Weighted $\ell_1$ Minimization: Stability, robustness, and some implications

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Collaboration

Joint work with:

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Outline

Part 1: Introduction and Overview

Part 2: Stability and Robustness of Weighted $\ell_1$ Minimization

Part 3: Experimental Results and Stylized Applications

Part 3: Some implications of the weighted $\ell_1$ result
Motivation

- We want to recover a $k$-sparse signal $x \in \mathbb{R}^N$.
- Given $n \ll N$ linear and noisy measurements $y = Ax + e$.
- If $A$ has the RIP with $\delta_{2k} < \sqrt{2} - 1$ or $\delta_{(a+1)k} < \frac{a-1}{a+1}, a > 1$,
- Suppose $k, n$ and $N$ are such that $\ell_1$-minimization fails to recover $x$, and we have prior information on the support of $x$.
- How do we incorporate this knowledge in the recovery algorithm while keeping the measurement process non-adaptive?
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**Definition: Restricted Isometry Property (RIP)**

The RIP constant $\delta_k$ is defined as the smallest constant such that $\forall x \in \Sigma_k^N$

$$
(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2,
$$
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Constrained $\ell_1$-minimization

- $\min_{u \in \mathbb{R}^N} \|u\|_1$ subject to $\|Au - y\|_2 \leq \|e\|_2$, $k \lesssim n/\log(N/n)$
- $\|x^* - x\|_2 \leq C_0\|e\|_2^2 + C_1 k^{-1/2}\|x - x_k\|_1$
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Failed recovery and prior information

- Eg. when $k > \hat{k} \approx n/\log(N/n)$
- Eg. indices 1, 3, and 6 are non-zero.
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- How do we incorporate this knowledge in the recovery algorithm while keeping the measurement process non-adaptive?
Signals with Prior Information

In many applications, it is possible to draw an estimate of the support of the signal, for example:

- Natural images have large DCT coefficients that are localized in the low frequency subbands.
- Video sequences are temporally correlated, resulting in a shared subset of their support.
- Other signals such as seismic data, ...
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But, the $\ell_1$ minimization formulation is non-adaptive, i.e., aside from sparsity, no prior information on $x$ is used in the recovery.
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Part 3: Some implications of the weighted $\ell_1$ result
Suppose that $x$ is a $k$-sparse signal supported on an unknown set $T_0$. Let $\tilde{T}$ be a known support estimate that is partially accurate. We want to:

1. Recover $x$ by incorporating $\tilde{T}$ in the recovery algorithm.
2. Obtain recovery guarantees based on the size and accuracy of $\tilde{T}$.

Our approach: weighted $\ell_1$ minimization.
Problem Setup

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Weighted $\ell_1$ Minimization

Given a set of measurements $y$, solve

$$\min_x \|x\|_{1,w} \text{ subject to } \|Ax - y\|_2 \leq \epsilon \quad \text{with} \quad w_i = \begin{cases} 1, & i \in \tilde{T}^c, \\ \omega, & i \in \tilde{T}. \end{cases}$$

where $0 \leq \omega \leq 1$ and $\|x\|_{1,w} := \sum_i w_i |x_i|$, $\|e\|_2^2 \leq \epsilon$. 

\[\text{Diagram showing intervals } T_0, \tilde{T}, T^c, \tilde{T} \cap T_0, \tilde{T} \cap T^c, T_0^c, \text{ and } w.\]
Contributions

- We adopt weighted $\ell_1$ minimization and derive stability and robustness guarantees for the recovery of a signal $x$ with partial support estimate $\tilde{T}$.

- We show that if at least 50% of $\tilde{T}$ is accurate, then weighted $\ell_1$ minimization guarantees recovery with
  - weaker RIP conditions
  - smaller recovery error bounds

- We demonstrate through extensive experiments that assigning weights $0 < \omega < 1$ on $\tilde{T}$ results in the best reconstruction performance, especially if $x$ is compressible.
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Related Work

- **Borries et al. '07**: empirically demonstrate that $x$ is recoverable with $s$ fewer measurements by setting $\omega = 0$ on a known subset of the support of size $s$.

- **Khajehnejad et al. '09**: find a class of signals $x$, defined by a probabilistic model on sparsity and by the weight vector, that can be recovered with high probability using weighted $\ell_1$ minimization.

- **Vaswani et al. '10**: propose weighted $\ell_1$ minimization with zero weights and find weaker sufficient recovery conditions in the noise-free case.

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Weighted $\ell_1$ Minimization

Find the vector $x$ from a set of measurements $y$ using the support estimate $\tilde{T}$ by solving

$$
\min_x \|x\|_{1,w} \text{ subject to } \|Ax - y\|_2 \leq \epsilon \quad \text{with} \quad w_i = \begin{cases} 
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where $0 \leq \omega \leq 1$ and $\|x\|_{1,w} := \sum_i w_i |x_i|$. 

Stability and Robustness

- Let \( x \) be in \( \mathbb{R}^N \) and let \( x_k \) be its best \( k \)-term approximation, supported on \( T_0 \).
- Let \( |\tilde{T}| = \rho k \) and define \( \alpha = \frac{|\tilde{T} \cap T_0|}{|\tilde{T}|} \), and \( 0 \leq \omega \leq 1 \).
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![Diagram showing sets and their intersections]
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**Theorem (Main Result)**

Suppose there exists an $a \in \frac{1}{k} \mathbb{Z}$, with $a \geq (1 - \alpha)\rho$, $a > 1$, and that $A$ satisfies

$$\delta_{ak} + a\gamma\delta_{(a+1)k} < a\gamma - 1.$$ 

Then the solution $x^*$ to the weighted $\ell_1$ problem obeys

$$\|x^* - x\|_2 \leq C'_0\epsilon + C'_1k^{-1/2}\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{\tilde{T}^c \cap T_0^c}\|_1\right).$$

$$\gamma = \frac{1}{(\omega+(1-\omega)\sqrt{1+\rho-2\alpha\rho})^2}$$
Sufficient Recovery Condition

It is sufficient to have:

\[ \delta(a+1)k < \hat{\delta}(\omega) := \frac{a-(\omega+(1-\omega)\sqrt{1+\rho-2\alpha\rho})^2}{a+(\omega+(1-\omega)\sqrt{1+\rho-2\alpha\rho})^2} \]

\[ \delta(a+1)k < \hat{\delta}(1) := \frac{a-1}{a+1} \]
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2. \( \delta(a+1)k < \hat{\delta}(1) := \frac{a-1}{a+1} \)

Take for example: \( \hat{\delta}(1) = 0.6667 \), and \( \omega = 0.5, \rho = 1 \),

- if \( \alpha = 0.7 \), then \( \hat{\delta}(\omega) = 0.7279 \).
- if \( \alpha = 0.3 \), then \( \hat{\delta}(\omega) = 0.6151 \).
Sufficient Recovery Condition

It is sufficient to have:

1. \( \tilde{\delta}(a+1)k < \hat{\delta}(\omega) := \frac{a-(\omega+(1-\omega)\sqrt{1+\rho-2\alpha\rho})^2}{a+(\omega+(1-\omega)\sqrt{1+\rho-2\alpha\rho})^2} \)

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Error Bound Constants

Measurement noise constant $C'_0$:

$$C'_0 = \frac{2 \left(1 + (\omega + (1 - \omega)\sqrt{1 + \rho - 2\alpha\rho}) / \sqrt{a}\right)}{\sqrt{1 - \delta_{(a+1)k}} - \frac{\omega + (1 - \omega)\sqrt{1 + \rho - 2\alpha\rho}}{\sqrt{a}} \sqrt{1 + \delta_{ak}}}$$

$$C_0 = \frac{2 \left(1 + 1/\sqrt{a}\right)}{\sqrt{1 - \delta_{(a+1)k}} - \frac{1}{\sqrt{a}} \sqrt{1 + \delta_{ak}}}$$
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Take for example: $C_0 = 5.6048$, and $\omega = 0.5$, $\rho = 1$,

- if $\alpha = 0.7$, then $C'_0 = 4.9178$.
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Error Bound Constants

Signal compressibility constant $C'_1$:

\[ C'_1 = \frac{2a^{-1/2} \left( \sqrt{1 - \delta_{(a+1)k}} + \sqrt{1 + \delta_{ak}} \right)}{\sqrt{1 - \delta_{(a+1)k}} - \frac{\omega(1-\omega)}{\sqrt{a}} \sqrt{1 + \delta_{ak}}} \]

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Part 3: Some implications of the weighted $\ell_1$ result
Recovery of Sparse Signals

- SNR averaged over 20 experiments for $k$-sparse signals $x$ with $k = 40$, and $N = 500$. 
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- The noise free case:

![Graphs showing SNR vs. number of measurements for different weights $\alpha$. The graphs demonstrate the impact of varying $\alpha$ on the recovery of sparse signals, with the noise free case highlighted.]
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Discussion

- Intermediate values of the weight $\omega \approx 0.5$ result in the highest SNR even when $\alpha < 0.5$.

- Recall the recovery error bound

$$\|x^* - x\|_2 \leq C'_0(\omega)\epsilon + C'_1(\omega)k^{-1/2} \left( \omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{\tilde{T}_c \cap T_0^c}\|_1 \right).$$

- As $\omega$ goes to zero,
  - the constant $C'_1(\omega)$ increases
  - the term $\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{\tilde{T}_c \cap T_0^c}\|_1$ decreases

- There exists $0 < \omega < 1$ that minimizes their product.
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Video Compressed Sensing Example

- A video sequence is a collection of images acquired at periodic instances in time.
- For each video frame $j$, collect $n_j$ CCD readings sampled randomly from the CCD array.
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![Image of video frames with corresponding weight maps]
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Video Compressed Sensing Results

- $n_0 = N/2$, $n_j = N/2.2$ for $j = 1, 2, \ldots$
Part 1: Introduction and Overview

Part 2: Stability and Robustness of Weighted $\ell_1$ Minimization

Part 3: Experimental Results and Stylized Applications

Part 3: Some implications of the weighted $\ell_1$ result
Some Implications

- Weighted $\ell_1$ minimization can recover less sparse signals than standard $\ell_1$ when enough prior information is available.
- We showed that the recovery is stable and robust.
- We also showed that if at least 50% of the support estimate is accurate, then the recovery is guaranteed with weaker RIP conditions and smaller error bounds.

Some questions:
- How/when can we find the support estimate $\tilde{T}$?
- Can we draw a more accurate $\tilde{T}$ after solving the weighted $\ell_1$ minimization problem?
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Work in Progress - Partial Support Recovery (1)

Let $x \in \mathbb{R}^N$ be $k$-sparse and suppose the measurement matrix $A$ is such that $\ell_1$ minimization cannot recover $x$.

- If for some $k_0 < k$, $A$ has $\delta_{(a+1)k_0} < \frac{a-1}{a+1}$
- And if $x$ decays such that there exists an $s_0 \leq k_0$ where

  $$|x(s_0)| \geq (\eta_0 + 1)\|x_{T_0^c}\|_1, \quad T_0 = \text{supp}(x|_{k_0})$$

- Then

  $$\text{supp}(x|_{s_0}) \subseteq \text{supp}(x_0^*|_{k_0}),$$

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- If $\alpha > 0.5$ and $\omega < 1$, then $s_1 \geq s_0$.
- Assuming $x$ decays according to weak $\ell_p$, the above condition requires $p \geq 3!$.
- More conditions on signal decay are required to ensure $s_1 > s_0$.
- The derived conditions are very pessimistic compared to the experimental results!
- But what if we keep iterating?
Let $x \in \mathbb{R}^N$ be $k$-sparse and suppose the measurement matrix $A$ is such that $\ell_1$ minimization cannot recover $x$.

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Iterative weighted $\ell_1$ algorithm (work in progress)

1. Solve an initial $\ell_1$ minimization problem to obtain a support estimate.
2. Solve weighted $\ell_1$ minimization with weight equal to 0.5 on the previous support estimate.
3. Obtain a new support estimate.
4. Solve weighted $\ell_1$ minimization with
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5. Iterate until convergence.
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Iterative weighted $\ell_1$ algorithm (work in progress)

1: **Input** $b = Ax$
2: **Output** $x^{(t)}$
3: **Initialize** $\hat{p} = 0.99, \hat{k} = n \log(N/n)/2, \omega_1 = 0.5, \omega_2 = 0,$
   $T_1 = \emptyset, T_2 = \emptyset, \Omega = \emptyset,$
   $l = 0, t = 0, s^{(0)} = 0, x^{(0)} = 0$
4: **while** $\|x^{(t)} - x^{(t-1)}\|_2 \leq Tol\|x^{t-1}\|_2$ **do**
5: $t = t + 1$
6: $W = \mathbf{1}$
7: $\Omega = \text{supp}(x^{(t-1)}|_{s^{(t-1)}})$
8: $T_2 = T_1 \cap \Omega$
9: $W_{T_1} = \omega_1, W_{T_2} = \omega_2$
10: $x^{(t)} = \arg\min_u \|u\|_{1,W}$ s.t. $Au = b$
11: $l = \min_{\Lambda} |\Lambda|$ s.t. $\|x^{(t)}_{\Lambda}\|_2 \geq \hat{p}\|x^{(t)}\|_2$
12: $s^{(t)} = \min\{l, \hat{k}\}$
13: $T_1 = \text{supp}(x^{(t)}|_{s^{(t)}})$
14: **end while**
Iterative weighted \( \ell_1 \) algorithm (work in progress)

\( N = 1000 \)
Iterative weighted $\ell_1$ algorithm (work in progress)

$N = 2000$
Conclusion

- It is not necessary to apply weights inversely proportional to the coefficient magnitude of the signal.
- Signal classes are very strict, experiments indicate more general classes are available.
- Consider compressible signals and noisy measurements.
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Thank you!

*Partial funding provided by NSERC DNOISE II CRD.*