Functor Calculus and Operads

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1 Overview

The goal of this workshop was to bring together experts in the various forms of functor calculus, as well as the theory of operads, in order to discuss recent connections being made between these fields. The main areas covered were: (i) Goodwillie's homotopy calculus of functors, its structure, and relationship to the Whitehead Conjecture; (ii) embedding spaces as studied by the Goodwillie-Weiss manifold calculus and the actions of operads on such spaces; (iii) operads, especially the little-discs operads and Koszul duality; (iv) André-Quillen (co)homology, its applications, and its relation to homotopy calculus for functors of operadic algebras.

The workshop consisted of sixteen talks by researchers in these fields, with a focus on making connections between different topics and bringing to light some recent developments. Since the number of participants was limited by the fact that this was a half-workshop, we focused, for the most part, on inviting established researchers rather than graduate students and postdocs. The aim then was for new directions for future research to be explored.

2 Background

2.1 Homotopy Calculus

The homotopy calculus of functors was developed by Goodwillie in the series of papers [G1, G2, G3]. For a functor F of to/from topological spaces or spectra, this theory systematically provides a sequence of approximations to F satisfying higher-order excision conditions. This sequence is known as the *Taylor tower of* F in reference to the analogy with ordinary Taylor series. The layers in this tower (i.e. the fibres of the maps between successive approximations) take a particularly simple form. Each layer is described by a single spectrum $\partial_n F$ (the nth *derivative* of F) with an action of the symmetric group Σ_n, and the spectrum $\partial_n F$ can often by computed.

The key example for the homotopy calculus seems to be the identity functor on based spaces. The Taylor tower for the identity functor converges at nilpotent spaces and the derivatives, and hence the layers, have been calculated by Johnson [J]. Arone and Mahowald [AM] used these calculations to give an explicit description of the p-local Taylor tower of the identity evaluated at a sphere. Their work also revealed interesting connections with chromatic homotopy theory, and new calculations of unstable v_n-periodic homotopy groups.

One way to understand homotopy calculus is as a tool for interpolating between stable and unstable homotopy theory. The layers of the tower are stable in the sense that their values on a space X depend only on the suspension spectrum of X. Piecing together the layers to form the tower then amounts to building in unstable data. One of the long-term question addressed by the workshop is how to understand the information needed to construct the entire tower from the layers which are relatively easy to describe.

The work of Arone-Mahowald [AM] and Arone-Dwyer [AD] has been highly influential in linking the homotopy calculus, especially for the identity functor, to other areas in homotopy theory. One topic that this workshop looked at in detail is the connection with the Whitehead Conjecture proved by Kuhn in [K1].

2.2 Manifold calculus and spaces of embeddings

The manifold calculus of Goodwillie and Weiss [W1, GW] concerns functors on the category of open subsets of a fixed manifold M and was developed primarily as a tool for studying spaces of embeddings of M in some other manifold N. The Taylor tower in this context interpolates between the space of embeddings $\mathrm{Emb}(M, N)$ and the space of immersions $\mathrm{Imm}(M, N)$, which plays the role of the linear approximation to $\mathrm{Emb}(M, N)$. As in the case of the homotopy calculus the layers of the Taylor tower have a fairly simple form and for space of embeddings, these layers are related to the configuration spaces of points in N.

The question of convergence of the Taylor tower in the manifold calculus is an interesting one. The tower for the space $\mathrm{Emb}(M, N)$ converges when $\dim(N) - \dim(M) \geq 3$. Note that this excludes the case of classical knots (i.e. embeddings of S^1 in S^3) but is effective for knots in dimensions 4 and higher. Considerable success has been had in applying the manifold calculus to calculate the homology of spaces of knots in higher codimension (see [LTV, ALTV, ALV]). For the classical case of knots in S^3 the approach of Hatcher that exploits the structure of the group of diffeomorphisms preserving a knot turned out to be very fruitful. Based on this approach Budney introduced different operad actions [Bu1, Bu2] on the space of long knots in $\mathbb{R}^3$ and showed that these actions are free. This almost completely determined the homotopy type of spaces of knots in $\mathbb{R}^3$ and S^3.

2.3 Operads and Koszul duality

It has recently become apparent that the theory of operads plays a significant role in both the homotopy and manifold versions of the calculus of functors. The fact that the layers of the Taylor tower for embedding spaces are related to configuration spaces, makes it no surprise that there is a connection to the little disc operads. This connection was realized in a striking way by Sinha [S], who showed that the Taylor tower for the space of long knots in $\mathbb{R}^m$ could be identified with the Tot-tower for a cosimplicial space associated with the little m-discs operad $\mathcal{C}_m$. Sinha's models have been exploited in many ways. For example, Dwyer and Hess [DH] have shown that the knot spaces can be described as double-loopings of spaces of operad maps $\mathcal{C}_1 \to \mathcal{C}_m$.

It turns out that operads also play a significant role in the homotopy calculus. Ching showed in [C] that the derivatives (that is, the coefficient spectra that determine the layers in the Taylor tower) of the identity functor possess an operad structure and Arone and Ching [AC] showed that this operad naturally acts on the derivatives of other functors. A central role in this theory is played by Koszul duality for operads. In particular, the operad formed by the derivatives of the identity is Koszul dual to the cooperad formed by the derivatives of the functor $\Sigma^\infty \Omega^\infty$ from spectra to spectra. This duality is a topological version of the rational Koszul duality between the Lie and commutative operads that lies at the heart of Quillen's models for rational homotopy theory [Q2].

2.4 André-Quillen homology and algebras over operads

The ideas of the calculus of functors are remarkably general and can easily be adapted to new contexts. For example, one can naturally describe analogues of the Taylor tower for functors on other categories. One of the most fruitful such generalizations is to functors of commutative S-algebras (that is, E_∞-ring spectra). Kuhn [K3] has used the Taylor tower of the identity functor on commutative S-algebras to study the Morava K-theory of infinite loop spaces. McCarthy, with various coauthors, has studied the Taylor tower of algebraic K-theory considered as a functor on the category of commutative S-algebras.

Recall that the homotopy calculus interpolates in some sense between stable and unstable homotopy theory. For commutative S-algebras, Basterra-Mandell [BM1] have shown that the notion of stable homotopy theory is essentially equivalent to topological André-Quillen homology (TAQ), that is, the derived functor of abelianization. It is therefore to be expected that a general theory of calculus for functors of commutative S-algebras will be closely related to TAQ and this relationship is clearly seen in the work of Kuhn and McCarthy.

Basterra-Mandell showed more generally that stabilization is given by Quillen homology for algebras over any suitable operad in the category of spectra (not just the commutative operad). This allows for a uniform approach to functor calculus for such categories of algebras. Given the extent to which various different operads already appear in the theory of calculus, this seems to be important.

3 Presentations on Current Work

Here we give an overview of how the talks presented in the workshop relate to the main areas of study.

Several talks focused on the theoretical side of the homotopy calculus. **Greg Arone** described, in two talks, work on the problem of reconstructing the Taylor tower of a functor from its derivatives. His first talk described the action of the operad ∂_I, formed by the derivatives of the identity on based spaces, on the derivatives of any functor to or from the category of based spaces. His second talk concerned new work, also joint with Ching, that aims to describe all the remaining structure on the derivatives of a functor beyond these operad actions. This new structure is a coaction by a certain cotriple and such a coaction is sufficient to recover the entire Taylor tower of the functor.

Eric Finster gave towers of approximations for the suspension spectrum for a homotopy limit. These towers are inspired by the homotopy calculus and, for example, yield spectral sequences for calculating the stable homotopy or homology of such a limit.

Brenda Johnson reported on joint work with Kristine Bauer, Rosona Eldred and Randy McCarthy. Previous work of Johnson and McCarthy has produced alternative models of the Taylor tower of a pointed functor using a cotriple associated to the cross-effect construction. These models are equivalent to Goodwillie's for analytic functors. Johnson spoke here about a generalization of this previous work to Taylor towers in an unbased setting.

It has recently been conjectured that the Taylor tower of the identity functor on based spaces can be used to give a new proof of the Whitehead Conjecture (proved by Kuhn in [K1]). Two different approaches to this were presented at the workshop. **Mark Behrens** gave two talks related to this. He described a new perspective on the calculations of Arone-Mahowald, and then showed how this perspective can be used to produce a contracting homotopy for the Whitehead sequence, thus giving a new of the Conjecture. This work is described in [Be2]. **Bill Dwyer** described joint work with Greg Arone and Kathryn Lesh on another way to approach this.

Ryan Budney gave two talks: the first introducing the manifold calculus and its relationship with pseudoisotopy theory. In his second talk, he presented various operad actions on embedding spaces [Bu1, Bu2], in particular he described a new operad formed from 'splicing diagrams' that plays a central role.

Michael Weiss reported on joint work with Rui Reis on Pontryagin classes for topological vector bundles controlled by the cohomology of the group of homeomorphisms of $\mathbb{R}^n$. This work uses ideas from both the manifold calculus and Weiss's orthogonal calculus [W2].

Operads appeared in many of the talks at the workshop but they were studied in particular in talks by **Benoit Fresse** and **Pascal Lambrechts**. Fresse described the space of homotopy-automorphisms of the little 2-discs operads. Lambrects presented joint work with Ismar Volic on formality for little N-discs operads.

The other main collection of talks were on topological André-Quillen homology. **Kathryn Hess** gave a series of two talks introducing this topic, focusing on the generality of Quillen homology for algebras over arbitrary operads. She concluded by describing joint work with John Harper [HH] on Whitehead and Hurewicz-type Theorems for Quillen homology.

Mike Mandell described the structure of Quillen homology for operadic algebras in greater detail, making the connection with Koszul duality. In particular the Quillen homology of a an algebra A should be a model for the Koszul dual coalgebra of A. He then described the structure of Postnikov towers for such algebra with an application to an E_4-structure on the Brown-Peterson spectrum.

Filling in for a last-minute withdrawal, **Dev Sinha** described recent detailed calculations in the cohomology of symmetric groups with relevance to operads.

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Abstracts

Speaker: **Gregory Arone** (Virginia)
Title: *Part 1: operads, modules and the chain rule*
Abstract: Let F be a homotopy functor between the categories of pointed topological spaces or spectra. By the work of Goodwillie, the derivatives of F form a symmetric sequence of spectra $\partial_* F$. This symmetric sequence determines the homogeneous layers in the Taylor tower of F, but not the extensions in the tower. In these two talks we will explore the following question: what natural structure does $\partial_* F$ possess, beyond being a symmetric sequence? Our ultimate goal is to describe a structure that is sufficient to recover the Taylor tower of F from the derivatives. Such a description could be considered an extension of Goodwillie's classification of homogeneous functors to a classification of Taylor towers.

By a theorem of Ching, the derivatives of the identity functor form an operad. In the first talk we will see that the derivatives of a general functor form a bimodule (or a right/left module, depending on the source and target categories of the functor) over this operad. Koszul duality for operads plays an interesting role in the proof. As an application we will show that the module structure on derivatives is exactly what one needs to write down a chain rule for the calculus of functors.

Title: *Part 2: beyond the module structure*
Abstract: The (bi)module structure on derivatives is not sufficient to recover the Taylor tower of a functor. In the second talk we will refine the structure as follows. We will see that there is a naturally defined comonad

on the category of (bi)-modules over the derivatives of the identity functor, and that the derivatives of a functor are a coalgebra over this comonad. From this coalgebra structure one can, in principle, reconstruct the Taylor tower of a functor. Thus this coalgebra structure seems to give a complete description of the structure possessed by the derivatives of a functor.

We give an explicit description (as explicit as permitted by our current understanding) of the structure possessed by the derivatives of functors from Spectra to Spectra, from Spaces to Spectra and from Spaces to Spaces. An interesting example is the functor $X \mapsto \Sigma^\infty\Omega^\infty(E \wedge X)$. Here E is a fixed spectrum, and the functor can be thought as a functor from either the category of Spaces or Spectra to the category of Spectra. The derivatives of this functor are given by the sequence $E, E^{\wedge 2}, \ldots, E^{\wedge n}, \ldots$. The fact that this sequence is the sequence of derivatives of a functor seems to tell us something about the structure possessed by spectra in general. In particular it tells us that spectra possess a natural structure of a restricted algebra over the Lie operad (an observation first made by Bill Dwyer).

Speaker: **Mark Behrens** (MIT)
Title: *Survey of the Goodwillie tower of the identity I*
Abstract: The Goodwillie tower of the identity functor from spaces to spaces is a powerful tool for understanding unstable homotopy from the stable point of view. I will describe the derivatives of this functor, which were studied by Johnson and Arone-Mahowald. I will also explain the Arone-Mahowald computation of the homology of the layers of the tower evaluated on spheres. These layers were shown by Arone-Dwyer to be equivalent to stunted versions of the $L(k)$ spectra studied by Kuhn, Mitchell, Priddy, and others.

Associated to the Goodwillie tower is a spectral sequence which computes the unstable homotopy groups of a space from the stable homotopy groups of the layers. I will explain consequences for unstable v_k-periodic homotopy discovered by Arone and Mahowald. I will also discuss relations with the EHP sequence at the prime 2. Specifically, differentials in the Goodwillie spectral sequence can often be computed in terms of Hopf invariants, and differentials in the EHP sequence can often be computed in terms of attaching maps in the $L(k)$-spectra.

Title: *Survey of the Goodwillie tower of the identity II*
Abstract: I will recall Ching's operad structure on the derivatives of the identity. Coupled with operadic structures recently discovered by Arone-Ching, this gives an action of a certain algebra of Dyer-Lashof-like operations on the layers of the Goodwillie tower of any functor from spaces to spaces. Specializing to the case of functors concentrated in degrees n and 2n, we will derive a formula for the homological behavior of the k-invariant of the associated Goodwillie tower.

The 2-primary Goodwillie tower of S^1 is closely related to the Whitehead conjecture (a.k.a. Kuhn's theorem). Work of Arone-Dwyer-Lesh shows that the k-invariants in the tower for S^1 deloop to give candidates for a contracting homotopy to Kuhn's Kahn-Priddy sequence, and conjectured this was indeed the case. The homological formulas for k-invariants will be used to compute these delooped k-invariants, verifying the conjecture (Arone-Dwyer-Lesh have simultaneously developed a different proof of their conjecture.)

Speaker: **Ryan Budney** (Victoria)
Title: *A primer on concordance or pseudoisotopy*
Abstract: I'll describe the context for concordance and pseudoisotopy, the Morlet Disjunction lemma, the pre-80's work on the subject. Tom's dissertation, and his work with Weiss and Klein on The Embedding Calculus, up to around Dev Sinha's early work on the embeddings.

Title: *Manifold embeddings and operads*
Abstract: This talk will be about "where we are and where we're going", describing recent work of Sinha, Salvatore, Turchin, Arone, Lambrechts, Dwyer and Hess on rational homotopy-type and iterated loop-space structures on embedding spaces. I'll also talk about other operads that act on embedding spaces and how they might fit into what is known.

Speaker: **Bill Dwyer** (Notre Dame)
Title: *Symmetric powers vs. the calculus filtration*
Abstract: The talk will discuss ideas which shed light on the relationship between the (increasing) filtration

of the Eilenberg-MacLane spectrum $H\mathbb{Z} = \mathrm{SP}^\infty(S^0)$ by the symmetric power spectra $\mathrm{SP}^n(S^0)$, and the (decreasing) filtration of the circle S^1 by the Goodwillie tower of the identity functor. Figuring in the picture is a peculiar object which is both simplicial and cosimplicial, and, in either guise, is both trivial and is not. Many Bredon (co)homology spectral sequences collapse, and the circle does double duty as the sphere S^1 and as the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$. Kuhn's Theorem (the Whitehead Conjecture) is in the background.

Speaker: **Eric Finster** (EPFL)
Title: *The Goodwillie tower for homotopy limits*
Abstract: In view of the many fruitful applications of the Goodwillie Calculus to the study of mapping spaces of various types, a natural question is the extent to which similar techniques can be applied to the study of their twisted cousins, that is, to spaces of sections. Homotopy theorists have two closely related convenient ways of modeling a space of this type: as the totalization of a cosimplicial space or as the homotopy limit of a diagram. These two models are essentially equivalent to each other in the sense that every homotopy limit can be realized as the total space of its cosimplicial replacement, and conversely, the total space of a fibrant cosimplicial space is equivalent to its homotopy limit when viewed as a diagram over Δ.

One advantage of a cosimplicial model, however, is that it comes with a canonical filtration, the totalization tower. Applying a generalized homology theory to the totalization tower yields the Bousfield-Kan spectral sequence, which under favorable conditions allows one to compute homological invariants of the associated total space. Viewed in another light, the Bousfield-Kan spectral sequence can be seen as a canonical filtration of the spectrum $\Sigma^\infty \operatorname{Tot} \mathcal{X}$ and we can discuss convergence in these terms: we say the Bousfield-Kan spectral sequence is convergent when the natural family of maps

$$\{\Sigma^\infty \operatorname{Tot}_n \mathcal{X}\}_{n\in\mathbb{N}} \longrightarrow \{\operatorname{Tot}_n \Sigma^\infty \mathcal{X}\}_{n\in\mathbb{N}}$$

constitutes a pro-equivalence of spectra from the constant tower at $\Sigma^\infty \operatorname{Tot} \mathcal{X}$ to the totalization tower of the cosimplicial spectrum $\Sigma^\infty \mathcal{X}$.

Homotopy limits, on the other hand, might be said to have a slight conceptual advantage in that they correspond nicely with our geometric intuition. Taking K to be a simplicial set, a diagram $F : \Delta K \to \mathcal{S}$ indexed by the simplex category of K can be thought of as a presheaf of spaces on K, and its homotopy limit as the space of global sections, which we will here denote by $\operatorname{holim}_K F$.

In this lecture, we present a stable filtration

$$\Sigma^\infty \operatorname{holim}_K F \to \operatorname{holim}_{P_n K} \Sigma^\infty F_n$$

which plays the role of the Goodwillie tower for homotopy limits, generalizing Arone's model for $\Sigma^\infty \operatorname{Map}(K, X)$ when X is the common value of a constant functor F. After outlining the basic construction, we discuss convergence issues, examine its relationship to the Bousfield-Kan spectral sequence, and provide examples.

Speaker: **Benoit Fresse** (Lille)
Title: *Homotopy automorphisms of* $\mathbf{E_2}$*-operads and Grothendieck-Teichmüller groups*
Abstract: The notion of E_2-operad usually refers to a topological operad which is weakly equivalent to Boardman-Vogt' operad of little 2-cubes C_2. This structure is used to model operations acting on 2-fold loop spaces. The structure of an E_2-operad is also used in simplicial and in differential graded algebra in order to model a first level of homotopy commutative structures.

In the algebraic setting, one considers the singular chain complex of the operad of little 2-cubes $Sing_\bullet(\mathsf{C}_2)$, which defines an operad in simplicial cocommutative coalgebras. For our purpose, we form a dual structure $Sing^\bullet(\mathsf{C}_2)$, which is a cooperad in cosimplicial commutative algebras. In what follows, we prefer to adopt the terminology of cosimplicial commutative Hopf cooperad (also commonly used in the literature) for the category of cooperads in cosimplicial commutative algebras and we use the notation $c\mathcal{H}opf\mathcal{O}p_1^c$ to refer to this category. The first objective of this talk is to explain that the category $c\mathcal{H}opf\mathcal{O}p_1^c$ gives a faithful algebraic model for the rational homotopy category of operads (assuming that $\mathbb{Q}$ is our ground ring). In the case of little 2-cubes, we will prove that the application of a space-like functor to a fibrant replacement of $Sing^\bullet(\mathsf{C}_2)$ yields a topological operad $(\mathsf{C}_2)^\wedge_\mathbb{Q}$ such that $\pi_*(\mathsf{C}_2)^\wedge_\mathbb{Q} = 0$ for $* > 1$ and $\pi_1(\mathsf{C}_2)^\wedge_\mathbb{Q}$ is the rational prounipotent completion (the Malcev completion) of the fundamental group of C_2.

Our main goal is to determine the homotopy of the space of homotopy automorphisms $\operatorname{haut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{C}_2)^\wedge_\mathbb{Q}$ associated to this topological operad $(\mathsf{C}_2)^\wedge_\mathbb{Q}$. Formally, we have a simplicial function space $\operatorname{Map}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P}, \mathsf{P})$

associated to any topological operad P. The space of homotopy automorphisms $\mathrm{haut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$ consists of the connected components of $\mathrm{Map}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P},\mathsf{P})$ which are attached to homotopy equivalences of operads $\phi : \mathsf{P} \xrightarrow{\sim} \mathsf{P}$. Basically, we can also identify $\pi_0(\mathrm{haut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P}))$ with the group of homotopy classes of these operad homotopy equivalences $\phi : \mathsf{P} \xrightarrow{\sim} \mathsf{P}$. Actually, the construction of $\mathrm{haut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$ makes sense if P is cofibrant as an operad only. Therefore, we tacitly apply the definition to a cofibrant replacement of P when the operad P is not itself cofibrant.

For basic reasons, we are mostly interested in a subspace $\mathrm{haut}^1_{\mathcal{T}op\mathcal{O}p}(\mathsf{C}_2)^\wedge_{\mathbb{Q}} \subset \mathrm{haut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{C}_2)^\wedge_{\mathbb{Q}}$, formed by restricting ourselves to the connected components of operad homotopy equivalences inducing the identity in homology. The operad $(\mathsf{C}_2)^\wedge_{\mathbb{Q}}$, which has no homotopy in degree $* \neq 1$, has a simple categorical model yielded by the Malcev completion of the operad of colored braid groupoids. The prounipotent version of the Grothendieck-Teichmüller group $GT(\mathbb{Q})$, defined by Drinfeld, can be identified with the group of automorphisms of this operad in groupoids. Accordingly, we can form a group morphism

$$\eta : GT(\mathbb{Q}) \to \pi_0(\mathrm{haut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{C}_2)^\wedge_{\mathbb{Q}}),$$

which is obviously injective (use fundamental groupoids). For our purpose, we consider a group $GT^1(\mathbb{Q}) \subset GT(\mathbb{Q})$, formed by removing a central factor $\mathbb{Q}^\times$ in $GT(\mathbb{Q})$. Our main result asserts:

Theorem. We have

$$\pi_*(\mathrm{haut}_{\mathcal{T}op\mathcal{O}p}((\mathsf{C}_2)^\wedge_{\mathbb{Q}}) = \begin{cases} GT^1(\mathbb{Q}), & \text{if } * = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Most of the talk will be devoted to the formulation of this theorem. The remaining time will be used to give an outline of the proof. In brief, our arguments rely on a study of the deformation complex of $Sing^\bullet(\mathsf{C}_2)$ in the category of cosimplicial Hopf cooperads.

Speaker: **Kathryn Hess** (EPFL)
Title: *What is André-Quillen (co)homology and why is it important?* Abstract: In this talk I will first describe classical André-Quillen homology, which was invented, essentially simultaneously, by Michel André and by Dan Quillen in the late 1960's. Quillen defined a very general homology theory, calculated as derived functors of abelienization, for objects in any category with all finite limits and enough projectives; special cases include both singular homology of spaces and (reduced) homology of groups. The Quillen homology of commutative rings is called *André-Quillen homology*.

I'll present Quillen's homology theory in the general setting and sketch proofs of its important elementary properties: the transitvity long exact sequence, invariance under base change and a sort of additivity. I'll then describe briefly the particular cases of commutative rings and of associative algebras.

In the late 1990's, Maria Basterra defined *topological André-Quillen cohomology (TAQ)* of commutative ring spectra and proved that homotopical analogues of the transitvity long exact sequence and invariance under base change held for this new theory. I will describe the construction of TAQ and explain why it is of interest to stable homotopy theorists. In particular, any connective, commutative ring spectrum admits a Postnikov tower decomposition, where the k-invariants correspond to classes in TAQ. Also, as shown by Basterra and Mike Mandell, TAQ is universal, in the sense that every cohomology theory of E_∞-ring spectra is TAQ with appropriate coefficients. Finally, as an aide to computation, I will state Mandell's theorem relating André-Quillen homology of E_∞-simplicial algebras and of commutative ring spectra to that of E_∞-differential graded algebras.

In the second part of this talk, I will briefly survey applications of André-Quillen (co)homology. In particular I will describe the role of André-Quillen homology in Haynes Miller's proof of the Sullivan conjecture. I will then sketch Bousfield's proof that André-Quillen homology is the natural home of obstructions to realizing a morphism $H_*(K;\mathbb{F}_p) \to H_*(L;\mathbb{F}_p)$ of cocommutative coalgebras with compatible Steenrod algebra structure as a map of spaces, as well as its generalization by Paul Goerss and Mike Hopkins, where $H_*(-;\mathbb{F}_p)$ is replaced by E_* for any commutative ring spectrum satisfying the Adams conditions and the commutative operad by any operad in spectra. If time permits, I will also talk about work of Natàlia Castellana, Juan Crespo and Jérôme Scherer on finite generation of mod p cohomology over the Steenrod algebra, in which André-Quille homology played an important role.

Title: *André-Quillen homology and towers*
Abstract: This second talk will be more closely related to the theme of this workshop, as I will describe

various towers related to (André-)Quillen homology that are, at the very least, highly analogous to the towers of Goodwillie calculus.

I will begin by presenting work of Nick Kuhn, who, for any commutative ring spectrum R, constructed a filtration of $\mathrm{TAQ}(R)$ such that the filtration quotients look just like the fibers of stages of a Goodwillie tower. The filtration gives rise, for any field $\mathbb{F}$, to a spectral sequence converging to $H^*\big(\mathrm{TAQ}(R);\mathbb{F}\big)$, which Kuhn applied to computing $\mathrm{TAQ}(R)$ for $R = \Sigma^\infty S^1_+, \Sigma^\infty \mathbb{Z}/2_+$, and $D(S^1_+)$, where D denotes the Spanier-Whitehead dual.

Let X be spectrum, and let $\mathbb{P}X$ denote the free commutative ring spectrum generated by X. To elucidate the relationship between $K(n)_*(\mathbb{P}X)$ and $K(n)_*(\Omega^\infty X)$, Kuhn constructed and studied the *André-Quillen tower* of an augmented, commutative R-algebra (in spectra) A, which is the Goodwillie tower of the identity functor on the category of R-algebras, evaluated on A. Its limit is essentially an I-adic completion of A, where I is the augmentation ideal of A. I will present Kuhn's results on properties of the André-Quillen tower, such as when it converges and its relation with localization.

I will conclude by talking about joint work with John Harper on a *homotopy completion tower* interpolating between Quillen homology and homotopy completion of algebras in symmetric spectra over a general operad $\mathcal{O}$, which generalizes Kuhn's André-Quillen tower. Our understanding of this homotopy completion tower enabled us to prove that, under reasonable conditions on the operad $\mathcal{O}$, if an $\mathcal{O}$-algebra map induces a weak equivalence on Quillen homology, then it also induces a weak equivalence on homotopy completions. Moreover, under mild connectivity conditions, an $\mathcal{O}$-algebra X is weakly equivalent to its homotopy completion, i.e., the homotopy completion tower for X converges to X. As almost immediate corollaries, we obtained Whitehead and Hurewicz theorems for Quillen homology. We have also established a Serre-type finiteness theorem for Quillen homology: under reasonable cofibrancy and connectivity conditions on the operad $\mathcal{O}$, if the homotopy groups of each level of the operad are finitely generated, then finiteness (respectively, finite generation) of the Quillen homology groups of an $\mathcal{O}$-algebra implies finiteness (respectively, finite generation) of its homotopy groups.

Speaker: **Brenda Johnson** (Union)
Title: *Cross-effects, cotriples and calculus: an update*
Abstract: In a series of papers published between 1998 and 2004, Randy McCarthy and I developed an analogue of Goodwillie's calculus of homotopy functors in an abelian setting using cross effects and cotriples. For functors $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ is a pointed category with finite coproducts and $\mathcal{B}$ is an abelian category, the $(n+1)$st cross effect $cr_{n+1}F$ is a symmetric functor of $n+1$ variables first used by Eilenberg and Mac Lane. The functor cr_{n+1} and the diagonal functor Δ^* form an adjoint pair between the categories of functors from $\mathcal{A}$ to $\mathcal{B}$ and functors from $\mathcal{A}^{\times n+1}$ to $\mathcal{B}$ which in turn yields a cotriple $\perp_{n+1} = \Delta^* \circ cr_{n+1}$ on the category of functors from $\mathcal{A}$ to $\mathcal{B}$. For $F : \mathcal{A} \to \mathcal{B}$ the cotriple produces an augmented simplicial object $\perp_{n+1}^{*+1} F \to F$ whose homotopy cofiber gives us a functor $\Gamma_n F := \mathrm{cofiber}(\perp_{n+1}^{*+1} F \to F)$. These $\Gamma_n F$ can be assembled into a tower of functors $\cdots \to \Gamma_{n+1}F \to \Gamma_n F \to \dots \Gamma_1 F \to \Gamma_0 F$. The nth term in this tower, $\Gamma_n F$, is a degree n functor, i.e., its $(n+1)$st cross effect is quasi-isomorphic to 0. This is a weaker condition than the n-excisive condition satisfied by the terms in Goodwillie's Taylor tower of a homotopy functor. In his thesis, Andrew Mauer-Oats extended this cotriple tower construction to endofunctors of the category of based spaces. He also proved that when F is a functor that commutes with geometric realization, $\Gamma_n F \simeq P_n F$, the nth term in Goodwillie's Taylor tower of the functor F.

In this talk I will discuss recent work with K. Bauer, R. Eldred, and R. McCarthy to extend these ideas to a more general context. We work with functors $F : \mathcal{C}_f \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are simplicial model categories, $f : A \to B$ is a morphism in $\mathcal{C}$, $\mathcal{C}_f$ is the full subcategory of objects $A \to X \to B$ that factor the morphism f, and $\mathcal{D}$ is stable. In this setting, the functors cr_{n+1} and Δ^* only form an adjoint pair up to homotopy. Mauer-Oats handled this problem for spaces by using combinatorial techniques to show directly that $\perp_{n+1}$ is a cotriple. In the current work, we identify two adjoint pairs whose composition yields $\perp_{n+1}$ as its associated cotriple. With this, the construction of the tower

$$\dots \Gamma^f_{n+1} F \to \Gamma^f_n F \to \dots \Gamma^f_1 F \to \Gamma^f_0 F$$

proceeds much as it did in the abelian case. The nth term is now a functor that is n-excisive relative to f, that is, it takes strongly cocartesian $(n+1)$-cubes of objects built by using the morphism f to cartesian diagrams. When F commutes with realizations, we generalize Mauer-Oats' results, showing that $\Gamma^f_n F$ is n-excisive and P_nF and $\Gamma_n^f F$ are weakly equivalent functors. More generally, for an arbitrary $F : \mathcal{C}_f \to \mathcal{D}$ (that does not necessarily commute with realizations) evaluation at the initial object A in $\mathcal{C}_f$ yields a weak equivalence $\Gamma_n^f F(A) \simeq P_nF(A)$.

Speaker: **Pascal Lambrechts** (Louvain-la-Neuve)
Title: *Formality of the little N-disks operad*
Abstract: This is joint work with Ismar Volic. The little N-disk operad is an important object in functor calculus, notably in the manifold calculus of embeddings of a manifold into a vector space. Since the work of Fred Cohen in the 1970's, the homology of this operad is well known to be a (generalized) Poisson operad. In the paper "Operads, motives and deformation quantization", M. Kontsevich stated and sketched the proof that this operad is formal. This means that there exists a zigzag of quasi-isomorphims of operads connecting the chains on the little disks operad with its homology (with coefficients in the field of real numbers). The special case of operads of little two-dimensional disks this has been proved by Tamarkin.

In this talk we will explain the proof of the formality of this operad. The main idea of the proof is to build a certain graph-complex and to show 1) that it is quasi-isomorphic to the chains of the little disks operad, thanks to a construction which is called the Kontsevich configuration integral; 2) that it is also quasi-isomorphic to its homology, which is proved by a simple algebraic argument. In order to properly interpret the integral we will replace the little disks operad by an equivalent one, namelythe operad of configuration spaces compactified la Fulton-MacPherson. We will recall some interesting geometry of this operad.

Actually the formality result is more precise than initially stated by Kontsevich: (i) formality holds in the category of CDGA's, which encode rational homotopy type; (ii) we consider that the little disk operad has an operation in arity 0; (iii) some relative version of the operad holds.

An important technical point in the proof is that we need to use a variant of deRham theory for semi-algebraic sets instead of smooth manifolds. We probably will not have the time to explain this subtelity. There is also a cyclic version of formality for the framed little 2-disks operad proved by J. Giansiracusa and P. Salvatore

Speaker: **Mike Mandell** (Indiana)
Title: *Quillen Cohomology of Operadic Algebras and Obstruction Theory*
Abstract: Quillen defined homology in terms of abelianization. For operadic algebras Quillen homology is the derived functor of indecomposibles, and the bar duality (or "derived Koszul duality") construction provides a model for the Quillen homology. The k-invariants of Postnikov towers for algebras over an operad $\mathcal{O}$ lie in the Quillen cohomology groups, and this leads to an obstruction theory for $\mathcal{O}$-algebra structures and $\mathcal{O}$-algebra maps. I will discuss my recent paper with Maria Basterra where we apply this obstruction theory to show that BP is an E_4 ring spectrum.

Speaker: **Michael Weiss** (Aberdeen)
Title: *Smooth maps to the plane and Pontryagin classes*
Abstract: This is joint work with Rui Reis.

The long-term goal is to prove that the Pontryagin classes in the $4i$-dimensional rational cohomology of $BTOP(n)$, due to Novikov and Thom, satisfy the standard relations that we expect from Pontryagin classes of vector bundles, for example: vanishing when $i > 2n$. Here $TOP(n)$ is the topological group of homeomorphisms from $\mathbb{R}^n$ to itself. It is not difficult to recast this as a problem in orthogonal calculus, for the functor $V \mapsto BTOP(V)$ from real vector spaces with inner product to based spaces. Throwing in smoothing theory as well, one can recast it as a problem about spaces of regular (nonsingular) smooth maps from $D^n \times D^2 \to D^2$, extending the standard projection on the boundary. Imitating the parametrized Morse theory of Cerf and K Igusa, we investigate these spaces of regular maps by thinking of them as being contained in spaces of smooth maps to the plane with only moderate singularities. Then we need homotopy formulae for spaces of smooth maps with only moderate singularities. This leads us to Vassiliev's discriminant method. We reformulate that as a method for proving theorems in manifold calculus (formerly "embedding calculus"). The reformulation gives us some generalizations, allowing us to impose good behavior conditions not just on singularities in the source, $D^n \times D^2$, but also on singularity sets in the target, D^2. That gives us a lot to play with. We hope it is enough.