Rearrangements and Isoperimetric Inequalities

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Theorem (Groemer, 1976)

Assume

- $K \subset \mathbb{R}^n$ *- convex body,* $\operatorname{vol}(K) = 1$.
- D_n *- Euclidean ball in* $\mathbb{R}^n$ *of* $\operatorname{vol}(D_n) = 1$.

Set

$$\mathbb{E}(K, N) := \int_K \dots \int_K \operatorname{vol}(\operatorname{conv}\{x_1, \dots, x_N\})\, dx_1 \dots dx_N.$$

Then

$$\mathbb{E}(K, N) \geq \mathbb{E}(D_n, N).$$

Equality holds only for ellipsoids.

[Blaschke, 1917], ..., [Christ, 1984], ..., [Pfiefer, 1990],...
[Giannopoulos-Tsolomitis, 2003], ...

Notation:

- $\mathcal{P}_{[n]}$ - probability measures on $\mathbb{R}^n$, abs. cont. w.r.t. Lebesgue measure
- For $N \geq n$, $x_1, \dots, x_N \in \mathbb{R}^n$, treat the $n \times N$ matrix

$$[x_1 \dots x_N]$$

as linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$.
- If $C \subset \mathbb{R}^N$ is a convex body, then

$$[x_1 \dots x_N]C = \left\{ \sum_{i=1}^{N} c_i x_i : (c_i) \in C \right\} \subset \mathbb{R}^n.$$

Theorem (Paouris, P. '11)

Suppose

- $N \geqslant n$ *and* $\mu_1, \dots, \mu_N \in \mathcal{P}_{[n]}$; $f_i = \frac{d\mu_i}{dx}$;
- $C \subset \mathbb{R}^N$ *is a convex body.*

Set

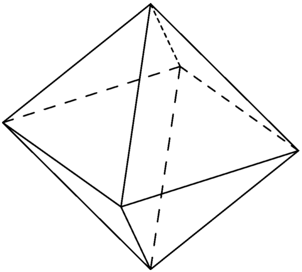
$$\mathcal{F}_C(f_1, \dots, f_N) = \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \mathrm{vol}\left([x_1 \dots x_N]C\right) \prod_{i=1}^{N} f_i(x_i) dx_N \dots dx_1.$$

If $\|f_i\|_\infty \leqslant 1$ *for* $i = 1, \dots, N$, *then*

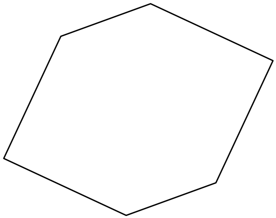
$$\mathcal{F}_C(f_1, \dots, f_N) \geqslant \mathcal{F}_C(\mathbb{1}_{D_n}, \dots, \mathbb{1}_{D_n}),$$

where $D_n \subset \mathbb{R}^n$ *is the Euclidean ball of volume one.*

$C = B_1^N \subset \mathbb{R}^N$



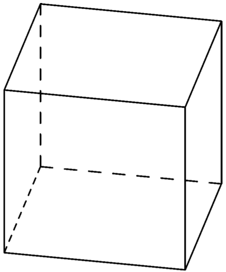
$[x_1 \dots x_N] B_1^N = \mathrm{conv}\{\pm x_1, \dots, \pm x_N\} \subset \mathbb{R}^n$



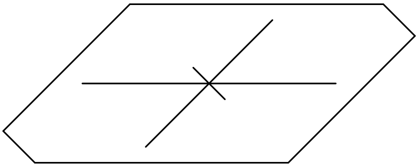
Symmetric analogue of Groemer's theorem when $f_i = \mathbb{1}_K$ with $K \subset \mathbb{R}^n$ is a convex body of $\mathrm{vol}(K) = 1$.

As before $f_i = \mathbb{1}_K$, where $K \subset \mathbb{R}^n$ is a convex body of $\mathrm{vol}\,(K) = 1$.

$C = B_\infty^N \subset \mathbb{R}^N$



$[x_1 \dots x_N] B_\infty^N = \{\sum_{i=1}^N c_i x_i : |c_i| \leq 1\}$



[Bourgain, Meyer, Milman, Pajor, '88]

Write $T_n = [x_1 \dots x_N]$. For $y \in S^{n-1}$,

$$\begin{aligned} h(T_N C, y) &= \sup\{\langle T_N x, y\rangle : x \in C\} \\ &= \sup\{\langle x, T_N^t y\rangle : x \in C\} \\ &= h(C, T_N^t y). \end{aligned}$$

Let $C = B_q^N$ with $1/p + 1/q = 1$.

$$h(T_N B_q^N, y)^p = \sum_{i=1}^{N} |\langle x_i, y\rangle|^p .$$

If $x_1, x_2, \dots$ i.i.d. $\sim \mu \in \mathcal{P}_{[n]}$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |\langle x_i, y\rangle|^p = \int_{\mathbb{R}^n} |\langle x, y\rangle|^p \, d\mu(x).$$

L_p-centroid bodies

Theorem (Lutwak, Yang, & Zhang, 2000)

Let $K \subset \mathbb{R}^n$ be compact, star-shaped with $\mathrm{vol}(K) = 1$. For $p \geq 1$, define $Z_p(K)$ by its support function

$$h(Z_p(K), y) := \left(\int_K |\langle x, y \rangle|^p \, dx \right)^{1/p} \quad (y \in S^{n-1}).$$

Then

$$\mathrm{vol}(Z_p(K)) \geq \mathrm{vol}(Z_p(D_n)).$$

Equality holds only for centered ellipsoids.

- Different proof: [Campi-Gronchi, 2002]
- Generalized to $\mu \in \mathcal{P}_{[n]}$ by [Paouris, '10].

L_p-centroid bodies in our framework:

- $\mu \in \mathcal{P}_{[n]}$ with density f.
- $X_1, X_2, \ldots$ independent random vectors $\sim f$.

Then

$$Z_p(f) = \lim_{N\to\infty} N^{-1/p}[X_1 \ldots X_N]B_q^N \quad (a.s.)$$

in the Hausdorff metric. Consequently, if $\|f\|_\infty \leq 1$.

$$\mathrm{vol}\,(Z_p(f)) \geq \mathrm{vol}\,(Z_p(\mathbb{1}_{D_n}))\,.$$

- Works also for Orlicz-centroid bodies ([LYZ, 2010]).

- SLLN in a more general setting: [Artstein & Vitale, '75]

Steps in the proof

Recall

$$\mathcal{F}_C(f_1, \dots, f_N) = \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \mathrm{vol}\left([x_1 \dots x_N]C\right) \prod_{i=1}^{N} f_i(x_i) dx_N \dots dx_1.$$

1. Rearrangement inequality:

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(f_1^*, \dots, f_N^*),$$

   where f_i^* is the symmetric decreasing rearrangement of f.
2. If $\|f_i\|_\infty \leq 1$ for $i = 1, \dots, N$, then

$$\mathcal{F}_F(f_1^*, \dots, f_N^*) \geq \mathcal{F}_F(\mathbb{1}_{D_n}, \dots, \mathbb{1}_{D_n}).$$

Rearrangements

If $A \subset \mathbb{R}^n$ is measurable with $\operatorname{vol}(A) < \infty$, then A^* is the (open) ball with the same volume as A:

$$A^* = rB_2^n \text{ with } \operatorname{vol}(A) = \operatorname{vol}(rB_2^n).$$

If $f : \mathbb{R}^n \to \mathbb{R}^+$ is integrable its *symmetric decreasing rearrangement* is

$$f^*(x) = \int_0^\infty \mathbb{1}_{\{f(x)>s\}^*}(x) ds$$

Compare with the "layer-cake representation"

$$f(x) = \int_0^\infty \mathbb{1}_{\{f>s\}}(x) ds$$

Properties of f^*:

- $f^*(x) = f^*(y)$ if $|x| = |y|$.
- $f^*(x) \geq f^*(y)$ if $|x| \leq |y|$.
- $\mathrm{vol}\left(\{f > s\}\right) = \mathrm{vol}\left(\{f^* > s\}\right)$ for each s.

1-d rearrangements If $\theta \in S^{n-1}$, fix a coordinate system such that $e_1 := \theta$. The Steiner symmetrization $f^*(\cdot|\theta)$ of f with respect to $\theta^\perp$: for $x_2, \dots, x_n \in \mathbb{R}$, set $h(t) = f(t, x_2, \dots, x_n)$ and define

$$f^*(t, x_2, \dots, x_n|\theta) := h^*(t). \tag{1}$$

Theorem (Brascamp, Lieb & Luttinger, '74)

Let

- $f_1, \ldots, f_M : \mathbb{R} \to \mathbb{R}^+$ *integrable*
- $u_1, \ldots, u_M \in \mathbb{R}^n$.

Then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{M} f_i(\langle x, u_i \rangle) dx \leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{M} f_i^*(\langle x, u_i \rangle) dx$$

Corollary

Let

- $K = -K \subset \mathbb{R}^n$ *be convex.*
- $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}^+$.

Then

$$\int_K \prod_{i=1}^n f_i(x_i)dx \leqslant \int_K \prod_{i=1}^n f_i^*(x_i)dx.$$

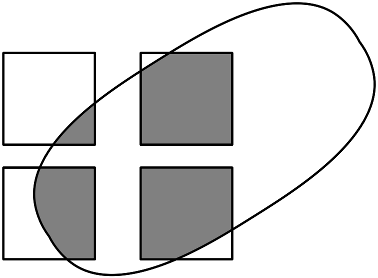
Proof: Take

$$K_m = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : |\langle x, u_i \rangle| \leqslant 1\}$$

$$\mathbb{1}_{K_m} = \prod_{i=1}^m \mathbb{1}_{[-1,1]}(\langle \cdot, u_i \rangle)$$

$$\int_K \prod_{i=1}^n f_i(x_i)dx \leqslant \int_K \prod_{i=1}^n f_i^*(x_i)dx.$$

- When $f_i = \mathbb{1}_{L(i)}$, $L(i) \subset \mathbb{R}$, compact: [Pfiefer, 1990]
- Extensions for K non-convex: [Draghici, 2006]



Recall $F : \mathbb{R}^N \to \mathbb{R}$ is

- **quasi-concave** if $\forall s$, $\{x : F(x) > s\}$ is convex

Corollary

Assume

- $F : \mathbb{R}^N \to \mathbb{R}^+$ *is quasi-concave and even*
- $g_1, \dots, g_N : \mathbb{R} \to \mathbb{R}^+$ *integrable*

Then

$$\int_{\mathbb{R}^N} F(t) g_1(t_1) \cdots g_N(t_N) dt \leqslant \int_{\mathbb{R}^N} F(t) g_1^*(t_1) \cdots g_N^*(t_N) dt.$$

Proof:

$$\int_{\mathbb{R}^N} F(t) g_1(t_1) \cdots g_N(t_N) dt = \int_{\mathbb{R}^N} \int_0^\infty \mathbb{1}_{\{F > s\}} g_1(t_1) \cdots g_N(t_N) ds dt$$

Recall $F : \mathbb{R}^N \to \mathbb{R}$ is

- **quasi-convex** if $\forall s$, $\{x : F(x) \le s\}$ is convex

Corollary

Assume

- $F : \mathbb{R}^N \to \mathbb{R}^+$ *is quasi-convex and even*
- $g_1, \dots, g_N : \mathbb{R} \to \mathbb{R}^+$ *integrable*

Then

$$\int_{\mathbb{R}^N} F(t) g_1(t_1) \cdots g_N(t_N) dt \geqslant \int_{\mathbb{R}^N} F(t) g_1^*(t_1) \cdots g_N^*(t_N) dt.$$

Proof: *Previous case* +

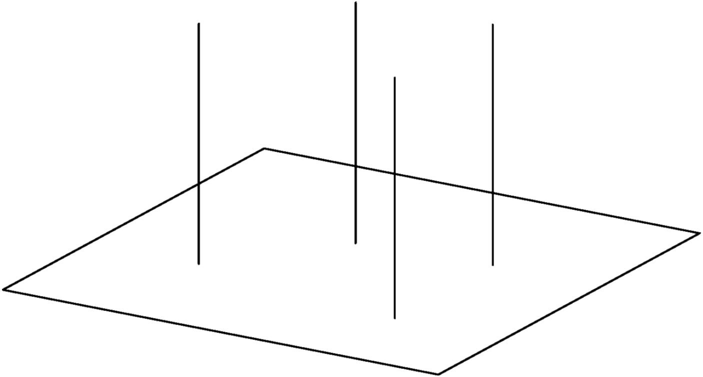
$$\mathbb{1}_{\{F \le s\}} = 1 - \mathbb{1}_{\{F > s\}}.$$

Groemer's Convexity Condition

$F : \otimes_{i=1}^{N}\mathbb{R}^n \to \mathbb{R}^+$ satisfies (**GCC**) if for every $z \in \mathbb{R}^n$ and for every $y_1 \dots, y_N \in z^\perp$ the function $F_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$F_Y(t) = F(y_1 + t_1 z, \dots, y_N + t_N z)$$

is even and convex.



Proposition

Suppose

- $F : \otimes_{i=1}^{N} \mathbb{R}^n \to \mathbb{R}^+$ *satisfies* (**GCC**);
- $f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}^+$ *integrable*

Then

$$\mathcal{F}_F(f_1, \dots, f_N) \geqslant \mathcal{F}_F(f_1^*, \dots, f_N^*).$$

Proof: For each $\theta \in S^{n-1}$,

$$\mathcal{F}_F(f_1, \dots, f_N) \geqslant \mathcal{F}_F(f_1^*(\cdot|\theta), \dots, f_N^*(\cdot|\theta)).$$

Approx f^* by successive $f^*(\cdot|\theta)$.

- [Christ, 1984]
- See also [Baernstein & Loss, 1997]

Verifying (**GCC**)

- Fix $\theta \in S^{n-1}$, $y_1, \ldots, y_N \in \theta^{\perp}$.
- Write $[x_1 \ldots x_N] = [x_i]$.

For $s, t \in \mathbb{R}^N$,

$$\mathrm{vol}\left(\left[y_i + \frac{s_i + t_i}{2}\theta\right] C\right) \leq 1/2\, \mathrm{vol}\left([y_i + s_i\theta]\, C\right) + 1/2\, \mathrm{vol}\left([y_i + t_i\theta]\, C\right).$$

- For $P = P_{\theta^{\perp}}$,

$$P[y_i + s_i\theta]C = P[y_i + t_i\theta]C.$$

From rotational invariance to the ball

Standard Lemma:

Lemma

Suppose

- $f : \mathbb{R}^n \to [0,1]$, $\int_{\mathbb{R}^n} f(x)dx = 1$.
- $g = \mathbb{1}_{D_n}$
- $\phi : \mathbb{R}^n \to \mathbb{R}^+$ *s.t.*
  - $\phi(x) = \phi(y)$ *whenever* $|x| = |y|$
  - $\phi(x) \leq \phi(y)$ *whenever* $|x| \leq |y|$

Then

$$\int_{\mathbb{R}^n} \phi(x) f^*(x)dx \geq \int_{\mathbb{R}^n} \phi(x) g(x)dx.$$

For $i = 1, \dots, N$, write

$$x_i = r_i \theta_i \qquad\qquad (0 \leq r_i < \infty,\ \theta_i \in S^{n-1})$$

Then

$$\mathrm{vol}\left([r_1\theta_1 \dots r_N\theta_N] C\right) = \mathrm{vol}\left([\theta_1 \dots \theta_N] \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_N \end{bmatrix} C \right)$$

Key Observation:

$$r_j \mapsto \int_{S^{n-1}} F(x_1, \dots, x_{j-1}, r_j\theta_j, x_{j+1}, \dots, x_N) d\sigma(\theta_j)$$

is increasing.

Averaging trick

Lemma

Suppose

- $\rho : \mathbb{R}^n \to \mathbb{R}^+$ *is such that*

$$\mathbb{R} \ni s \mapsto \rho(sx)$$

  is convex for each $x \in \mathbb{R}^n$;
- X *symmetric random vector with values in* $\mathbb{R}^n$.

Then

$$\mathbb{R}^+ \ni s \mapsto \mathbb{E}\rho(sX)$$

is an increasing function.

Lemma

Suppose $F : \otimes_{i=1}^{N} \mathbb{R}^n \to \mathbb{R}^+$ *satisfies* (**GCC**)*. Then for any* $x_1, \dots, x_N \in \mathbb{R}^n$ *and any* $1 \leqslant j \leqslant N$*,*

$$\mathbb{R} \ni s \mapsto F(x_1, \dots, sx_j, \dots, x_N)$$

is convex.