How often is a random quantum state \( k \)-entangled?

Stanislaw Szarek
Elisabeth Werner
Karol Zyczkowski

May 19, 2011
Goal

Determine the “size” of certain convex sets that appear naturally in quantum information theory.

Some notation

\( M_d = B(C^d) \) space of \( d \times d \)-matrices with complex entries.

Let \( \Phi : M_d \rightarrow M_d \) be linear.

• \( \Phi \) is positive if \( \Phi(\rho) \geq 0 \) for all \( \rho \in M_d, \rho \geq 0 \), i.e. positive semidefinite.

• Let \( 1 \leq k \leq d \). \( \Phi \) is called \( k \)-positive if \( \Phi \otimes I_k : M_d \otimes M_k \rightarrow M_d \otimes M_k \) is positive.

\( P_k(M_d) = \) set of \( k \)-positive maps on \( M_d \)

This set is a positive cone.
Goal
Determine the “size” of certain convex sets that appear naturally in quantum information theory.

Some notation
\[ \mathcal{M}_d = \mathcal{B}(\mathbb{C}^d) \] space of \( d \times d \)-matrices with complex entries.

Let \( \Phi : \mathcal{M}_d \to \mathcal{M}_d \) be linear.

- \( \Phi \) is positive if \( \Phi(\rho) \geq 0 \) for all \( \rho \in \mathcal{M}_d, \rho \geq 0 \), i.e. positive semidefinite

Stanislaw Szarek  Elisabeth Werner  Karol Zyczkowski
How often is a random quantum state \( k \)-entangled?
Goal

Determine the “size” of certain convex sets that appear naturally in quantum information theory.

Some notation

\( \mathcal{M}_d = \mathcal{B}(\mathbb{C}^d) \) space of \( d \times d \)-matrices with complex entries.

Let \( \Phi : \mathcal{M}_d \to \mathcal{M}_d \) be linear.

- \( \Phi \) is positive if \( \Phi(\rho) \geq 0 \) for all \( \rho \in \mathcal{M}_d, \rho \geq 0 \), i.e. positive semidefinite

- Let \( 1 \leq k \leq d \). \( \Phi \) is called \( k \)-positive if

  \[
  \Phi \otimes I_k : \mathcal{M}_d \otimes \mathcal{M}_k \to \mathcal{M}_d \otimes \mathcal{M}_k
  \]

  is positive.
Goal

Determine the “size” of certain convex sets that appear naturally in quantum information theory.

Some notation

\( \mathcal{M}_d = \mathcal{B}(\mathbb{C}^d) \) space of \( d \times d \)-matrices with complex entries.

Let \( \Phi : \mathcal{M}_d \to \mathcal{M}_d \) be linear.

- \( \Phi \) is positive if \( \Phi(\rho) \geq 0 \) for all \( \rho \in \mathcal{M}_d, \rho \succeq 0 \), i.e. positive semidefinite
- Let \( 1 \leq k \leq d \). \( \Phi \) is called \( k \)-positive if
  \[
  \Phi \otimes I_k : \mathcal{M}_d \otimes \mathcal{M}_k \to \mathcal{M}_d \otimes \mathcal{M}_k
  \]
  is positive.

\( \mathcal{P}_k(\mathcal{M}_d) = \) set of \( k \)-positive maps on \( \mathcal{M}_d \)

This set is a positive cone.
The **Jamiolkowski-Choi isomorphism** lets us relate maps $\Phi$ and matrices $C_\Phi$: 

$$\Phi \leftrightarrow C_\Phi$$
The Jamiolkowski-Choi isomorphism lets us relate maps $\Phi$ and matrices $C_\Phi$:

$$\Phi \longleftrightarrow C_\Phi$$

$$C_\Phi = \begin{pmatrix}
\Phi(E_{11}) \ldots \Phi(E_{1d}) \\
\ldots \\
\Phi(E_{d1}) \ldots \Phi(E_{dd})
\end{pmatrix}$$
The **Jamiolkowski-Choi isomorphism** lets us relate maps $\Phi$ and matrices $C_\Phi$:

$$\Phi \longleftrightarrow C_\Phi$$

$$C_\Phi = \begin{pmatrix} 
\Phi(E_{11}) & \ldots & \Phi(E_{1d}) \\
\ldots & \ldots & \ldots \\
\Phi(E_{d1}) & \ldots & \Phi(E_{dd}) 
\end{pmatrix}$$

f.i. via Jamiolkowski-Choi isomorphism

$$\mathcal{P}_k(\mathcal{M}_d) \longleftrightarrow \mathcal{B}\mathcal{P}_k(\mathbb{C}^d \otimes \mathbb{C}^d),$$

the space of $k$-block positive $d^2 \times d^2$ matrices.
The set of \textit{k-entangled} operators on $\mathbb{C}^d \otimes \mathbb{C}^d$ is

\[
\text{Ent}_k(\mathbb{C}^d \otimes \mathbb{C}^d) = \text{conv} \left( \left\{ \xi \langle \xi | : \xi = \sum_{j=1}^{k} u_j \otimes v_j, u_j, v_j \in \mathbb{C}^d, j = 1, \ldots, k \right\} \right)
\]
The set of $k$-entangled operators on $\mathbb{C}^d \otimes \mathbb{C}^d$ is

$$Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) = \text{conv} \left( \left\{ |\xi\rangle\langle\xi| : \xi = \sum_{j=1}^{k} u_j \otimes v_j, u_j, v_j \in \mathbb{C}^d, j = 1, \ldots, k \right\} \right)$$

$\xi = \sum_{j=1}^{k} u_j \otimes v_j \in \mathbb{C}^d \otimes \mathbb{C}^d$ is called a $k$-entangled vector, i.e. $k$-entangled states have rank $\leq k$. 
\( k = 1 \): separable vector, separable state
- $k = 1$: separable vector, separable state

- $k = d$: $\text{Ent}_d(\mathbb{C}^d \otimes \mathbb{C}^d)$ is the set of all states on a bipartite system $\mathbb{C}^d \otimes \mathbb{C}^d$. 
• $k = 1$: separable vector, separable state

• $k = d$: $Ent_d(C^d \otimes C^d)$ is the set of all states on a bipartite system $C^d \otimes C^d$

• For all $k$: $Ent_k(C^d \otimes C^d) \subset Ent_{k+1}(C^d \otimes C^d)$
Via Jamiolkowski-Choi isomorphism \( k \)-entangled states on \( \mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2} \) are in correspondence with maps on \( \mathcal{M}_d \)

\[
SP_k(\mathcal{M}_d) \leftrightarrow \text{Ent}_k(\mathbb{C}^d \otimes \mathbb{C}^d)
\]

\[
\Phi \leftrightarrow C_\Phi
\]
Via Jamiolkowski-Choi isomorphism $k$-entangled states on $\mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2}$ are in correspondence with maps on $\mathcal{M}_d$

$$SP_k(\mathcal{M}_d) \leftrightarrow Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d)$$

$$\Phi \leftrightarrow C_\Phi$$

$SP_k(\mathcal{M}_d)$ is the convex cone of $k$-superpositive operators $\Phi$ on $\mathcal{M}_d$:

$$\Phi(\rho) = \sum A_i^\dagger \rho A_i$$

such that each $A_i$ has rank $\leq k$. 
For $d = 3$

**a)** dual cones

**b)** compact, convex sets
Normalizations

- states $\rho \in \mathcal{M}_n$ get normalized such that $Tr(\rho) = 1$. 

Here:

$$\text{Ent}_k(C_d \otimes C_d) = \text{Ent}_k(C_d \otimes C_d) \cap \{ M \in B(C_d \otimes C_d) : Tr(M) = 1 \}$$

maps $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ get normalized such that $\Phi$ is trace preserving: for all states $\rho$

$$Tr(\Phi(\rho)) = Tr(\rho)$$
Normalizations

- states $\rho \in \mathcal{M}_n$ get normalized such that $Tr(\rho) = 1$.

Here:

$$Ent^1_k(\mathbb{C}^d \otimes \mathbb{C}^d) =$$

$$Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \left\{ M \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) = \mathcal{M}_{d^2} : Tr(M) = 1 \right\}$$
Normalizations

- states $\rho \in \mathcal{M}_n$ get normalized such that $Tr(\rho) = 1$.

Here:

$$Ent^1_k(\mathbb{C}^d \otimes \mathbb{C}^d) = Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) \cap \left\{ M \in B(\mathbb{C}^d \otimes \mathbb{C}^d) = \mathcal{M}_{d^2} : Tr(M) = 1 \right\}$$

- maps $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ get normalized such that $\Phi$ is trace preserving: for all states $\rho$

$$Tr(\Phi(\rho)) = Tr(\rho)$$
Here:

\[
\mathcal{S}\mathcal{P}_{k}^{TR}(\mathcal{M}_{d}) = \\
\mathcal{S}\mathcal{P}_{k}(\mathcal{M}_{d}) \cap \{ \Phi : \mathcal{M}_{d} \rightarrow \mathcal{M}_{d} : \text{Tr}(\Phi(\rho)) = \text{Tr}(\rho), \forall \rho \}\n\]
Geometrically, this means that we intersect the cones with hyperplanes and get convex sets.
Geometrically, this means that we intersect the cones with hyperplanes and get convex sets.

For $d = 3$, we have:

a) dual cones

b) compact, convex sets
Measure “size” of a convex body $K$ via volume radius

$$v_{\text{rad}}(K) = \left( \frac{|K|}{|B_n^2|} \right)^{\frac{1}{n}}$$
Measure “size” of a convex body $K$ via \textit{volume radius}

$$v_{\text{rad}}(K) = \left( \frac{|K|}{|B^n_2|} \right)^{\frac{1}{n}}$$

and \textit{mean width}

$$w(K) = 2 \int_{S^{n-1}} h_K(u) du$$
Measure “size” of a convex body $K$ via **volume radius**

$$v_{\text{rad}}(K) = \left( \frac{|K|}{|B^n_2|} \right)^{\frac{1}{n}}$$

and **mean width**

$$w(K) = 2 \int_{S^{n-1}} h_K(u) du$$

**Urysohn inequality**

$$v_{\text{rad}}(K) \leq \frac{1}{2} w(K)$$
Upper Bound

\[ w\left(\text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) = w\left(\text{ext}\left(\text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right)\right) = w\left(\{|\xi\rangle\langle\xi| : \xi \text{ k entangled, } |\xi| = 1\}\right) \leq \sqrt{2} w\left(\{|\xi\rangle\langle\xi| : \xi \text{ k entangled, } |\xi| = 1\}\right) = \sqrt{2} w\left(\text{Ent}_v^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) \]
Upper Bound

\[ \begin{align*}
    w \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) &= w \left( \text{ext} \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \right) \\
    &= w \left( \{|\xi\rangle\langle\xi| : \xi \text{ k entangled, } |\xi| = 1 \} \right) \\
    &\leq \sqrt{2} w \left( \{\xi : \xi \text{ k entangled, } |\xi| = 1 \} \right)
\end{align*} \]
Upper Bound

\[ w \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) = w \left( \text{ext} \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \right) = w \left( \left\{ |\xi\rangle\langle\xi| : \xi \text{ } k \text{ entangled, } |\xi| = 1 \right\} \right) \leq \sqrt{2} w \left( \left\{ \xi : \xi \text{ } k \text{ entangled, } |\xi| = 1 \right\} \right) = \sqrt{2} w \left( \text{Ent}_k^v(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \]
$$w(K) = 2 \int_{S^{n-1}} \max_{x \in K} \langle x, u \rangle du$$

$$= \gamma_n \int_{\mathbb{R}^n} \max_{x \in K} \langle x, y \rangle d\mu_n(y)$$
\[ w(K) = 2 \int_{S^{n-1}} \max_{x \in K} \langle x, u \rangle du \]

\[ = \gamma_n \int_{\mathbb{R}^n} \max_{x \in K} \langle x, y \rangle d\mu_n(y) \]

Dudley

\[ \leq C \gamma_n \int_0^\infty \sqrt{\log N(K, \varepsilon)} d\varepsilon \]
\[ w(K) = 2 \int_{S^{n-1}} \max_{x \in K} \langle x, u \rangle du \]
\[ = \gamma_n \int_{\mathbb{R}^n} \max_{x \in K} \langle x, y \rangle d\mu_n(y) \]

Dudley

\[ \leq C \gamma_n \int_0^\infty \sqrt{\log N(K, \varepsilon)} d\varepsilon \]

\( N(K, \varepsilon) \) is the smallest \( N \) such that there are points \( x_1, \ldots, x_N \) s.t.

\[ K \subset \bigcup_{i=1}^N x_i + \varepsilon B_2^n \]
\[ w(K) = 2 \int_{S^{n-1}} \max_{x \in K} \langle x, u \rangle \, du \]

\[ = \gamma_n \int_{\mathbb{R}^n} \max_{x \in K} \langle x, y \rangle \, d\mu_n(y) \]

Dudley

\[ \leq C \gamma_n \int_0^\infty \sqrt{\log N(K, \varepsilon)} \, d\varepsilon \]

\( N(K, \varepsilon) \) is the smallest \( N \) such that there are points \( x_1, \ldots, x_N \) s.t.

\[ K \subset \bigcup_{i=1}^N x_i + \varepsilon B_2^n \]

\[ \gamma_n \sim \frac{1}{\sqrt{n}} \]
\[
\text{vrad} \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \leq \frac{1}{2} w \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \\
\leq \frac{1}{\sqrt{2}} w \left( \text{Ent}_k^\gamma(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \\
\leq C \gamma_{d^4} \int_0^\infty \sqrt{\log N(\text{Ent}_k^\gamma(\mathbb{C}^d \otimes \mathbb{C}^d), \varepsilon)} \, d\varepsilon
\]
\[ E_{\text{ent}}^v_k(\mathbb{C}^d \otimes \mathbb{C}^d) \leftrightarrow \quad G_{d,k} \times G_{d,k} \times S_{\text{HS}}(F, E) \]

\[ \tau = \sum_{j=1}^{k} t_j |u_j\rangle \langle v_j| \leftrightarrow (E, F, T) \]
$\text{Ent}_k^v(\mathbb{C}^d \otimes \mathbb{C}^d) \leftrightarrow G_{d,k} \times G_{d,k} \times S_{\text{HS}}(F, E)$

$$\tau = \sum_{j=1}^{k} t_j |u_j\rangle \langle v_j| \leftrightarrow (E, F, T)$$

$$E = E_\tau = \text{span}\{u_j : 1 \leq j \leq k\}$$

$$F = F_\tau = \text{span}\{v_j : 1 \leq j \leq k\}$$

and $T \in S_{\text{HS}}(F, E)$ such that

$$\tau = TP_F$$
Szarek

\[ N(G_{d,k}, \varepsilon) \leq \left( \frac{C}{\varepsilon} \right)^{4k(d-k)} \]

\[ N(S_{HS}(F, E), \varepsilon) \leq \left( \frac{\tilde{C}}{\varepsilon} \right)^{2k^2} \]
Szarek

\[ N(G_{d,k}, \varepsilon) \leq \left( \frac{C}{\varepsilon} \right)^{4k(d-k)} \]

\[ N(S_{\text{HS}}(F, E), \varepsilon) \leq \left( \frac{\tilde{C}}{\varepsilon} \right)^{2k^2} \]

\[ N(\text{Ent}_k^v(\mathbb{C}^d \otimes \mathbb{C}^d)) \leq \left[ \left( \frac{C}{\varepsilon} \right)^{4k(d-k)} \right]^2 \left( \frac{\tilde{C}}{\varepsilon} \right)^{2k^2} \]

\[ \leq \left( \frac{C'}{\varepsilon} \right)^{8kd} \]
\[ \text{vrad} \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \leq C \gamma d^4 \int_0^1 \sqrt{8kd} \sqrt{\log \left( \frac{C'}{\varepsilon} \right)} \frac{1}{2} d\varepsilon \]

\[ \leq C \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}} \]
\[
\text{vrad} \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \leq C \gamma d^4 \int_0^1 \sqrt{8kd} \sqrt{\log \left( \frac{C'}{\varepsilon} \right)} \frac{1}{2} \, d\varepsilon \\
\leq C \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}
\]

- For \( k = 1 \) (separable states) and \( k = d \) (all states) this gives the right order
\[
\text{vrad} \left( \text{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d) \right) \leq C \gamma d^4 \int_0^1 \sqrt{8kd} \sqrt{\log \left( \frac{C'}{\varepsilon} \right)^{\frac{1}{2}}} \, d\varepsilon
\]

\[
\leq C \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}
\]

- For \( k = 1 \) (separable states) and \( k = d \) (all states) this gives the right order

- Lower bound: \( C \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}} \)