$\ln(1 - cx)'' = \frac{-c^2}{(1 - cx)^2} < 0 \quad \text{implies that}$$

$$(\ln f(\sqrt{\cdot}))'(x) = \frac{f'(\sqrt{x})}{f(\sqrt{x})} \frac{1}{2\sqrt{x}} \quad \text{is decreasing in } x > 0.$$

Hence $\displaystyle \frac{f'(t)}{f(t)} \frac{1}{2t} \leq \lim_{t \to 0} \frac{f'(t)}{f(t)} \frac{1}{2t} = \frac{1}{2} f''(0) = -d'$ and

$$(\exp(d' t^2) f(t))' = (f'(t) + 2d' t\, f(t)) \exp(-d' t^2) \leq 0.$$

Therefore $f(t) \leq \exp(-d' t^2)$, $t \leq t_1$. $\square$

Distribution functions

Hence for $d' \leq d$, i.e. condition (1), $f(t) \leq g(t) = \exp(-dt^2)$, $t \leq \bar{t_1}$,

and $\int_0^1 |f(t)|^2 dt = \int_0^1 |g(t)|^2 dt$, which means that

$$\int_0^1 x\ F(x)\ dx = \int_0^1 x\ G(x)\ dx.$$

Since $F < G$ near $x = 1$, $F - G$ has at least one zero $0 < x_0 < 1$. It will have exactly one such zero if $F - G$ is strictly decreasing, i.e. $F' < G'$ or $|F'| > |G'|$. Note $F' < 0$, $G' < 0$. Hence (3) of Lemma 1 for f, g and Proposition 2 will follow from

$$|F'(x)\ /\ G'(x)| > 1, \quad x \in [0, 1]\ . \tag{4}$$

Since $G(x) = g^{-1}(x) = \sqrt{\ln \frac{1}{x}}\ /\ \sqrt{d}$,

$$1/|G'(x)| = 2\sqrt{d} x \sqrt{\ln \frac{1}{x}}\ .$$

Distribution function of f

Let $0 < \bar{t_1} < \bar{t_2} < \cdots$ denote the zeros of f and
$x_i = \max_{t \in [\bar{t_i}, \bar{t}_{i+1}]} |f(t)| =: |f(\tilde{t_i})|, \quad i \in \mathbb{N}.$
The maxima x_i are decreasing in $i \in \mathbb{N}$ under the conditions of the Theorem, as seen by estimates based on integration by parts.
Since $f < g$ on $(0, \bar{t_1}]$, $|F'| > |G'|$ on $[x_1, 1)$. For $x \in (0, x_1)$, there is $m \in \mathbb{N}$ with $x \in [x_{m+1}, x_m)$. Considering level sets one finds

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Thus (4) means

$$2\sqrt{d} \sum_{|f(t)|=x} \frac{|f(t)|}{|f'(t)|} \sqrt{\ln \frac{1}{|f(t)|}} > 1, \quad x \in [x_{m+1}, x_m) \qquad (5)$$

For $x \in [x_{m+1}, x_m)$, there is one such $t = t_0 \in (0, \bar{t_1})$ and two t-values with $t_i < t_i'$ in $(\bar{t}_i, \bar{t}_{i+1}) \quad$ for $\quad i = 1, \cdots, m$.

Reduction to the size of f

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Lemma 4

Under the assumptions of the Polya-Szegö Proposition, when $h' \leq 0$

$$\frac{|f(t)|}{|f'(t)|} \geq t \, \frac{|f(t)|}{1 - \varepsilon |f(t)|}, \quad \varepsilon = \mathrm{sgn}(f(t) f'(t)), \quad t > 0 \, .$$

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Proof. Integration by parts yields with $h' \leq 0$

$$f'(t) = \frac{h(1)}{\int_0^1 \ h(r)dr}\ \frac{\cos t}{t} - \frac{f(t)}{t} + \frac{1}{x}\ \frac{\int_0^1 \cos(tr)r|h'(r)|dr}{\int_0^1 h(r)dr}\,.$$

If $f'(t) \geq 0$ use $\cos(tr) \leq 1$, if $f'(t) \leq 0$ use $\cos(tr) \geq -1$, and $\int_0^1 \ r|h'(r)|dr = \int_0^1 \ h(t)dt - h(1)$ to conclude

$$|f'(t)| \leq \frac{1}{t}\,[1 - \operatorname{sgn}(f'(t))\ f(t)]\,.$$

Dividing by $|f(t)|$ and forming reciprocals yields Lemma 4.

Estimate of $|F'(x)/G'(x)|$

Using that h is decreasing and concave with $h(1) \geq \frac{2}{3} h(0) = \frac{2}{3}$, we find

$$2\sqrt{d} = \sqrt{\frac{2}{\pi}} \left(\int_0^1 h(r) dr \right)^2 \Big/ \left(\int_0^1 h(r)^2 dr \right)$$

$$\geq \sqrt{\frac{2}{\pi}} \int_0^1 h(r) dr \geq \sqrt{\frac{2}{\pi}} \int_0^1 (1 - \frac{r}{3}) dr = \frac{5}{6} \sqrt{\frac{2}{\pi}} \,.$$

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Therefore (5) is satisfied if

$$\frac{5}{6}\sqrt{\frac{2}{\pi}} \sum_{f(t)=x} t \, \frac{|f(t)|}{1 - \varepsilon |f(t)|} \sqrt{\ln \frac{1}{|f(t)|}} > 1, \quad x \in [x_{m+1}, x_m) \tag{6}$$

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For $m = 1$ this means, with $f(t_i) = x$,

$$\frac{5}{6}\sqrt{\frac{2}{\pi}} \left((t_0 + t_1')\frac{x}{1-x} + t_1 \frac{x}{1+x} \right) \sqrt{\ln \frac{1}{x}}$$
$$= \frac{5}{6}\sqrt{\frac{2}{\pi}} \left((t_0 + t_1 + t_1') \, \frac{x}{1-x} - t_1 \, \frac{2x^2}{1-x^2} \right) \sqrt{\ln \frac{1}{x}} > 1, \quad x \in [x_2, x_1).$$

Since $\frac{x}{1-x}\sqrt{\ln\frac{1}{x}}$ is increasing in x, (6) is satisfied provided that

$$\frac{5}{6}\sqrt{\frac{2}{\pi}}\left((t_0+t_1+t_1')\,\frac{x_2}{1-x_2}-t_1\frac{2x_2^2}{1-x_2^2}\right)\sqrt{\ln\frac{1}{x_2}}>1 \tag{7}$$

with $f(t_i)=x_2=$ second maximum of $|f|$. We need lower estimates of t_0, t_1, t_1' and x_2.

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Lemma 5

Let $x_1=|f(\tilde{t}_1)|$ *be the first maximum of* $|f|$. *Then* $\tilde{t}_1\geq T_1\simeq 4.4934$, *the first maximum of* $\left|\frac{\sin t}{t}\right|$ *and* $t_1+t_1'\geq 2\tilde{t}_1\geq 2T_1$.
Further $t_0\geq\pi(1-x_2)$.

Lower estimate of x_2

Since $x_2 = |f(\tilde{t}_2)|$ with $2\pi < \bar{t}_2 < \tilde{t}_2 < \bar{t}_3$, we know $x_2 \geq \left|f\left(\frac{5}{2}\pi\right)\right|$.

Lemma 6

Assume that $h \in C^3[-1,1]$ with $h''' \geq 0$. Then

$$x_2 \geq f\left(\frac{5}{2}\pi\right) \geq \frac{h(1)}{\int_0^1 h(r)dr}\,\frac{1}{\frac{5}{2}\pi} \geq \frac{4}{15\pi}\ .$$

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Proof. Integration by parts yields with $\sin(tr)h'''(r) \leq h'''(r)$

$$f(t)\cdot\left(\int_0^1 h(r)dr\right) \geq h(1)\frac{\sin t}{t} + h'(1)\,\frac{\cos t}{t^2} - h'(0)\frac{1}{t^2} + h''(0)\frac{1}{t^3} - h''(1)\frac{1+\sin t}{t^3}\ .$$

Since $h \in C^3$ is even and $h''' \geq 0$, $h'(0) = h''(0) = 0$, $-h''(1) \geq 0$.
Put $t = \frac{5}{2}\pi$. □

Estimate of $|F'(x)/G'(x)|$ for $m = 1$

We have to prove (7). Using Lemma 5 and Lemma 6, (7) follows from

$$\frac{5}{6}\sqrt{\frac{2}{\pi}}\left[\pi + \frac{2T}{1-x_2} - \frac{4\pi x_2}{1-x_2^2}\right] x_2 \sqrt{\ln\frac{1}{x_2}} > 1 \ , \ 2T = 8.9868$$

which is true for $x_2 \geq \frac{4}{15\pi} > \frac{1}{12}$; The left side is increasing in x_2.

Estimate of $|F'(x)/G'(x)|$ for $m = 1$
We have to prove (7). Using Lemma 5 and Lemma 6, (7) follows from

$$\frac{5}{6}\sqrt{\frac{2}{\pi}}\left[\pi + \frac{2T}{1-x_2} - \frac{4\pi x_2}{1-x_2^2}\right] x_2 \sqrt{\ln \frac{1}{x_2}} > 1 \ , \ 2T = 8.9868$$

which is true for $x_2 \geq \frac{4}{15\pi} > \frac{1}{12}$; The left side is increasing in x_2.
The estimate for $m \geq 2$ is easier. $\square$