Compressive Sensing –
The Best or the Worst of Two Worlds?

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Banff March, 7 2011
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(which does not mean that Ben endorses all my statements ...)

Research supported by DARPA and NSF
Parametric vs Nonparametric World

**Parametric world**
- "Superresolution"
- No discretization issues
- Not robust vs model mismatch
- Lack of quantitative theory
- Numerous assumptions

Example: MUSIC Algorithm

**Non-parametric world**
- Robust
- No discretization issues
- Computationally efficient
- Cannot detect "weak" targets
- Requires "metaknowledge"
- Limited Resolution

Example: Spectrogram
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Example: Sparse MIMO Radar
Example: Spectrogram
Caught between two worlds

**Parametric world:**
Maximum Likelihood

\[ \min f(y; x_1, \ldots, x_s) \]

**Non-Parametric world:**
Spectrogram + “Thresholding”

\[ A^*y \]
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- **Compressive Sensing**
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  - Superresolution
  - Can detect weak targets
  - Example: Sparse MIMO Radar
- $N_T$ transmit antennas, $N_R$ receive antennas
- Co-located antennas (monostatic setup)
- Coherent propagation scenario
- $k$-th antenna sends signal $s_k$ of bandwidth $B$ and period $T$
Assume we take $N_s$ samples of the received radar signal. Let $Z(t; \theta, r)$ denote the received $N_R \times N_s$ signal matrix from a unit-strength target at direction $\theta$ and range $r$. Then

$$Z(t; \theta, r) = a_R(\theta)a_T(\theta)S(t - \tau),$$

where $S$ is an $N_T \times N$ matrix whose rows contain the circularly delayed signals $s_k(t - \tau), t = 1, \ldots, N$; and $\tau = 2r/c$ with $c$ denoting the speed of light.

$a_T(\theta)$ and $a_R(\theta)$ are the transmit- and receive array manifolds, which for uniformly spaced linear arrays can be written as

$$a_R(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi d_R \sin \theta} \\ \vdots \\ e^{j2\pi d_R(N_R-1) \sin \theta} \end{bmatrix}, \quad a_T(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi d_T \sin \theta} \\ \vdots \\ e^{j2\pi d_T(N_T-1) \sin \theta} \end{bmatrix}$$

where $d_R$ and $d_T$ are the normalized spacings (distance divided by wavelength) between antenna elements.
We discretize range/azimuth domain with step-sizes $\Delta_r, \Delta_\theta$ and obtain a range/azimuth grid $(\theta_i, r_j), 1 \leq i \leq N_\theta, 1 \leq j \leq N_r$. Here, $N_r, N_\theta$ denote the number of grid points in each axis.
We construct the response matrix $A$, whose columns are the vectors $z(t; \theta_i, r_j) := \text{vec}\{Z(t; \theta_i, r_j)\}$. Each $z$ has length $N_R N$, hence $A$ is an $N_R N_s \times N_\theta N_r$ matrix.

Assume the radar scene consists of $s$ scatterers located on $s$ points of the $(\theta_i, r_j)$-grid. Let $x$ be the $N_\theta N_r \times 1$ vector, whose non-zero elements are the amplitudes of the scatterers. That means $x$ has $s$ non-zero elements (but we do not know their location!).

The received radar signal $y$ is now given by

$$y = Ax + v,$$

where $v$ is Gaussian noise with variance $\sigma$.

Note: Unless we use crude discretization we have $N_R N_s < N_\theta N_r$. Hence the system is underdetermined.
In presence of Doppler shift $f_d$, we need to replace $Z(t; \theta, r)$ by

$$Z(t; \theta, r, f_d) = a_R(\theta)a_T^T(\theta)S(t - \tau, f_d),$$

where the entries of $S$ are the circularly delayed and Doppler shifted signals $s_k(t - \tau)e^{j2\pi f_d t}$.

Discretizing the “Doppler domain” with $N_f$ grid points and setting up the response matrix $A$ analogously to before, we obtain the system of equations

$$y = Ax + v,$$

where $A$ is now an $N_R N_S \times N_\theta N_r N_f$ matrix. Thus the system is even more underdetermined than before.
Waveforms: $s_k$ is a periodic, continuous-time white-noise signal of duration $T$ seconds, filtered by an ideal lowpass filter with cutoff frequency $B$ Hertz.

Antennas: Let $d_T = \frac{N_R}{2}$, $d_R = \frac{1}{2}$ (or $d_T = \frac{1}{2} d_R = \frac{N_T}{2}$).

Discretization:

Azimuth is discretized as $\beta = n \Delta \beta$ where $\Delta \beta = \frac{2}{N_R N_T}$, $n = -\frac{N_R N_T}{2}, \ldots, \frac{N_R N_T - 1}{2}$ and $\beta = \sin \theta$.

Range is discretized as $\tau = m \Delta \tau$ where $\Delta \tau = \frac{1}{2B}$, $m = 0, \ldots, N_s - 1$.

Generic sparse scatterer model: Location of the $S$ scatterers is selected uniformly at random, amplitudes of scatterers have random phases.

**LASSO:**

$$\min_x \frac{1}{2} \|Ax - y\|_2^2 + \lambda \|x\|_1.$$
Theorem (no Doppler): [B.Friedlander, T.S].

Assume that \( \mathbf{x} \) is drawn from the generic \( S \)-sparse scatterer model with

\[
S \leq \frac{c_0 N_r N_R}{4 \log(N_r N_R N_T)}
\]  

(1)

for some constant \( c_0 > 0 \). Furthermore, suppose that

\[
\log^3(N_r N_R N_T) \leq N_s.
\]  

(2)

If

\[
\min_k |x_k| > 8\sigma \sqrt{2 \log N_r N_R N_T},
\]  

(3)

then with probability at least \( P \) the Lasso estimate computed with \( \lambda = 2 \sqrt{2 \log(N_r N_R N_T)} \) obeys

\[
\text{supp}(\hat{x}) = \text{supp}(x), \quad \text{and} \quad \frac{\|\hat{x} - x\|_2}{\|x\|_2} \leq \frac{3\sigma \sqrt{N_r N_R N_T}}{\|y\|_2}
\]
Probability $P$ is given by

$$P > (1 - p_1 - p_2)(1 - p_3)(1 - p_4 - O((N_r N_R N_T)^{-2\log^2})), $$

where

$$p_1 = \max \{ 2(N_r N_R N_T \sqrt{2\pi \log N_r N_R N_T})^{-1}, 4e^{-\frac{N_T}{2}(t^2/2 - t^3/3)} \}, $$

with $t = 2\sqrt{\frac{\log(N_r N_R N_T)}{N_s}}$,

$$p_2 = e^{-\frac{N_s(\frac{3}{2} - \sqrt{2})}{2}}, $$

$$p_3 = 2(N_s \sqrt{2\pi \log N_s})^{-1} + 2e^{-N_s N_T/(2\alpha)} + e^{-\frac{N_s \sqrt{3/2 - 1}}{2}}, $$

$$p_4 = 2(N_r N_R N_T)^{-1}(2\pi \log(N_r N_R N_T) + S(N_r N_R N_T)^{-1}). $$
Proof-sketch:

Proof is based on careful analysis of structure of $A$ and a theorem by Candes-Plan.

Key steps:
- Need bound on $\|A\|_{\text{op}}$.
- Need bound on coherence $\mu(A)$.

Difficulty: $A$ is a mixture of random and deterministic matrix.

Key tools:
- Under the right conditions $AA^*$ is a block-Toeplitz matrix with circulant blocks (but $A$ is not!)
- Incoherence of $S$ comes into play
- Use bounds for quadratic forms (a’la Wright-Hanson)
- Concentration of measure
- Exploit specific choice for transmit/receive antenna spacing
Optimality of estimates

- Bounds on norm and coherence are optimal (up to small constants and probability)

- Coherence: \( \mu(A) \leq 2 \sqrt{\frac{1}{N_s}} \log(N_r N_R N_T) \). Why does \( \mu(A) \) only scale with \( N_s \) and not with the number of rows, \( N_R N_s \)?
  Comes from “decoupling”: \( A_{\tau,\beta} = a_R \otimes (S_\tau a_T) \)

- What about constants? For instance the condition

  \[
  \min_k |x_k| > 8\sigma \sqrt{2 \log N_r N_R N_T}
  \]

  implies for typical real-world parameters \( (N_r = 1024, N_R = N_T = 8) \) that

  \[
  \frac{|x_k|^2}{\sigma^2} > 64 \times 22
  \]

  thus we need an SNR of 31dB per antenna! The constant 8 moves us from medium-SNR range (13dB) to high-SNR range (31dB)!

- Can reduce constant from 8 to \( 1 + \epsilon \) (but also reduces \( P \)).
We assumed that scatterers lie exactly on discretized grid.

**Gridding error:** pointed out and partially analyzed by Pezeshki, Calderbank et al., Rauhut et al., Herman-S.

Well known: Using ideal low-pass filter yields significant “leakage”, sparse signal turns into signal with $1/t$-decay.

In absence of Doppler effect, we can reduce the gridding error to a “nuisance” via raised-cosine filter (gives cubic decay) or Gevrey-class filter (subexponential decay).

But pulseshaping implies that entries of $s_k$ become correlated and thus coherence of $A$ increases. Hence number of resolvable targets decreases.

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• A little pulseshaping goes a long way in the Doppler-free case.
Recall: Want (sampled) transmission pulses $s_k$ to be non-localized and non-smooth, otherwise would get large $\mu(A)$. For instance Gaussian would be a bad choice!

Let $\pi(\tau, \omega)$ denote the time-frequency shift operator. To analyze gridding error look at $\langle \pi(\tau, \omega)s_k, \pi(t, f)s_k \rangle$ for $(t, f) \neq \Lambda$ where $\Lambda$ is the grid.
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Can write $s_k$ as

$$s_k = \sum_{k, l} c_{k,l} \pi(k \Delta \tau, l \Delta f) \varphi$$

where $\varphi$ is the pulse shaping function. Then

$$|\langle \pi(\tau, \omega)s_k, \pi(t, f)s_k \rangle| = |\sum_{k,l,k',l'} c_{k,l} c_{k',l'} \langle \pi(\tau - t, \omega - f)\varphi, \varphi \rangle|$$

For this expression to be small for non-grid values $(t, f)$, $\varphi$ must be very localized and very smooth – like a Gaussian.
A Conundrum

Fundamental problem in presence of Doppler:

- Reducing gridding error via pulsesshaping means larger coherence, which kills ability for resolution of close targets.
- Not using pulsesshaping means large gridding error, which kills ability for resolution of close targets.

**Conclusion:** Standard CS approach does not cut it. We need some form of adaptive sensing (DARPA project) or other modifications of CS.
Can we exploit MIMO to solve discretization problem?

**At the transmitter:**
Should we use “staggered transmission” across transmit antennas? I.e., send $s_k$ from $k$-th antenna with offset $\frac{k}{N_T} \frac{1}{2B}$? Should we send pulses with different time-frequency localization from different antennas? Use time-localized pulses (good sparsity in time) to detect delays, and frequency-localized pulses (good sparsity in frequency) to detect Doppler.

**At the receiver:** Recall block structure of $A$

$$A = \begin{bmatrix} 
B_1 \\
B_2 \\
\vdots \\
B_{N_R}
\end{bmatrix},$$

where the $B_i$ are block matrices of size $N_S \times N_\theta N_r$. Can we exploit receive antennas by using a different grid offset for each receive antenna, i.e., for each $B_i$?
Other possibilities?

Combine model-based recovery (a’la Baraniuk et al.) with non-convex $\ell_p$-minimization:

- Model: A scatterer manifests itself in form of a cluster with certain decay properties
- We do not want to exploit model to allow for less sparsity, but for higher resolution.
- Usually we cannot prove convergence of $\ell_p$-minimization for $p < 1$. But maybe in this model-based setting we can?
- Initial numerical simulations via reweighted $\ell_1$-minimization seem promising.
- Can we use polarized transmission signals? This opens up a new dimension, but would require (as first step) CS theory over the quaternions.
- But can we really achieve superresolution?
Idea: Clutter is stationary, while targets move. Stack each “target scene” as column vector in a huge matrix $M$. Can model clutter as low-rank matrix $L$ and targets as sparse matrix $S$, $M = L + S$. Separate targets and clutter via Robust PCA

$$\text{minimize} \quad \|L\|_* + \lambda \|S\|_1 \quad \text{subject to} \quad L + S = M.$$
No Free Lunch With Compressive Sensing

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