Lattice codes for Gaussian relay channels

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Y. Song, N. Devroye, ``List decoding for nested lattices and applications to relay channels,'' Allerton 2010.
Y. Song, N. Devroye, ``Structured interference-mitigation in two-hop networks,'' ITA 2011.
Y. Song , N. Devroye, ``A lattice Compress-and-Forward strategy for canceling known interference in Gaussian multi-hop channels,'' CISS 2011.
Y. Song, N. Devroye, ``A Lattice Compress-and-Forward Scheme,'' ITW Paraty, 2011.

Y. Song , N. Devroye, ``Lattice codes for relay channels: DF and CF,''" IEEE Trans. on IT, in preparation, 2011.

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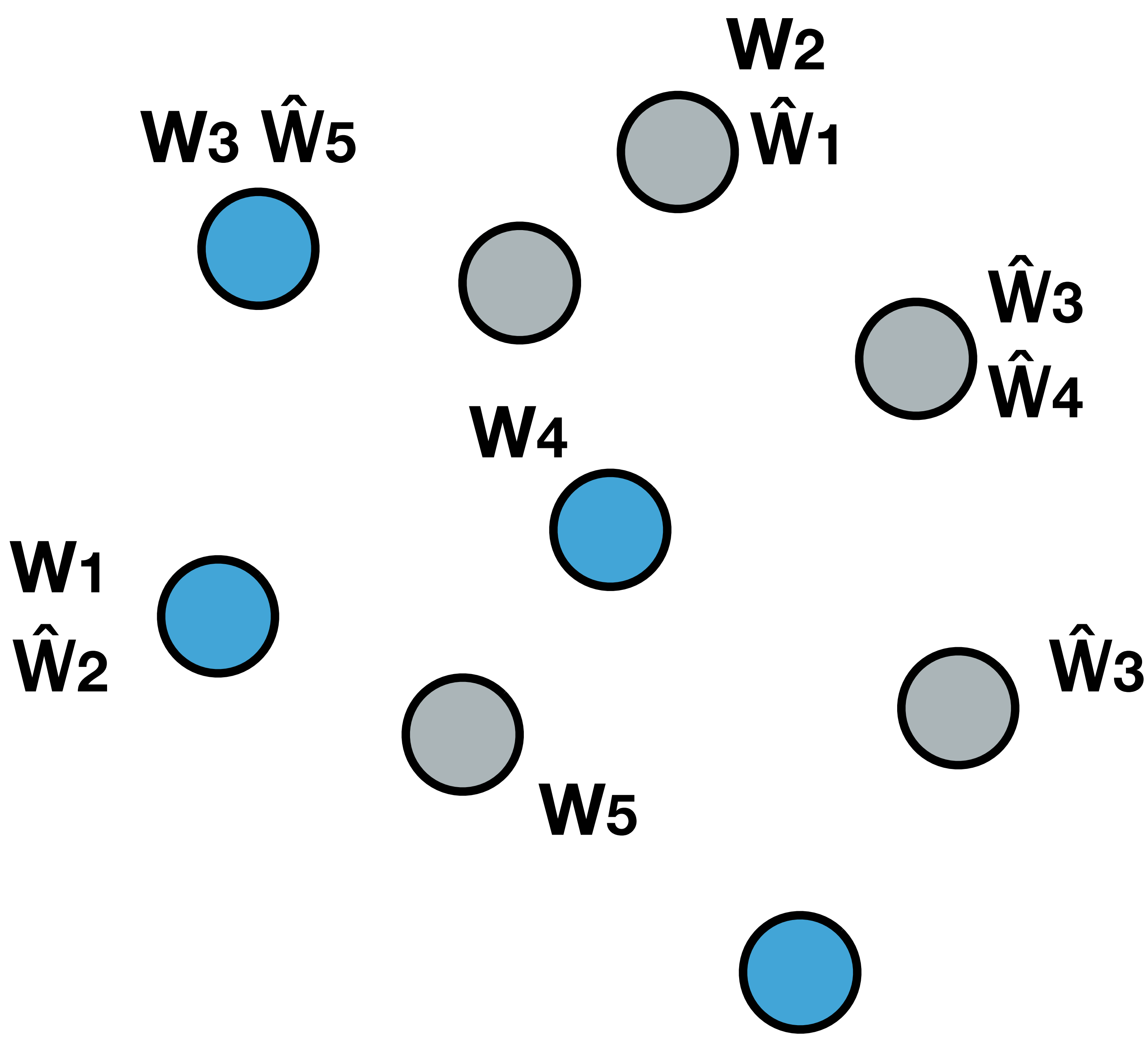
Structured codes for Gaussian networks



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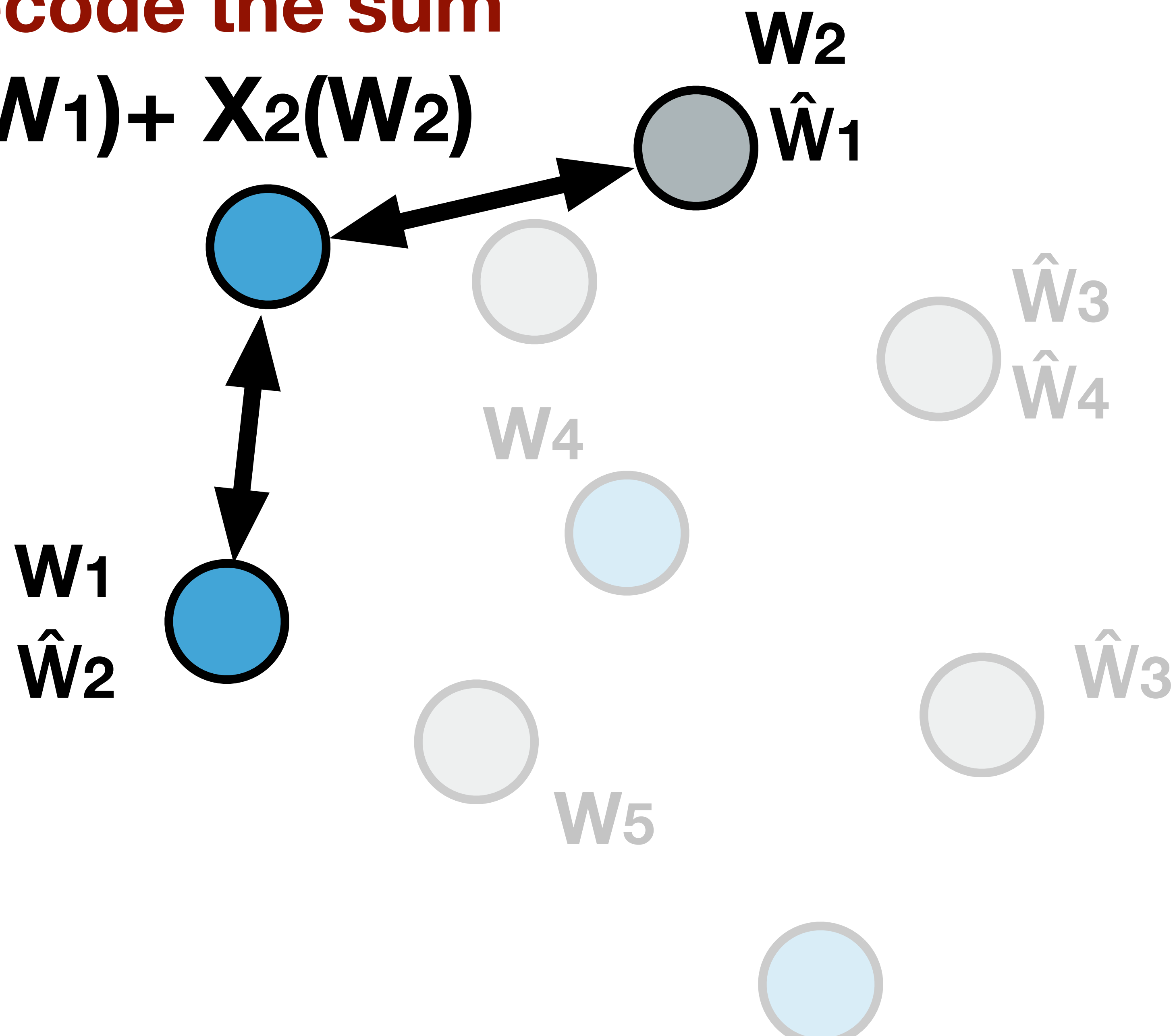
Structured codes for Gaussian networks

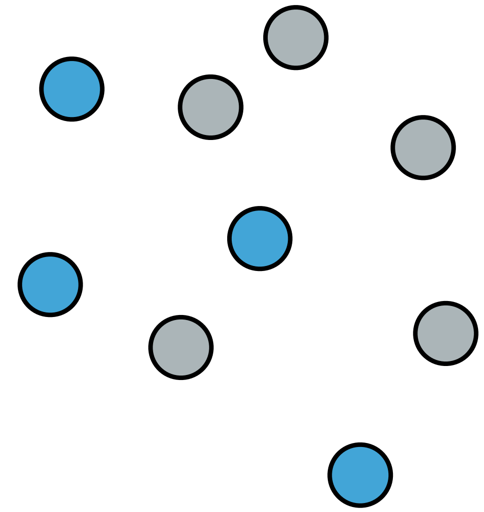
→ **decode the sum**

$$X_1(W_1) + X_2(W_2)$$



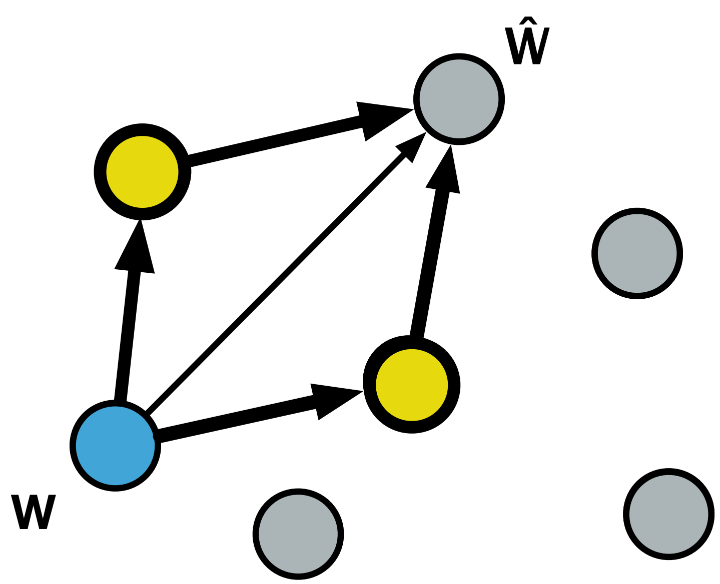
General "decode the sum" → Compute-and Forward

[Nazer, Gastpar, Trans IT, 2011]



Random codes for Gaussian networks

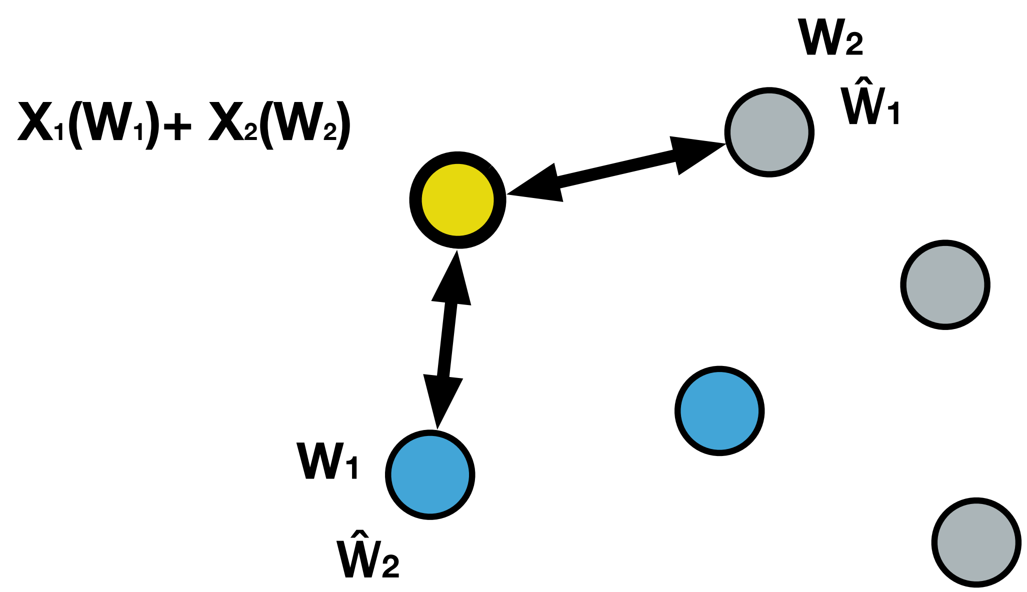
- **have:** cooperation



- **missing:** “decode the sum”

Structured codes for Gaussian networks

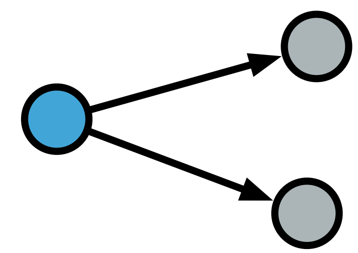
- **have:** “decode the sum”



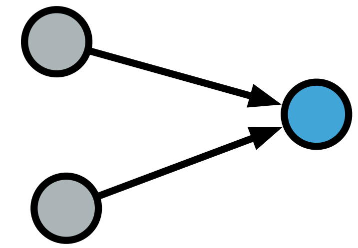
- **missing:** cooperation

Lattice codes for Gaussian relay networks?

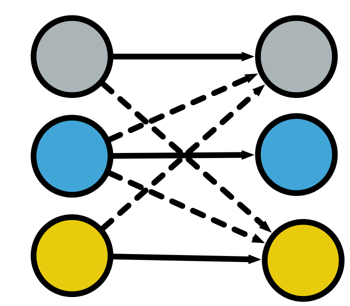
- demonstrated utility for **single-hop** networks:
  - AWGN channel *[Erez, Zamir, Trans. IT, 2004]*



  - AWGN broadcast channel *[Zamir, Shamai, Erez, Trans. IT, 2002]*



  - AWGN multiple-access *[Nazer, Gastpar, TransIT 2011]* and ``dirty" multiple-access channels *[Philosof, Khisti, Erez, Zamir, ISIT 2007]*
  - Distributed source coding *[Krithivasan, Pradhan, TransIT 2009]*

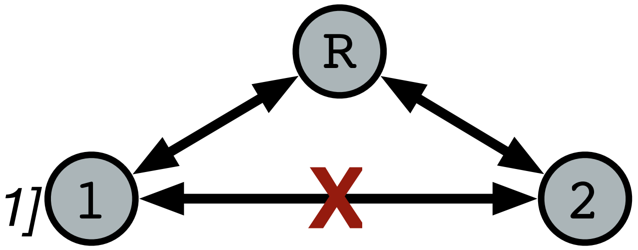


  - AWGN interference channel: interference decoding / interference alignment in K>2 interference channels *[Bresler, Parekh, Tse, TransIT, 2010] [Sridharan, Jafarian, Jafar, Shamai, arXiv 2008]*

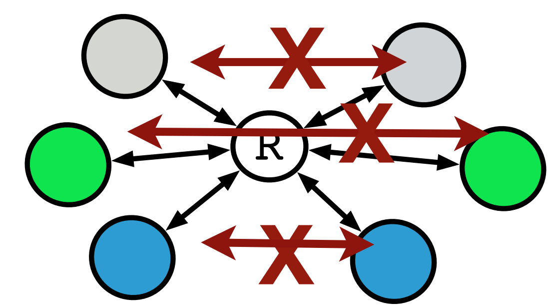
What about multi-hop networks?

Lattice codes for Gaussian relay networks?

- demonstrated utility for **specific two-hop** networks



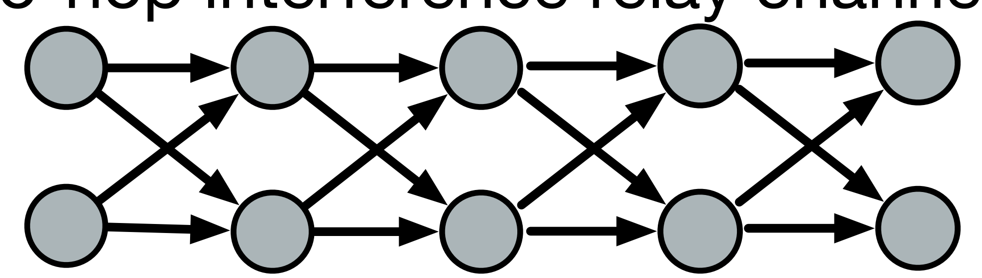
  - AWGN two-way relay channels *[Nazer, Gastpar, TransIT 2011]*
    [Wilson, Narayanan, Pfister, Sprintson, Trans. IT, 2010] [Nam, Chung, Lee, Trans. IT, 2010]



  - AWGN multi-way relay channels
    [Gunduz, Yener, Goldsmith, Poor, arXiv 2010], [Sezgin, Avestimehr, Khajehnejad, Hassibi, arXiv 2010][Kim, Smida, D, ISIT 2011]

- demonstrated utility for **specific multi-hop** networks

  - AWGN two-hop interference relay channel *[Mohajer, Diggavi, Fragouli, Tse, arxiv 2010]*



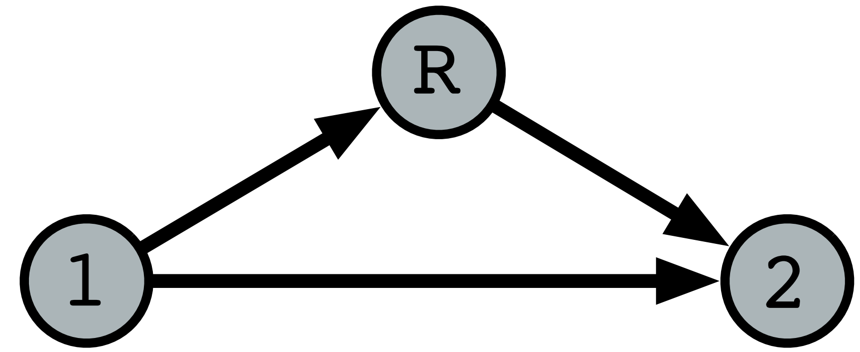
  - finite-field multi-hop interference relay channel *[Jeon, Chung, arxiv 2011]*

Are these techniques enough for general relay networks?

Missing ``cooperation" - combining of direct and relayed links

General relay network theorems

- AWGN relay channel DF and CF schemes first considered in *[Cover, El Gamal, 1979]*



  - DF extension to arbitrary # of relays and sources in *[Xie, Kumar, 2004]*
  - CF ... sources in *[Kr ... 010]*

All based on RANDOM coding

  - Quantize-and-map scheme for arbitrary # of relays and sources in *[Avestimehr, Diggavi, Tse, 2011] (finite gap)*
  - Noisy network coding *[Lim, Kim, El Gamal, Chung, 2010] (finite gap)*

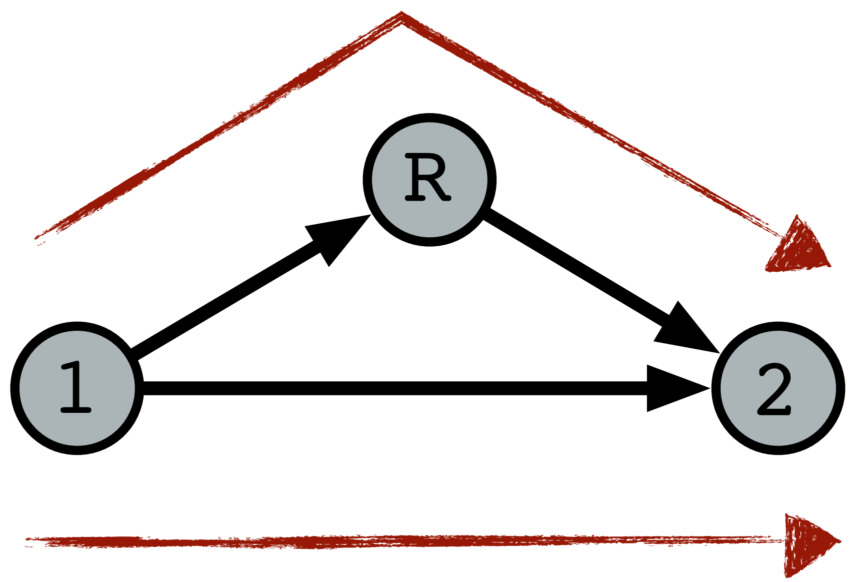
- Lattice-based schemes?
  - Quantize-and-map extended to lattice codes in *[Ozgur, Diggavi, 2011]*
  - Compute-and-forward framework *[Nazer, Gastpar, TransIT, 2011], [Niesen, Whiting, 2011]*

Lattice codes missing in?

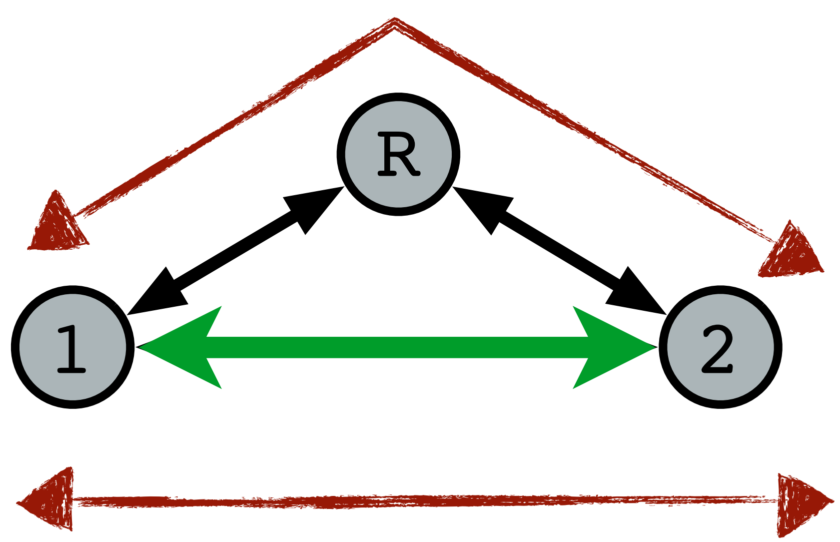
- AWGN relay channel ?

"Cooperation"

Various links carry same message!

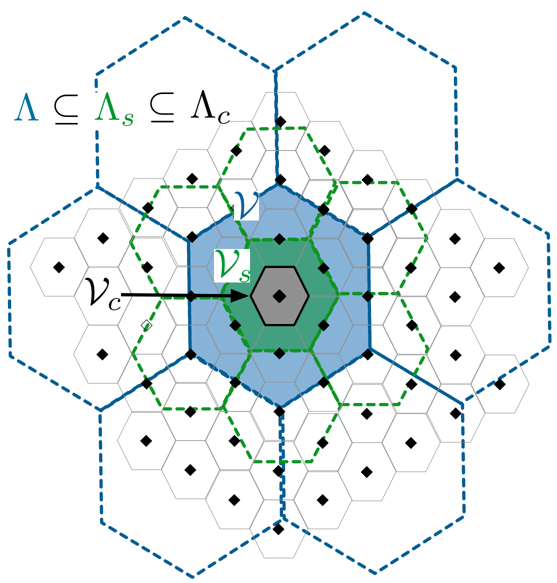


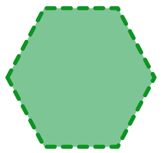
- Two-way relay channel in **presence of direct links?**



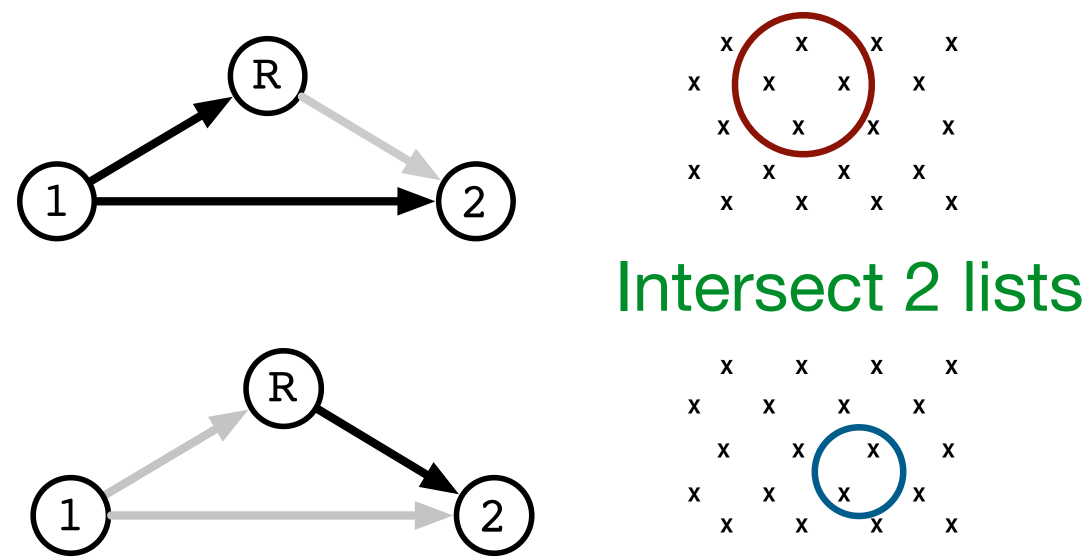
Enabling lattice "Cooperation"

Lattice list decoder

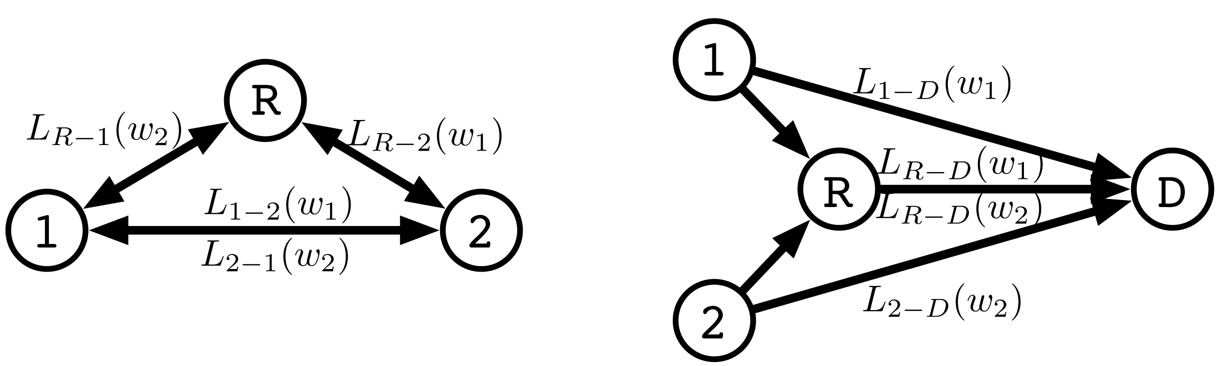


decode to 
rather than 

Lattices achieve DF rate

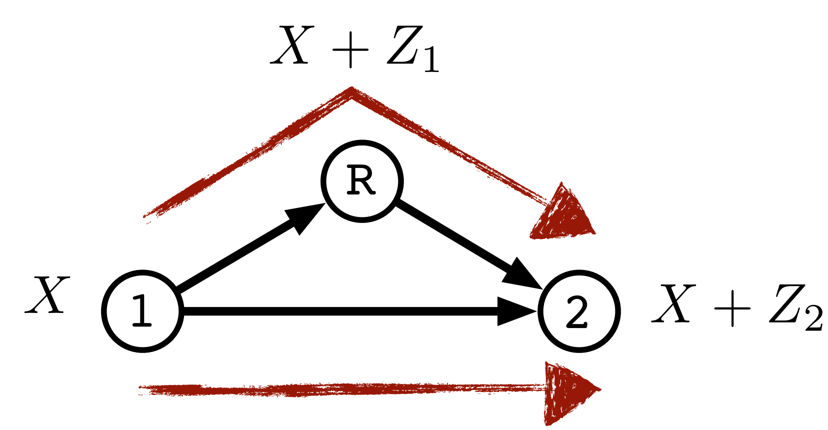


Lattices in multi-source networks



combine "decode sum"
and direct-link cooperation

Lattices achieve CF rate



lattices good source and channel
codes, special Wyner-Ziv

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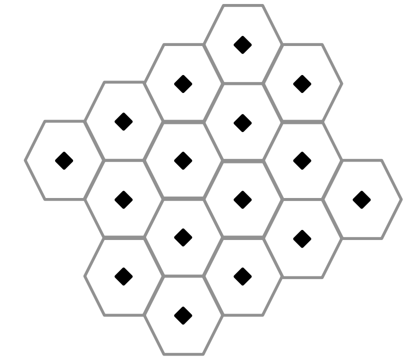
Outline - enabling *cooperation* via lattices

- Lattice notation
- Lattice list decoder
- Single source DF applications:
    - Lattices achieve DF rate for AWGN relay
    - Lattices achieve DF rate for AWGN multi-relay
- Multi-source DF applications:
    - Lattices for two-way relay channel with direct links
    - Lattices for multiple-access relay channel
- Lattices achieve CF rate for AWGN relay



Lattice notation

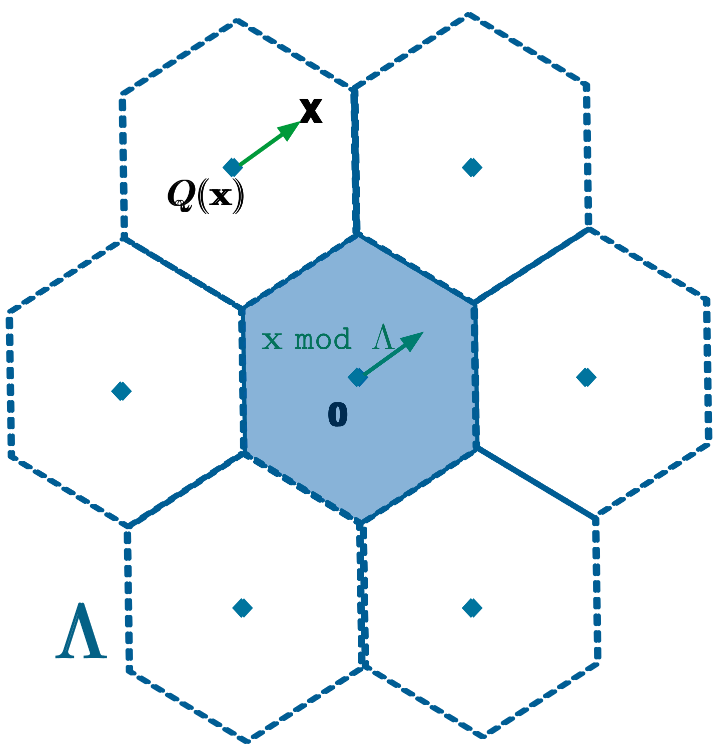
- $\Lambda = \{\lambda = G\,\mathbf{i} : \mathbf{i} \in \mathbb{Z}^n\}$, G the generator matrix

- *lattice quantizer* of Λ:

$$Q(\mathbf{X}) = \arg\min_{\lambda \in \Lambda} ||\mathbf{X} - \lambda||$$

- $\mathbf{x} \mod \Lambda := \mathbf{x} - Q(\mathbf{x})$

- *fundamental region* $\mathcal{V} := \{\mathbf{x} : Q(\mathbf{x}) = \mathbf{0}\}$ of volume V

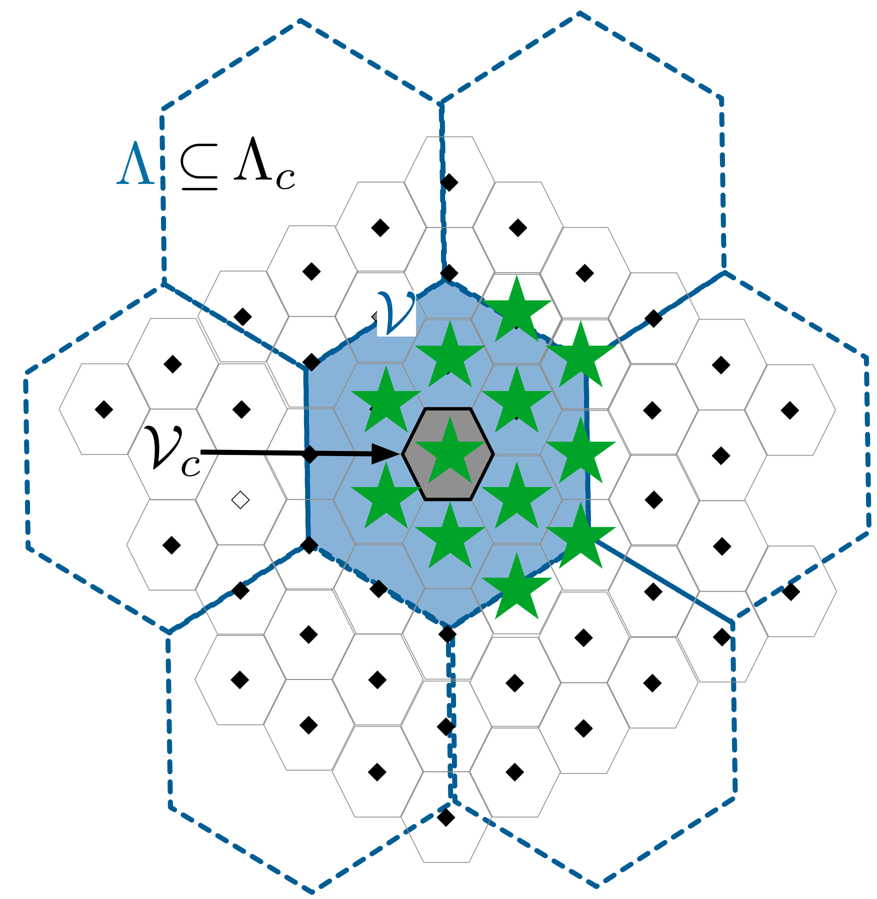


- *second moment per dimension of a uniform distribution over* $\mathcal{V}$:

$$\sigma^2(\Lambda) := \frac{1}{V} \cdot \frac{1}{n} \int_{\mathcal{V}} ||\mathbf{x}||^2 d\mathbf{x}$$

Nested lattice codes

- Nested lattice pair : $\Lambda \subseteq \Lambda_c$ (Λ is Rogers-good and Poltyrev-good, Λ_c is Poltyrev-good)



- The code book ★ $\mathcal{C} = \{\Lambda_c \cap \mathcal{V}(\Lambda)\}$ ★ is used to achieve the capacity of AWGN channel *[Erez+Zamir, Trans. IT, 2004]*

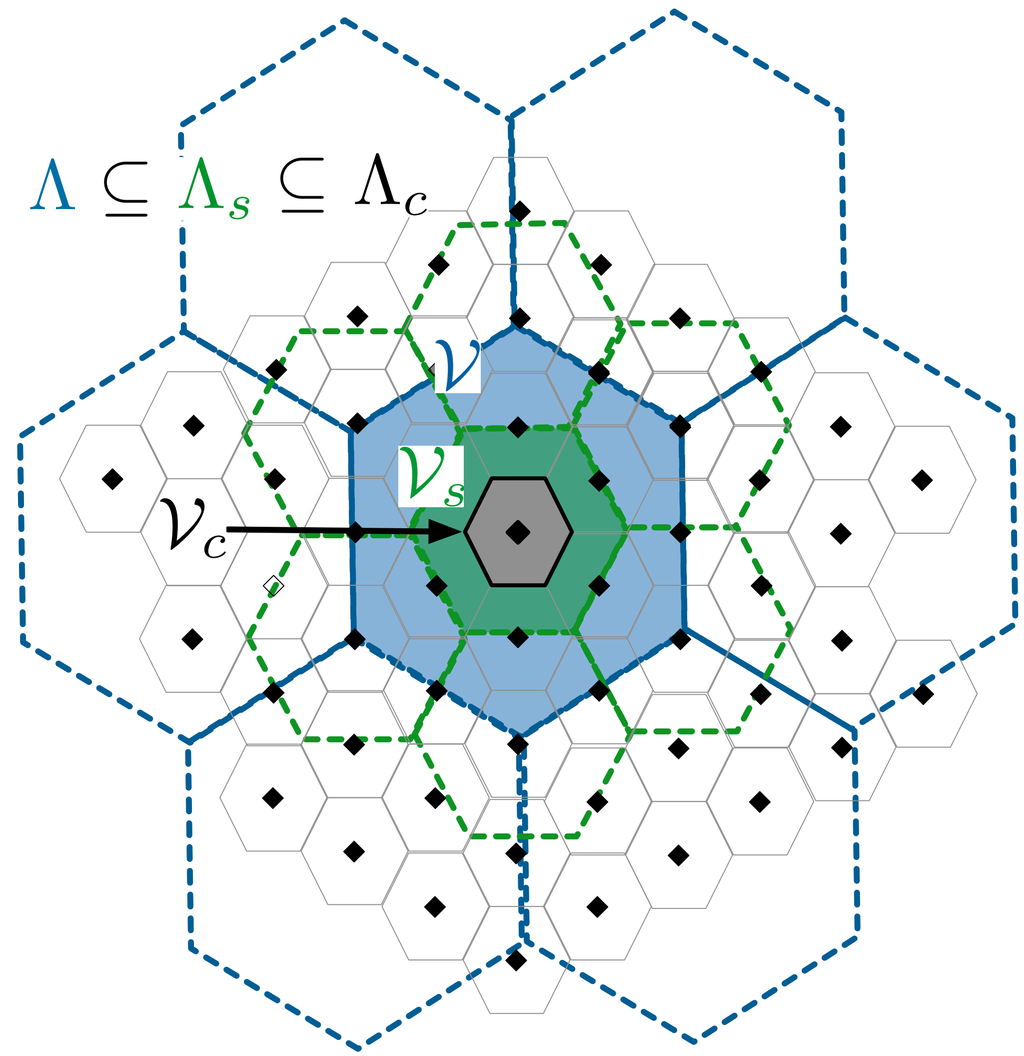
- Coding rate: $R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log \frac{V(\Lambda)}{V(\Lambda_c)}$ arbitrary *(# of ★)*

Nested lattice chains

- $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_K$ ($\Lambda_1, \Lambda_2 \ldots \Lambda_{K-1}$ are Rogers-good and Poltyrev-good, Λ_K is Poltyrev-good). The nesting rates between any pairs in the chain can attain any arbitrary values as the dimension $n \to \infty$.
  [Krithivasan, Pradhan, 2007] [Nam, Chung, Lee, TransIT, 2010]

- A "good" lattice chain with length 3 is used in our list decoding scheme :

$$\Lambda \subseteq \Lambda_s \subseteq \Lambda_c$$



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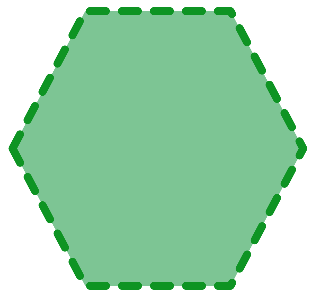
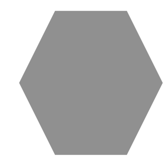
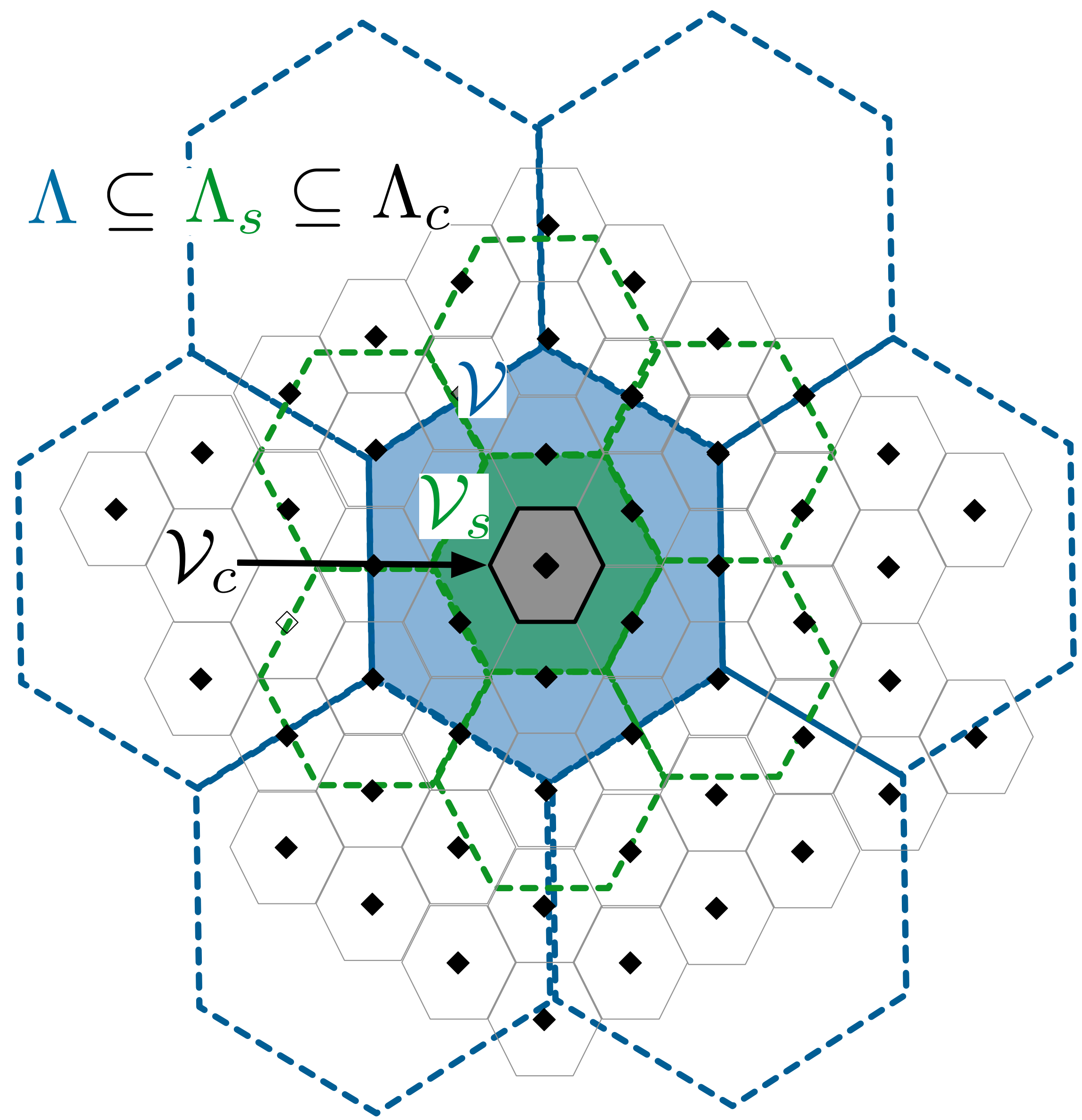
Lattice list decoder

- IDEA: decode to  rather than to 



- results in a **list** of codewords

- require **correct** codeword to be in list

- how many (lower bound) are in list?

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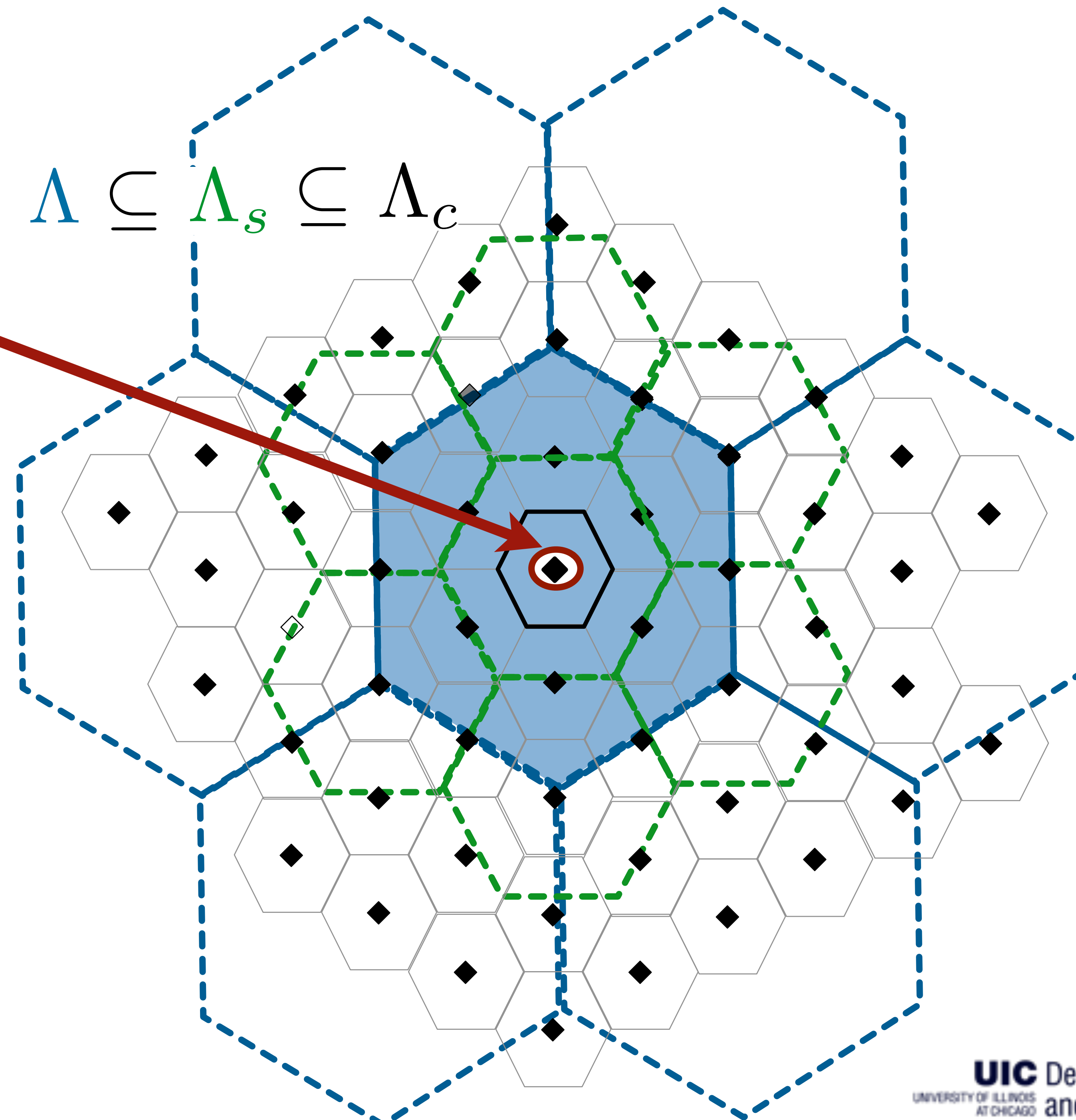
Encoding

• message of rate R over the AWGN channel $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ subject to the average power constraint P

• **Encoding:** take $\mathbf{t} \in \mathcal{C}_{\Lambda_c, \mathcal{V}}$ associated with message of rate R and $\mathbf{X} = (\mathbf{t} - \mathbf{U}) \mod \Lambda$

• $\mathbf{U}$ is a dither signal uniformly distributed over $\mathcal{V}$.



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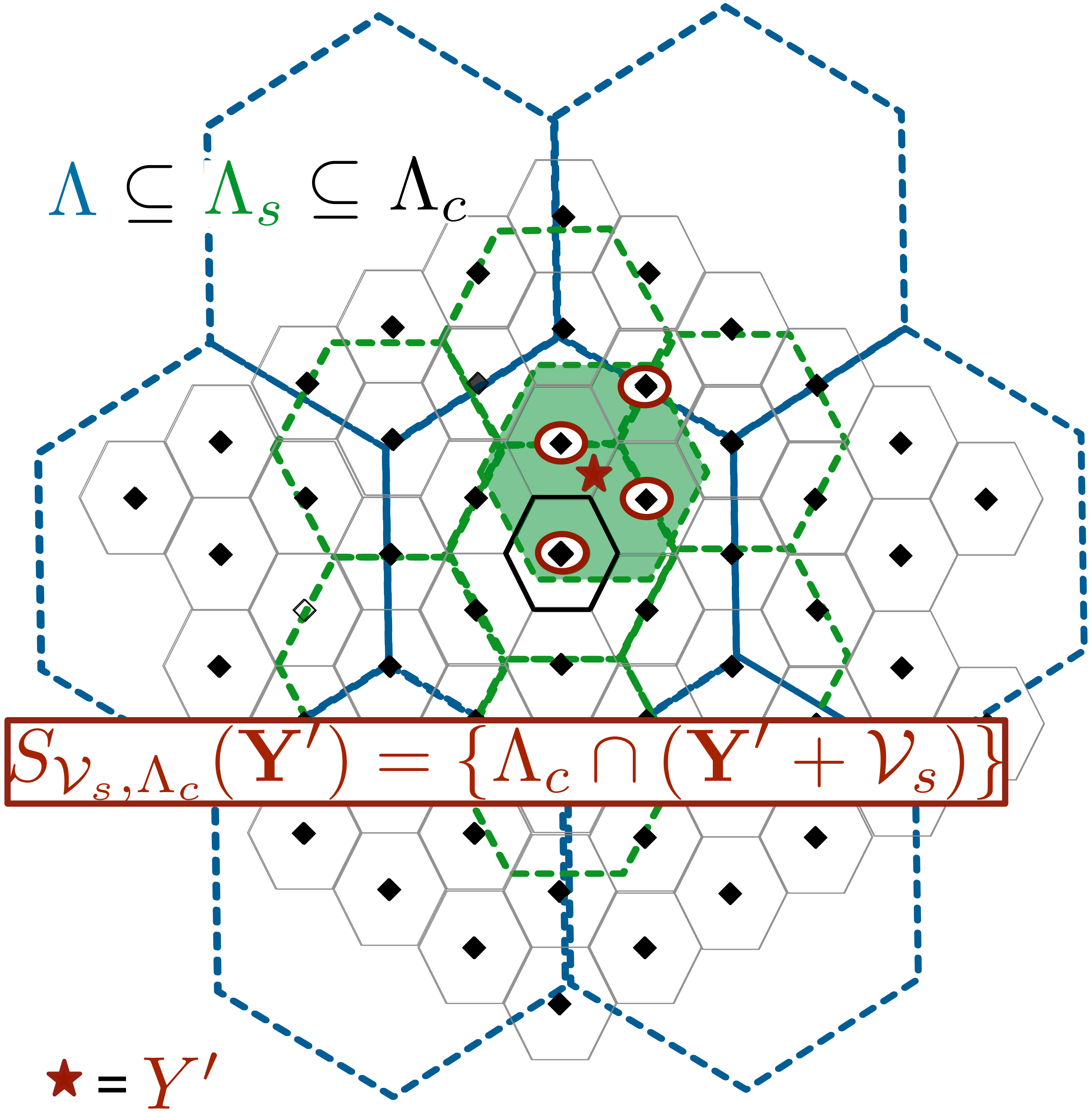
Decoding

- Receiver first computes

$$\begin{aligned}\mathbf{Y}' &= (\alpha \mathbf{Y} + \mathbf{U}) \mod \Lambda \\ &= (\mathbf{t} - (1-\alpha)\mathbf{X} + \alpha \mathbf{Z}) \mod \Lambda \\ &= (\mathbf{t} + \mathbf{Z}') \mod \Lambda\end{aligned}$$

- Receiver then decodes the *list* of codewords $\hat{\mathbf{t}}$:

$$L(\hat{\mathbf{t}}) := S_{\mathcal{V}_s,\Lambda_c}(\mathbf{Y}') \mod \Lambda$$



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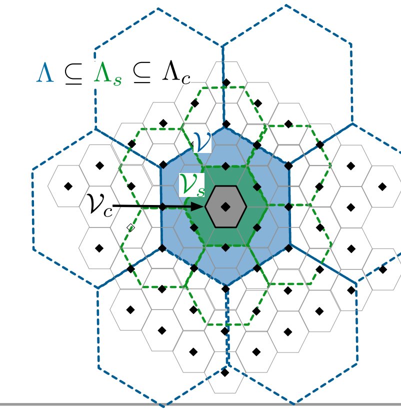
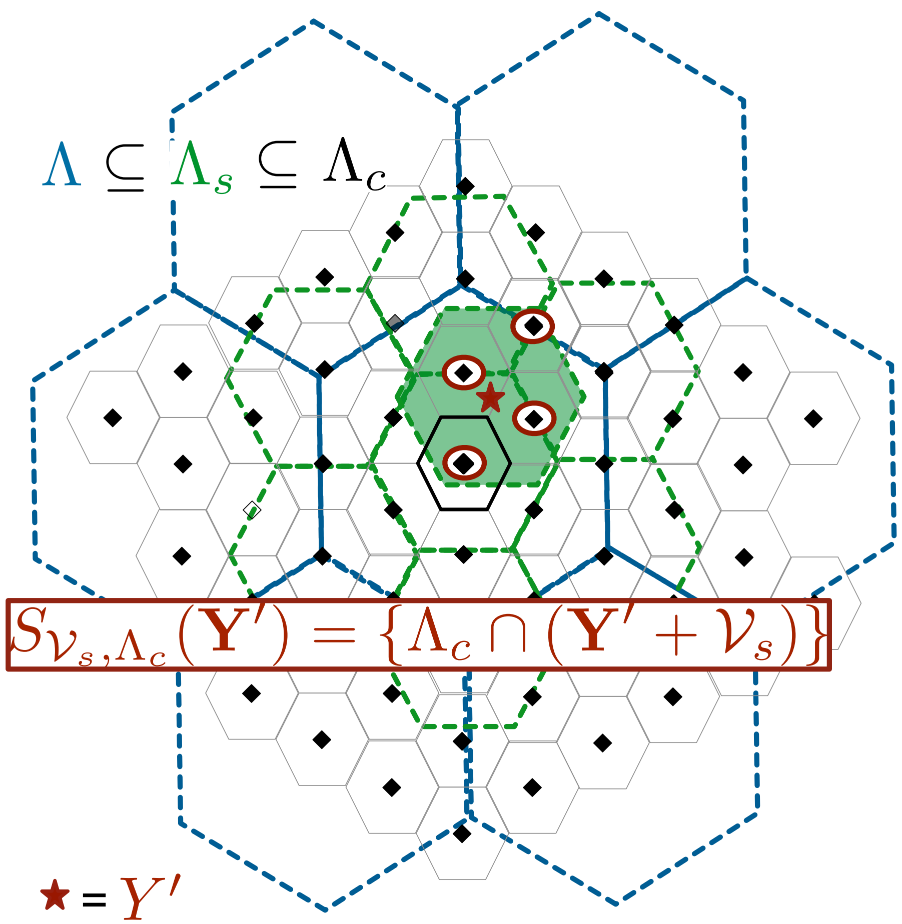


Lattice list decoder

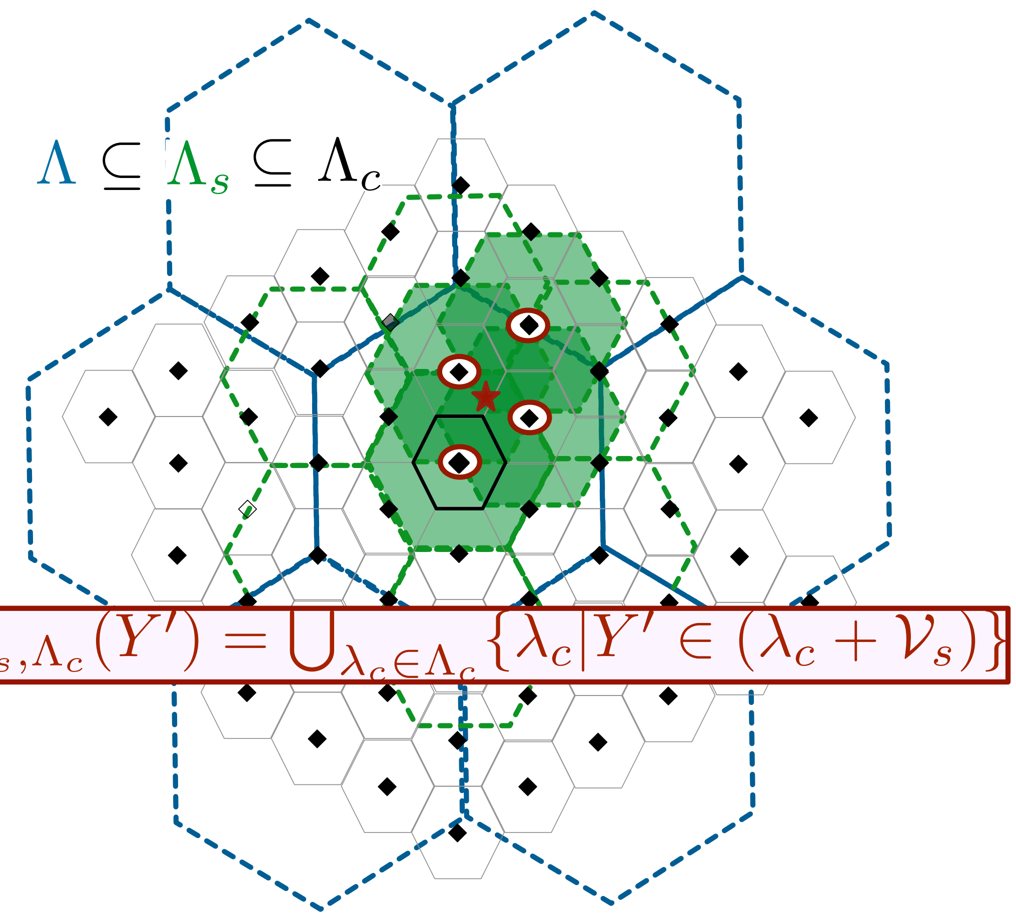
- **Probability of error for list decoding:** $P_e := \Pr\{\mathbf{t} \notin L(\hat{\mathbf{t}})\}$



$$S_{\mathcal{V}_s,\Lambda_c}(\mathbf{Y}') = \{\Lambda_c \cap (\mathbf{Y}' + \mathcal{V}_s)\}$$

SAME list as

$$Q_{\mathcal{V}_s,\Lambda_c}(Y') = \bigcup_{\lambda_c \in \Lambda_c} \{\lambda_c | Y' \in (\lambda_c + \mathcal{V}_s)\}$$



- easy to count # in list
- easy to bound probability of error

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Lattice list decoder

- Theorem 1: Using the encoding and decoding scheme defined above, the receiver decodes a list of codewords of size $2^{n(R-C(P/N))}$ with probability of error $P_e \rightarrow 0$ as $n \rightarrow \infty$

Outline - enabling *cooperation* via lattices

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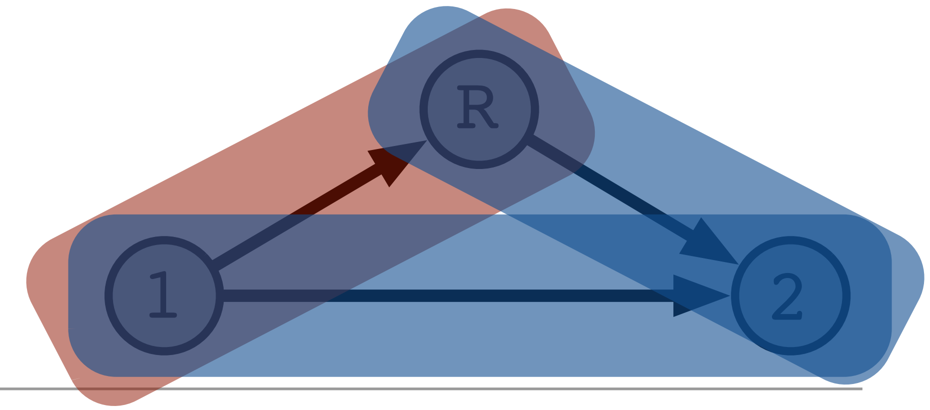
- Lattices achieve CF rate for AWGN relay

Decode and forward relaying



$$R_{DF} = \max_{p(x_1,x_R)} \{\min\{I(X_1;Y_R|X_R), I(X_1,X_R;Y_2)\}\}$$

- Irregular Markov Encoding with Successive Decoding
  [Cover, El Gamal 1979]
- Regular Encoding with Backward Decoding
  [Willems 1992]
- Regular Encoding with Sliding Window Decoding
  [Xie, Kumar 2002]
- Nice survey
  [Kramer, Gastpar, Gupta 2005]

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Single source: lattice DF

Lattices achieve the DF rate for the relay channel. The following Decode-and-Forward rates can be achieved using nested lattice codes for the Gaussian relay channel:

$$R < \max_{\alpha \in [0,1]} \min \left\{ \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_R} \right), \frac{1}{2} \log \left(1 + \frac{P + P_R + 2\sqrt{\bar{\alpha} P P_R}}{N_2} \right) \right\}.$$

Achieved using NESTED LATTICE CODES!

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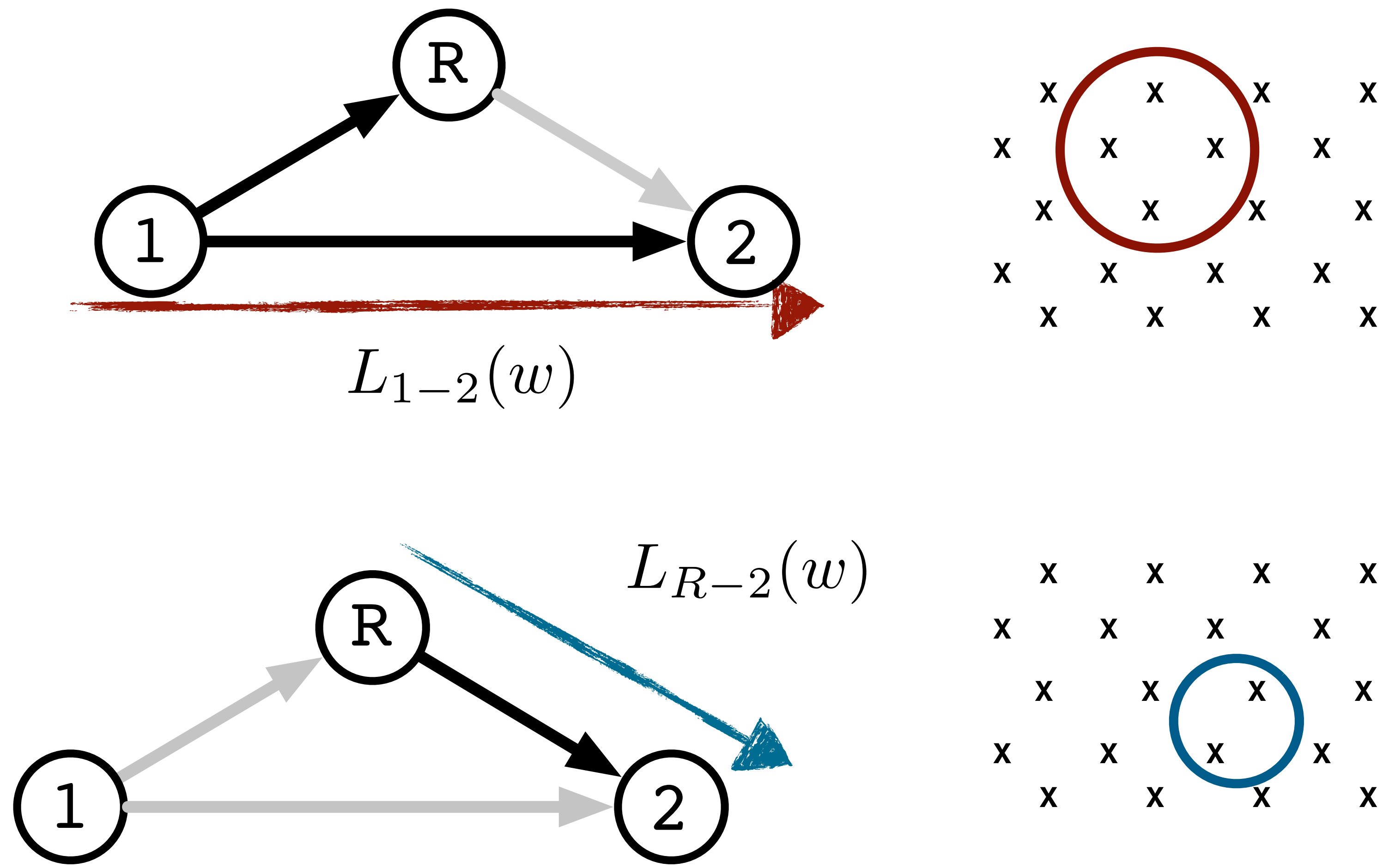
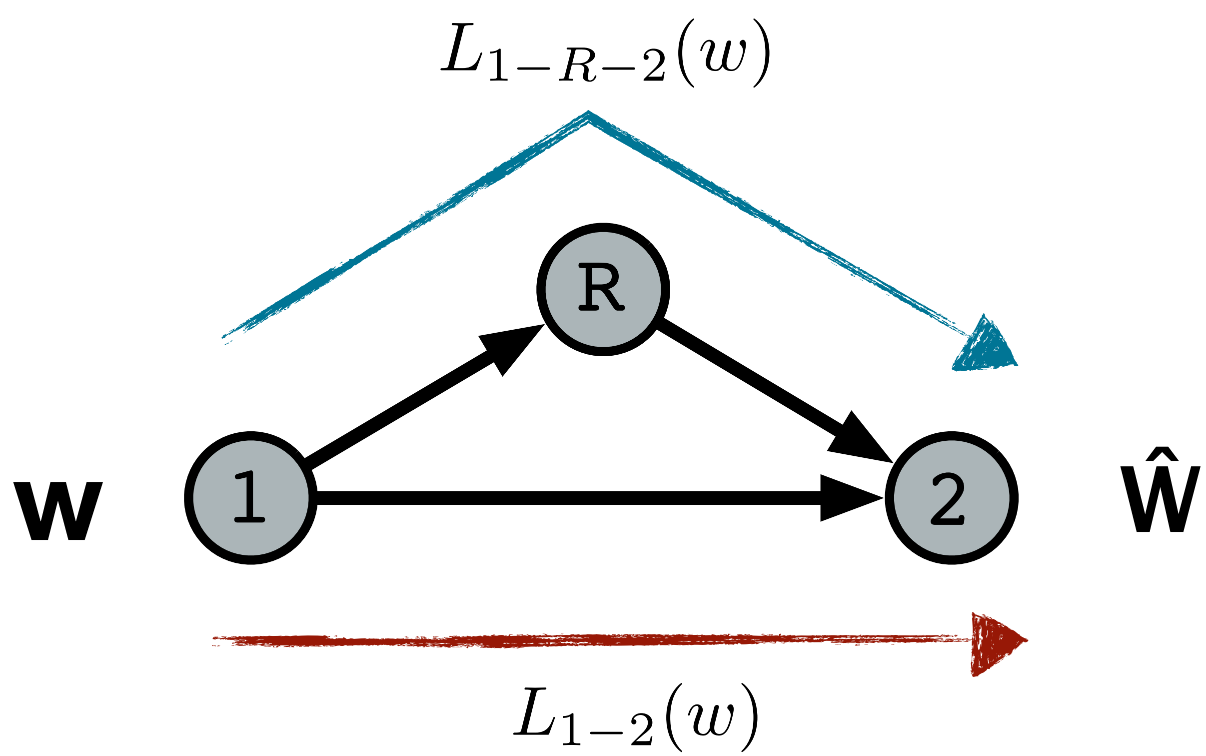
Central idea behind using lists



- **view cooperation between links as intersection of independent lists**

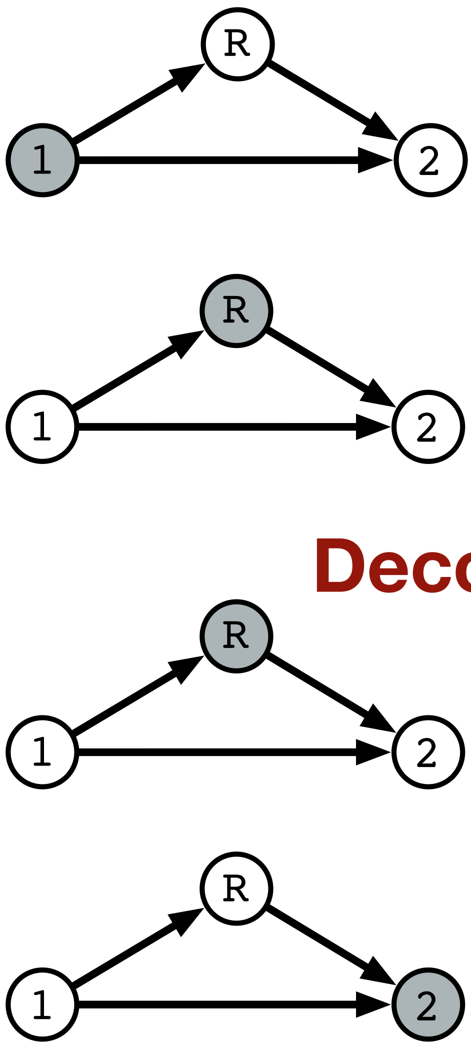
$$L_{1-2}(w) \cap L_{R-2}(w) \Rightarrow \text{UNIQUE } w$$

An aside.....



- **ideally would want this list, rather than forcing a decode.....**

Mimic all steps with lattice codes



Encoding

	Block 1	Block 2	Block 3	Block 4
	$x_1(w_1, 1)$	$x_1(w_2, w_1)$	$x_1(w_3, w_2)$	$x_1(1, w_3)$
	$x_R(1)$	$x_R(w_1)$	$x_R(w_2)$	$x_R(w_3)$

Decoding

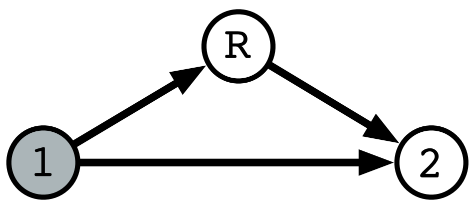
	Block 1	Block 2	Block 3	Block 4
	w_1	w_2	w_3	
	$L_{1-2}(w_1)$	$L_{R-2}(w_1)$ $L_{1-2}(w_2)$	$L_{R-2}(w_2)$ $L_{1-2}(w_3)$	$L_{R-2}(w_3)$

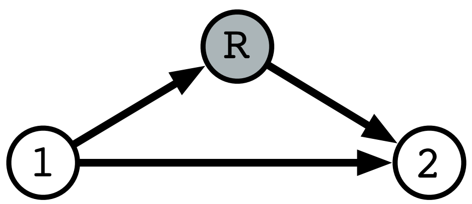
Intersect ($L_{1-2}(w_1)$, $L_{R-2}(w_1)$) → w_1

Intersect ($L_{1-2}(w_2)$, $L_{R-2}(w_2)$) → w_2

Intersect ($L_{1-2}(w_3)$, $L_{R-2}(w_3)$) → w_3

Encoding





Block b

$x_1(w_b, w_{b-1})$ $= x'_1(w_b) + x'_2(w_{b-1})$
$x_R(w_{b-1})$ $= x'_2(w_{b-1})$

$\longleftrightarrow$ $\Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1}$ + $\Lambda_2 \subseteq \Lambda_{s2} \subseteq \Lambda_{c2}$

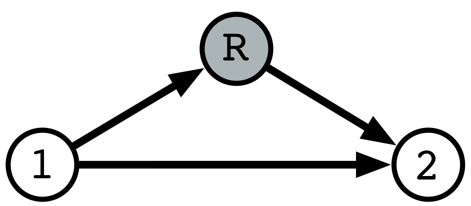
$\sigma^2(\Lambda_1) = \alpha P$ $\updownarrow$ Λ_1; $\sigma^2(\Lambda_2) = \bar{\alpha} P$ $\updownarrow$ Λ_2

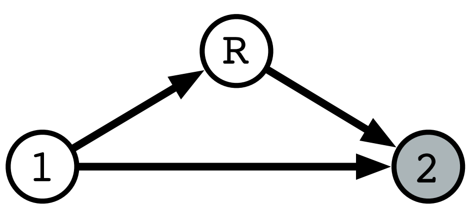
$\longleftrightarrow$ $\sqrt{\frac{P_R}{\bar{\alpha}P}}\Lambda_2 \subseteq \sqrt{\frac{P_R}{\bar{\alpha}P}}\Lambda_{c2}$

$\updownarrow$

$\sigma^2(\sqrt{\frac{P_R}{\bar{\alpha}P}}\Lambda_2) = P_R$

Decoding





Block b	*Block b+1*
w_b	w_{b+1}
$L_{R-2}(w_{b-1})$ $L_{1-2}(w_b)$	$L_{R-2}(w_b)$ $L_{1-2}(w_{b+1})$

$\longleftrightarrow$ Lattice decoder

$\longleftrightarrow$ **Intersection of independent lists**

Intersect

$\Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1}$ $\qquad$ $\Lambda_2 \subseteq \Lambda_{s2} \subseteq \Lambda_{c2}$

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- At relay block b:

Knows and cancels

$$Y_R(b) = X_1'(w_b) + X_2'(w_{b-1}) + X_R(w_{b-1}) + Z_R$$

$$R < \frac{1}{2}\log\left(1 + \frac{\alpha P}{N_R}\right)$$

- At destination block b:

$$Y_2(b) = X_1'(w_b) + X_2'(w_{b-1}) + X_R(w_{b-1}) + Z_2$$
$$= X_1'(w_b) + \left(1 + \sqrt{\frac{P_R}{\bar{\alpha} P}}\right) X_2'(w_{b-1}) + Z_2$$

- Decodes $L_{R-2}(w_{b-1})$ of size $2^{n(R-R_R)}$:

$$R_R < \frac{1}{2}\log\left(1 + \frac{(\sqrt{\bar{\alpha}P} + \sqrt{P_R})^2}{\alpha P + N_2}\right)$$

Lists independent by independent mappings

$$\Lambda_1 \subseteq \Lambda_{s1} \subseteq \Lambda_{c1}$$
$$\Lambda_2 \subseteq \Lambda_{s2} \subseteq \Lambda_{c2}$$

- intersect $L_{R-2}(w_{b-1})$ and $L_{1-2}(w_{b-1})$ from previous block

$$R < \frac{1}{2}\log\left(1 + \frac{\alpha P}{N_2}\right) + R_R$$
$$< \frac{1}{2}\log\left(1 + \frac{P + P_R + 2\sqrt{\bar{\alpha}PP_R}}{N_2}\right).$$

- subtract $X_2'(w_{b-1})$ and decode list $L_{1-2}(w_b)$ of size $2^{n(R-C(\alpha P/(N_2)))}$

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Outline - enabling *cooperation* via lattices

- Lattice notation
- Lattice list decoder
- Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay
- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel
- Lattices achieve CF rate for AWGN relay

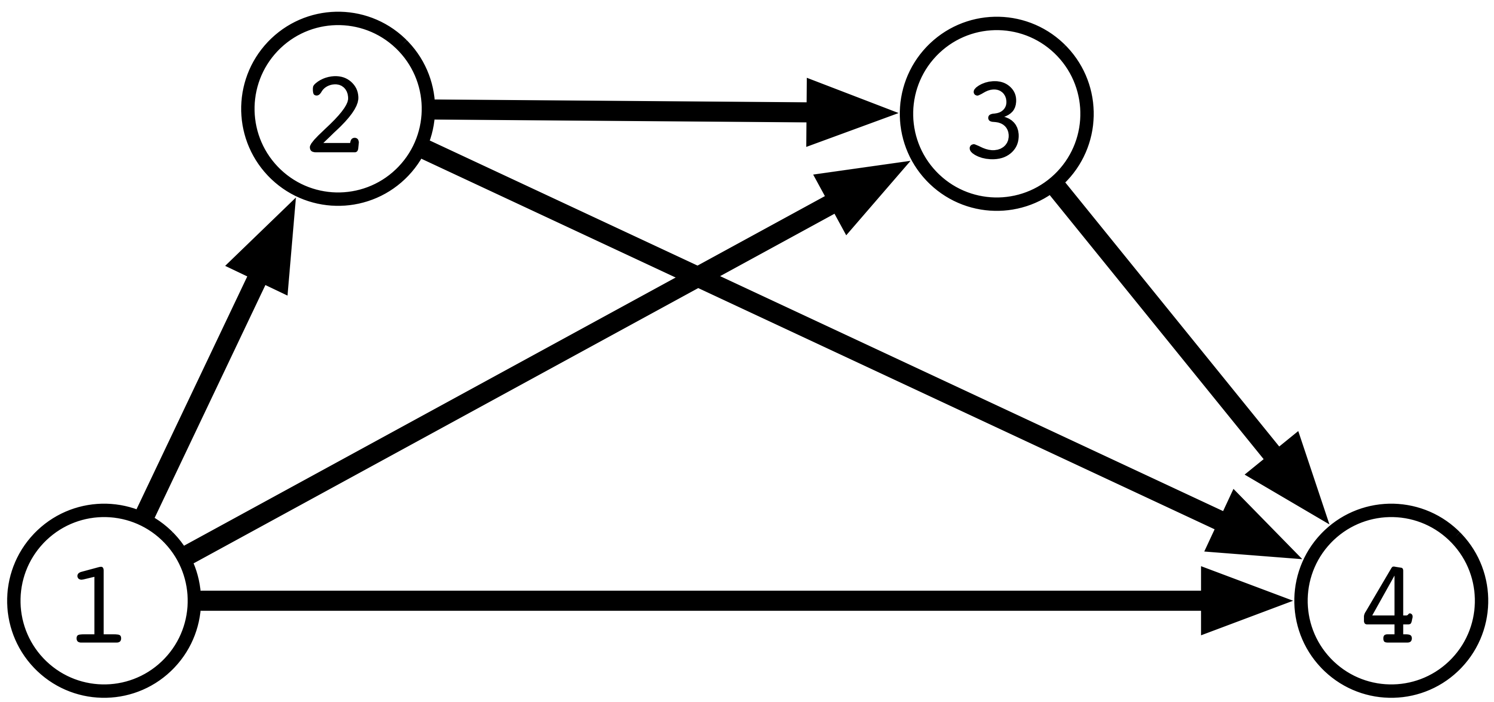
Single-source, multiple relay

[Xie, Kumar 2004]

[Kramer, Gastpar, Gupta 2005]

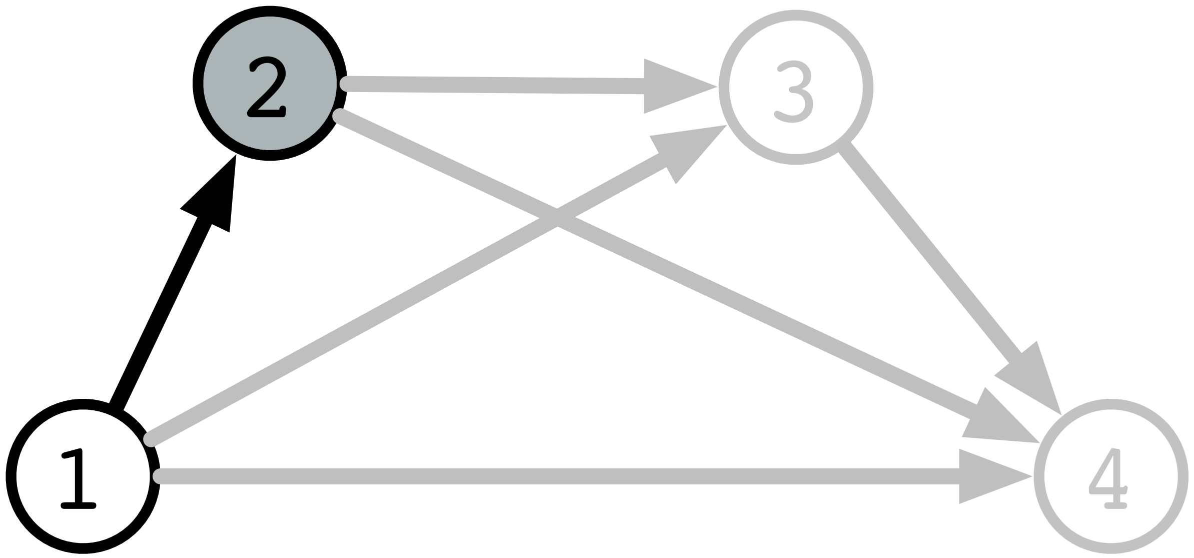
$$Y_2 = X_1 + Z_2, \quad Z_2 \sim \mathcal{N}(0, N_2)$$

$$Y_3 = X_1 + X_2 + Z_3, \quad Z_3 \sim \mathcal{N}(0, N_3)$$

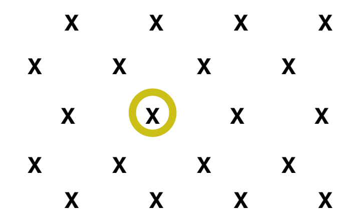


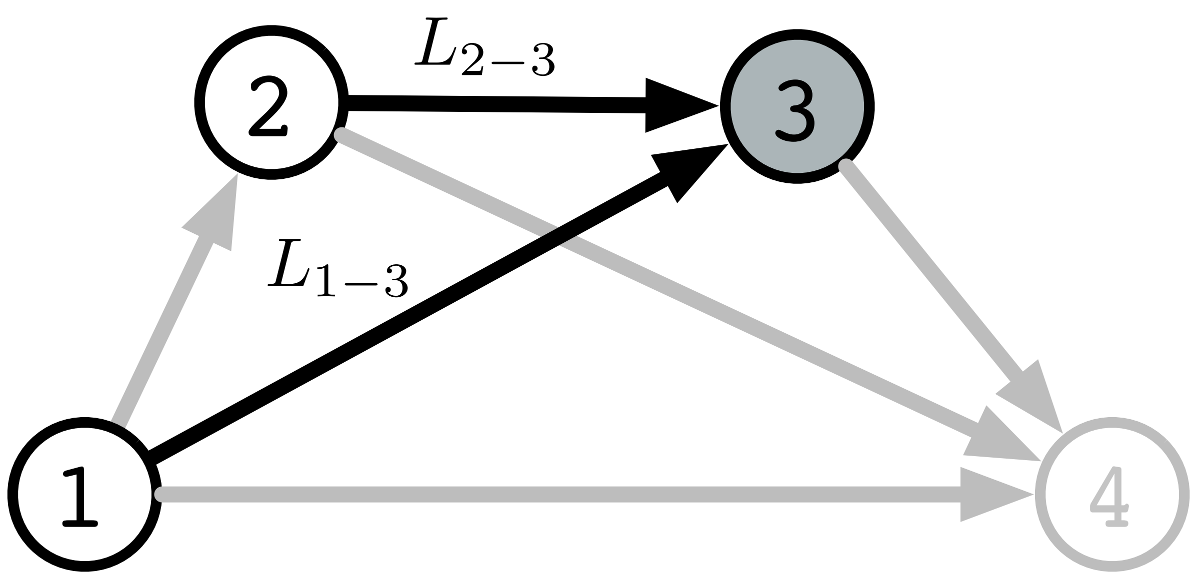
$$Y_4 = X_1 + X_2 + X_3 + Z_4, \quad Z_4 \sim \mathcal{N}(0, N_4)$$

NESTED LATTICE CODES can mimic the regular encoding / sliding window decoding DF rate

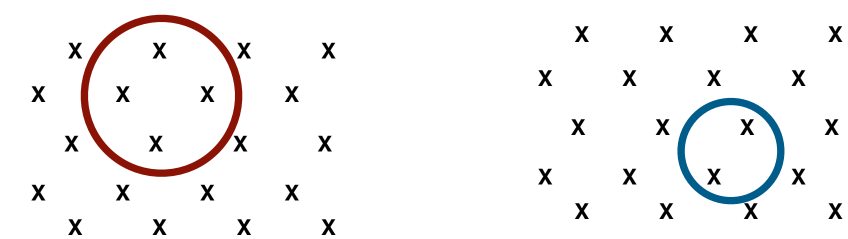


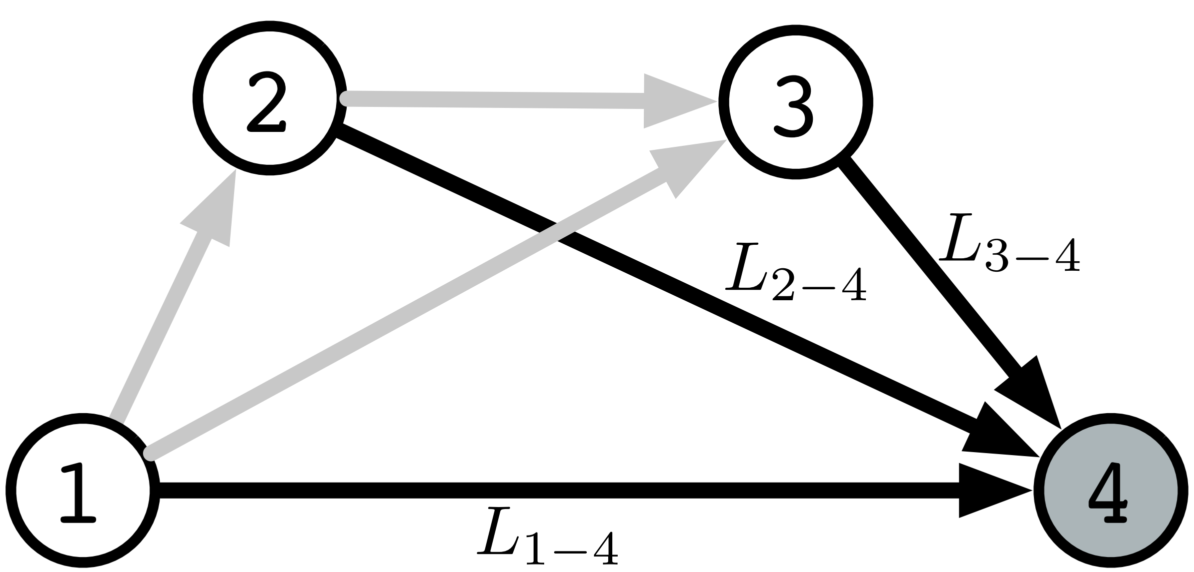
- Unique decoding



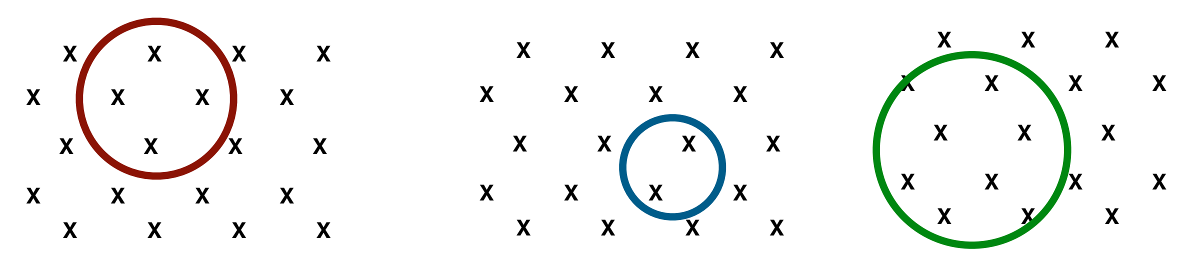


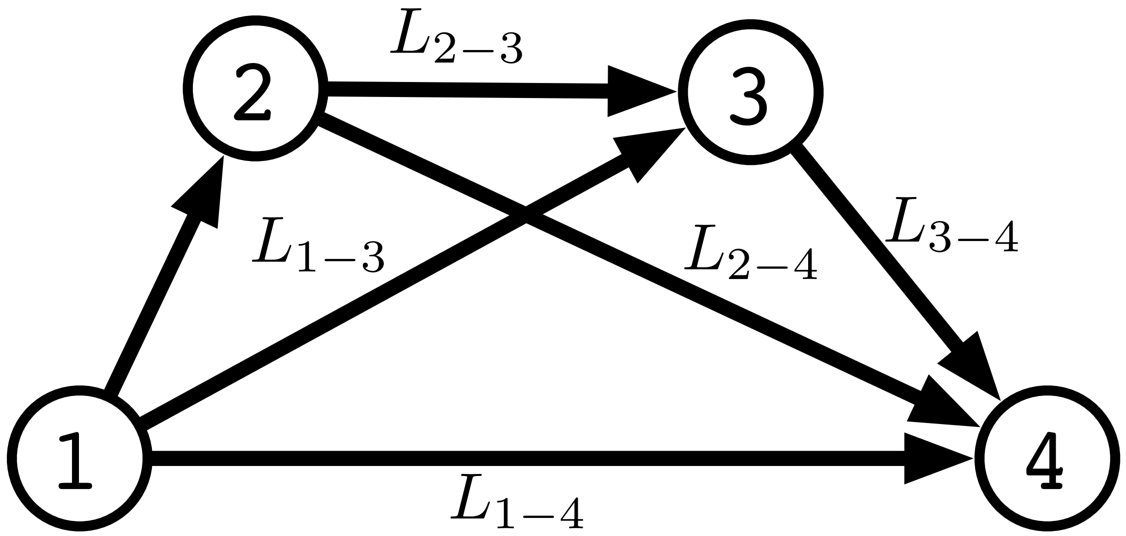
- Intersection 2 lists





- Intersection 3 lists





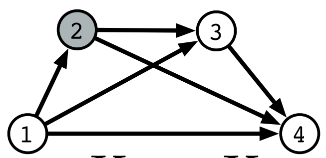
$x_1' \leftrightarrow \quad \Lambda_1 \subseteq \Lambda_{s(1-3)} \subseteq \Lambda_{s(1-4)} \subseteq \Lambda_{c1}$

$x_2' \leftrightarrow \quad \Lambda_2 \subseteq \Lambda_{s(2-3)} \subseteq \Lambda_{s(2-4)} \subseteq \Lambda_{c2}$

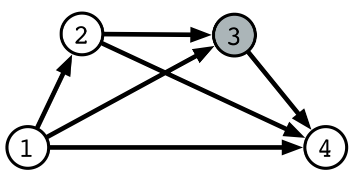
$x_3' \leftrightarrow \quad \Lambda_3 \subseteq \Lambda_{s(3-4)} \subseteq \Lambda_{c3}$

Lists independent by independent mappings

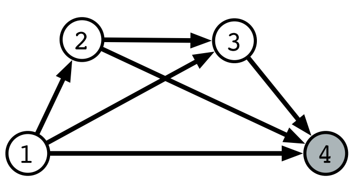
Decoding



$Y_2 = X_1 + X_3 + Z_2$



$Y_3 = X_1 + X_2 + Z_3$



$Y_4 = X_1 + X_2 + X_3 + Z_4$

Block b-1	*Block b-1*	*Block b*
		$\{w_{b-1}, w_{b-2}\}$ w_b
	Intersect $L_{1-3}(w_{b-1})$	$\{w_{b-2}\}$ w_{b-1} $L_{2-3}(w_{b-1})$ $L_{1-3}(w_b)$
Intersect $L_{1-4}(w_{b-2})$	$L_{2-4}(w_{b-2})$ $L_{1-4}(w_{b-1})$	$\{\}$ w_{b-2} $L_{3-4}(w_{b-2})$ subtracts w_{b-2} and forms $L_{2-4}(w_{b-1})$

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(as known random)

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- Single source DF applications:
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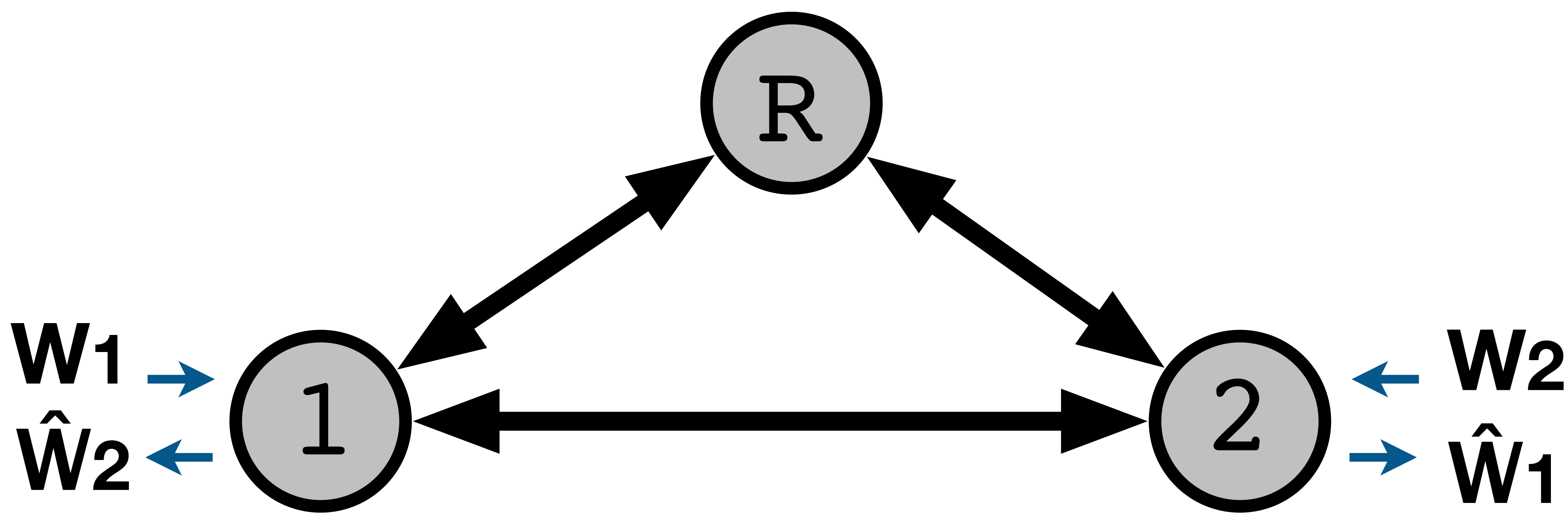
  - Lattices achieve DF rate for AWGN multi-relay

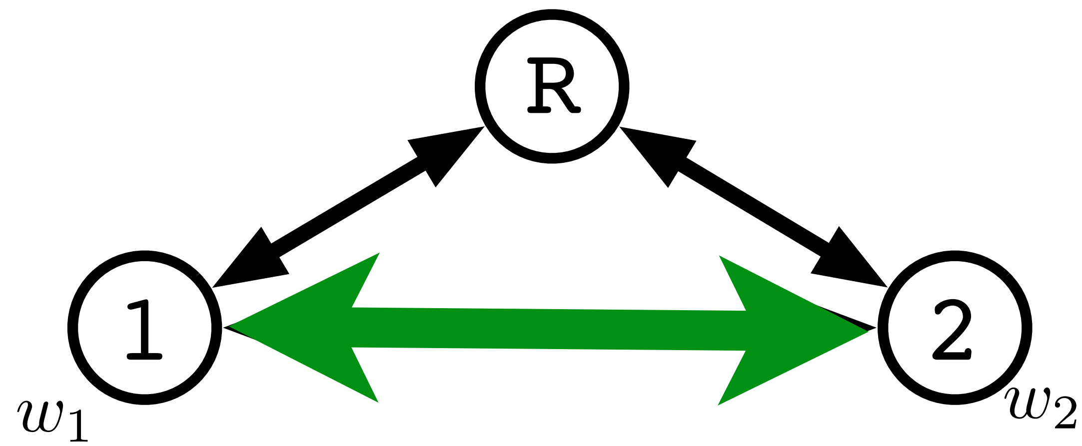
  } **Same rates as random**
- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel
- Lattices achieve CF rate for AWGN relay

Two-way relay channel (with direct links)



Two-way relay channel (with direct links)

$$Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R)$$



$$Y_1 = X_1 + X_2 + X_R + Z_1 \quad Y_2 = X_1 + X_2 + X_R + Z_2$$

$$Z_1 \sim \mathcal{N}(0, N_1) \qquad Z_2 \sim \mathcal{N}(0, N_2)$$

- we derive a new achievable rate region using **nested lattices**, **with direct link**
- this region attains **constant gaps** for certain degraded channels

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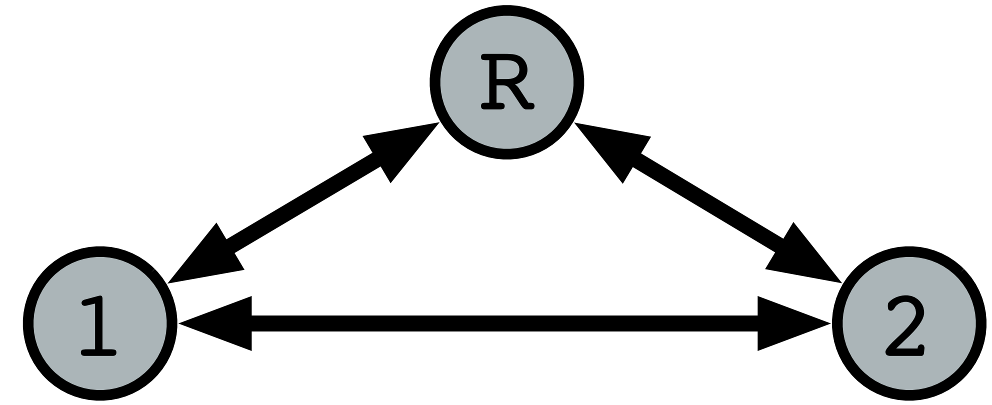
Rate region



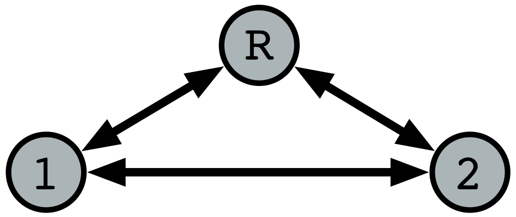
- Theorem: For the two-way relay channel with direct links, we may achieve:

$$R_1 \leq \min\left(\left[\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+\frac{P_1}{N_R}\right)\right]^+, \frac{1}{2}\log\left(1+\frac{P_1+P_R}{N_2}\right)\right)$$

$$R_2 \leq \min\left(\left[\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+\frac{P_2}{N_R}\right)\right]^+, \frac{1}{2}\log\left(1+\frac{P_2+P_R}{N_1}\right)\right)$$

$$R_1 \le \min\left(\left[\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+\frac{P_1}{N_R}\right)\right]^+, \frac{1}{2}\log\left(1+\frac{P_1+P_R}{N_2}\right)\right)$$

$$R_2 \le \min\left(\left[\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+\frac{P_2}{N_R}\right)\right]^+, \frac{1}{2}\log\left(1+\frac{P_2+P_R}{N_1}\right)\right)$$



- eliminates “MAC”-like constraints at relay *[Xie, CWIT, 2007]*

$$R_1 \le \min\left(\frac{1}{2}\log\left(1+\frac{P_1}{N_R}\right), \frac{1}{2}\log\left(1+\frac{P_1+P_R}{N_2}\right)\right)$$

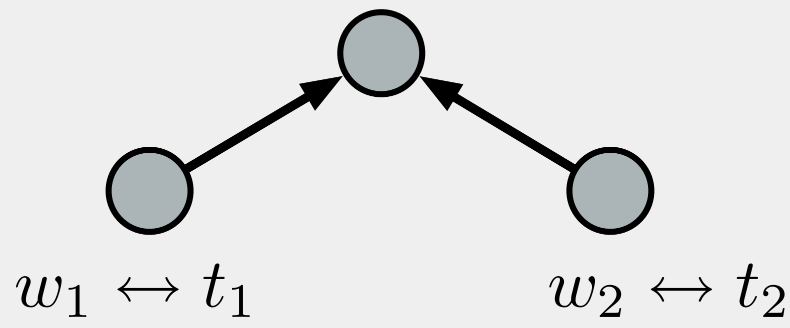
$$R_2 \le \min\left(\frac{1}{2}\log\left(1+\frac{P_2}{N_R}\right), \frac{1}{2}\log\left(1+\frac{P_2+P_R}{N_1}\right)\right)$$

$$R_1 + R_2 \le \frac{1}{2}\log\left(1+\frac{P_1+P_2}{N_R}\right)$$

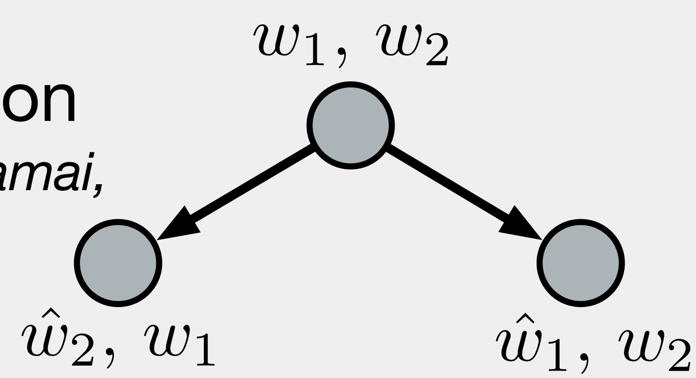
- combines direct and relayed information using lattice list decoder

- **Decoding sum**
  [Nam, Chung, Lee Trans. IT 2010]

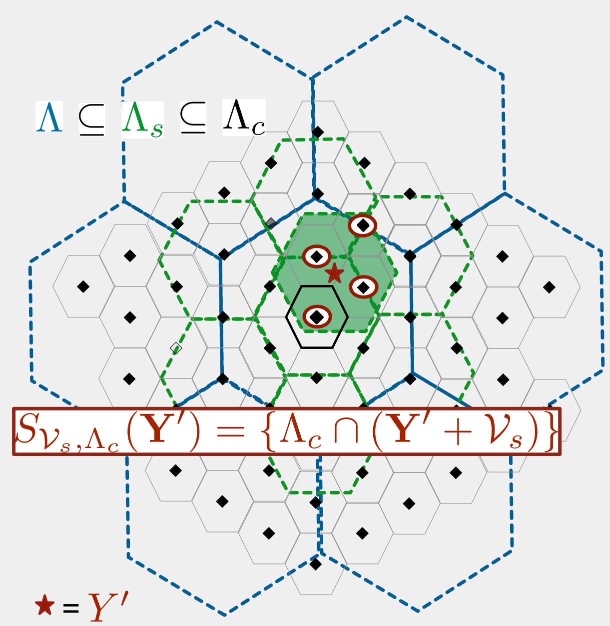
$$\hat{T} = (t_1 + t_2 - Q_2(t_2 + U_2)) \mod \Lambda_1$$



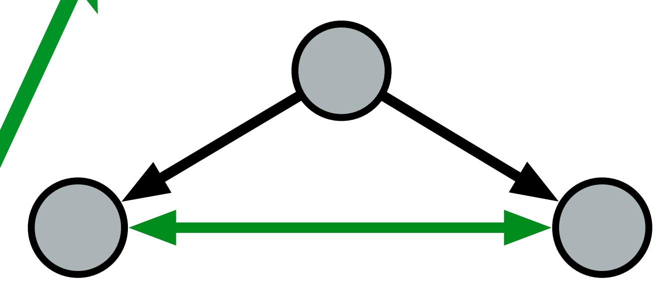
- **BC with side-information**
  [Xie, CWIT 2007][Kramer, Shamai, ITW 2007] [Wu ISIT 2007] [Tuncel 2006]



- **List decoding**
  [Song, D 2010]



New achievable rate region



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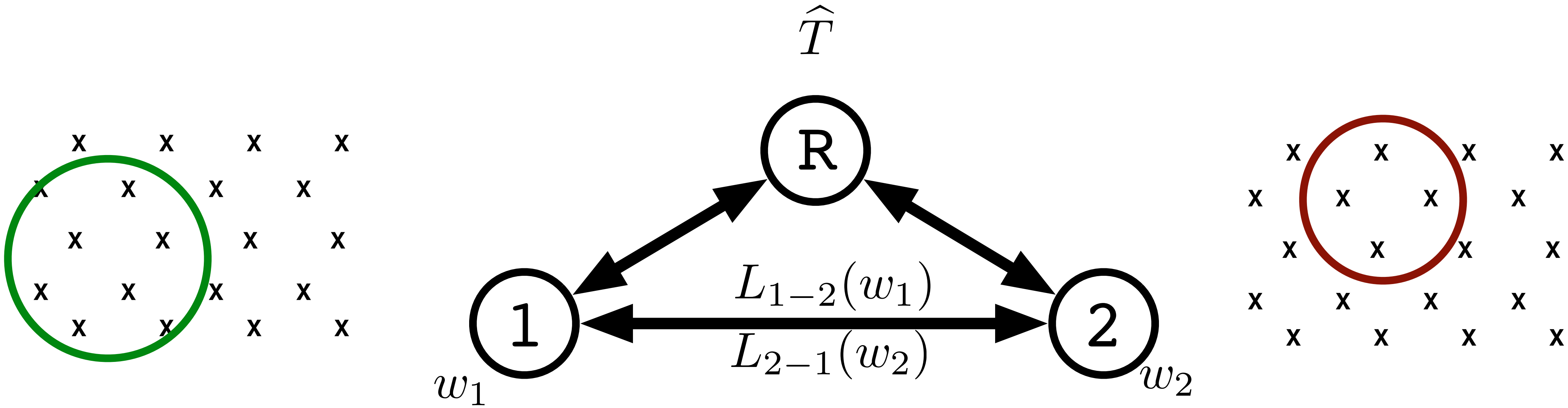
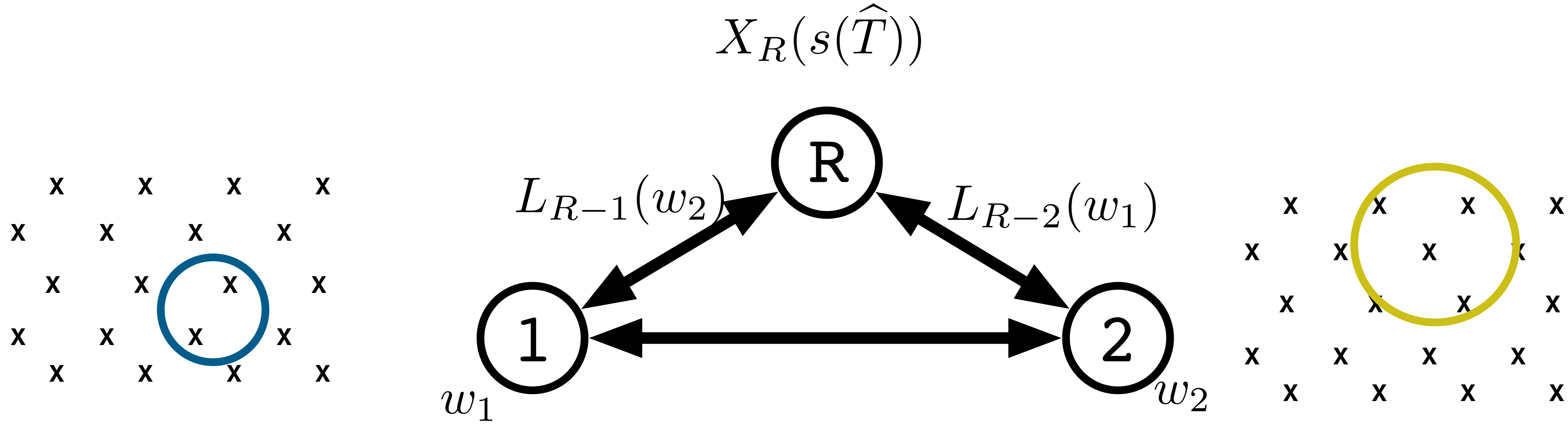


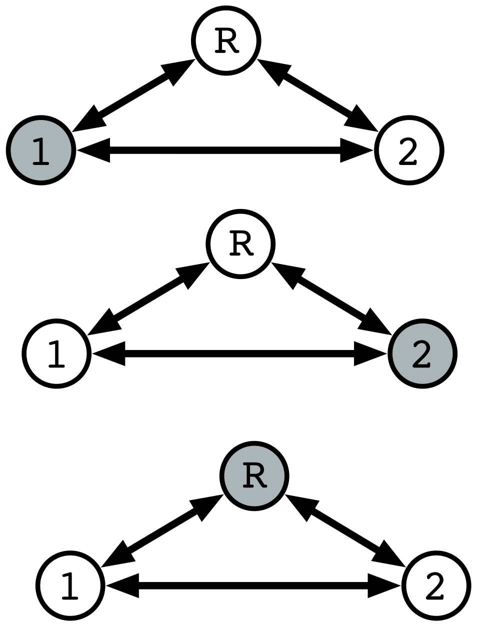
intersect 2 lists **intersect 2 lists**



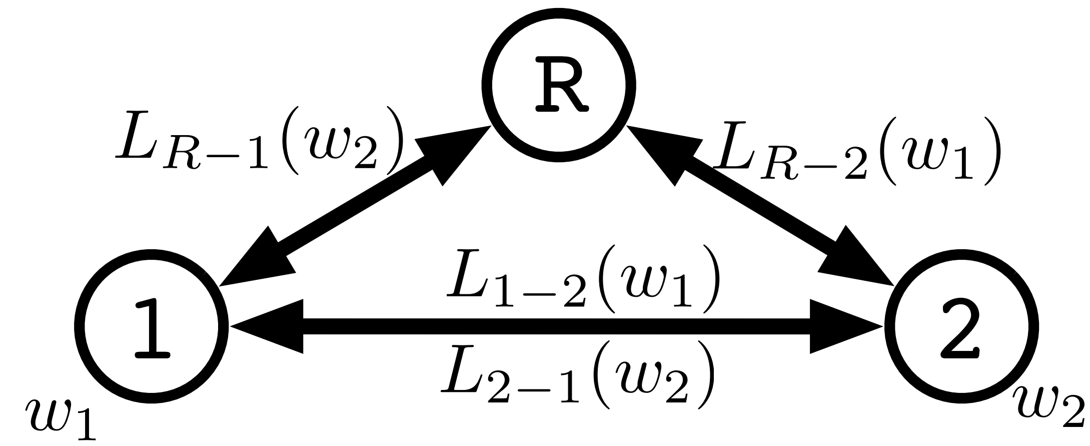
$L_{R-1}(w_{2(b-1)}) := \{w_{2(b-1)} : (x_R(s(T(b-1))), X_1(w_{1b}), Y_1(b)) \in A_\epsilon^N\}$
$L_{R-2}(w_{1(b-1)}) := \{w_{1(b-1)} : (x_R(s(T(b-1))), X_2(w_{2b}), Y_2(b)) \in A_\epsilon^N\}$

Encoding

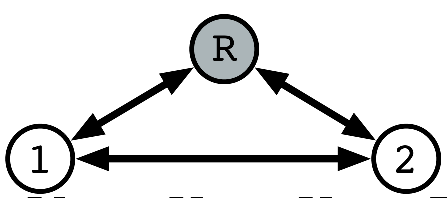
Block b



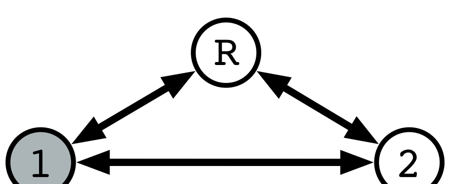
$\mathbf{X_1}(w_{1b}) = (\mathbf{t_1}(w_{1b}) - \mathbf{U_1}) \mod \Lambda_1$
$\mathbf{X_2}(w_{2b}) = (\mathbf{t_2}(w_{2b}) - \mathbf{U_2}) \mod \Lambda_2$
$\mathbf{X_R}(s(\hat{\mathbf{T}}(b-1)))$



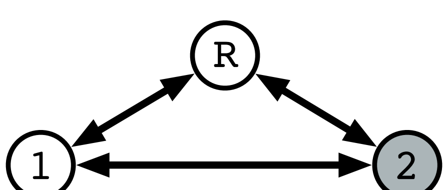
Decoding



$Y_R = X_1 + X_2 + Z_R$



$Y_1 = X_1 + X_2 + X_R + Z_1$



$Y_2 = X_1 + X_2 + X_R + Z_2$

Block b-1	*Block b*
$\hat{\mathbf{T}}(b-1) = (\mathbf{t_1}(w_{1(b-1)}) + \mathbf{t_2}(w_{2(b-1)}) - Q_2(\mathbf{t_2}(w_{2(b-1)})) + \mathbf{U_2})) \mod \Lambda_1$	$\hat{\mathbf{T}}(b) = (\mathbf{t_1}(w_{1(b)}) + \mathbf{t_2}(w_{2(b)}) - Q_2(\mathbf{t_2}(w_{2(b)})) + \mathbf{U_2})) \mod \Lambda_1$
Intersect $L_{2-1}(w_{2b-1})$	$L_{R-1}(w_{2b-1})$ subtracts off decoded w_{2b-1} $L_{2-1}(w_{2b})$
Intersect $L_{1-2}(w_{1b-1})$	$L_{R-2}(w_{1b-1})$ subtracts off decoded w_{1b-1} $L_{1-2}(w_{1b})$

Outline of achievability scheme

- Relay node: $Y_R = X_1 + X_2 + Z_R$, decodes $\hat{T}$:

$$R_1 < \frac{1}{2}\log(\frac{P_1}{P_1+P_2} + \frac{P_1}{N_R})$$
$$R_2 < \frac{1}{2}\log(\frac{P_2}{P_1+P_2} + \frac{P_2}{N_R})$$

- Node 2: $Y_2 = X_1 + X_R + Z_2$, decodes $\hat{w_1}$:

$$\begin{aligned} R_1 &< I(X_R;Y_2|X_2) + C(P_1/N_2) \\ &= \frac{1}{2}\log\left(1+\frac{P_R}{P_1+N_2}\right) + \frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right) \\ &= \frac{1}{2}\log\left(1+\frac{P_R+P_1}{N_2}\right). \end{aligned}$$

- Analogous for node 1

Outline - enabling *cooperation* via lattices

(than known random)

- Lattice notation

- Lattice list decoder

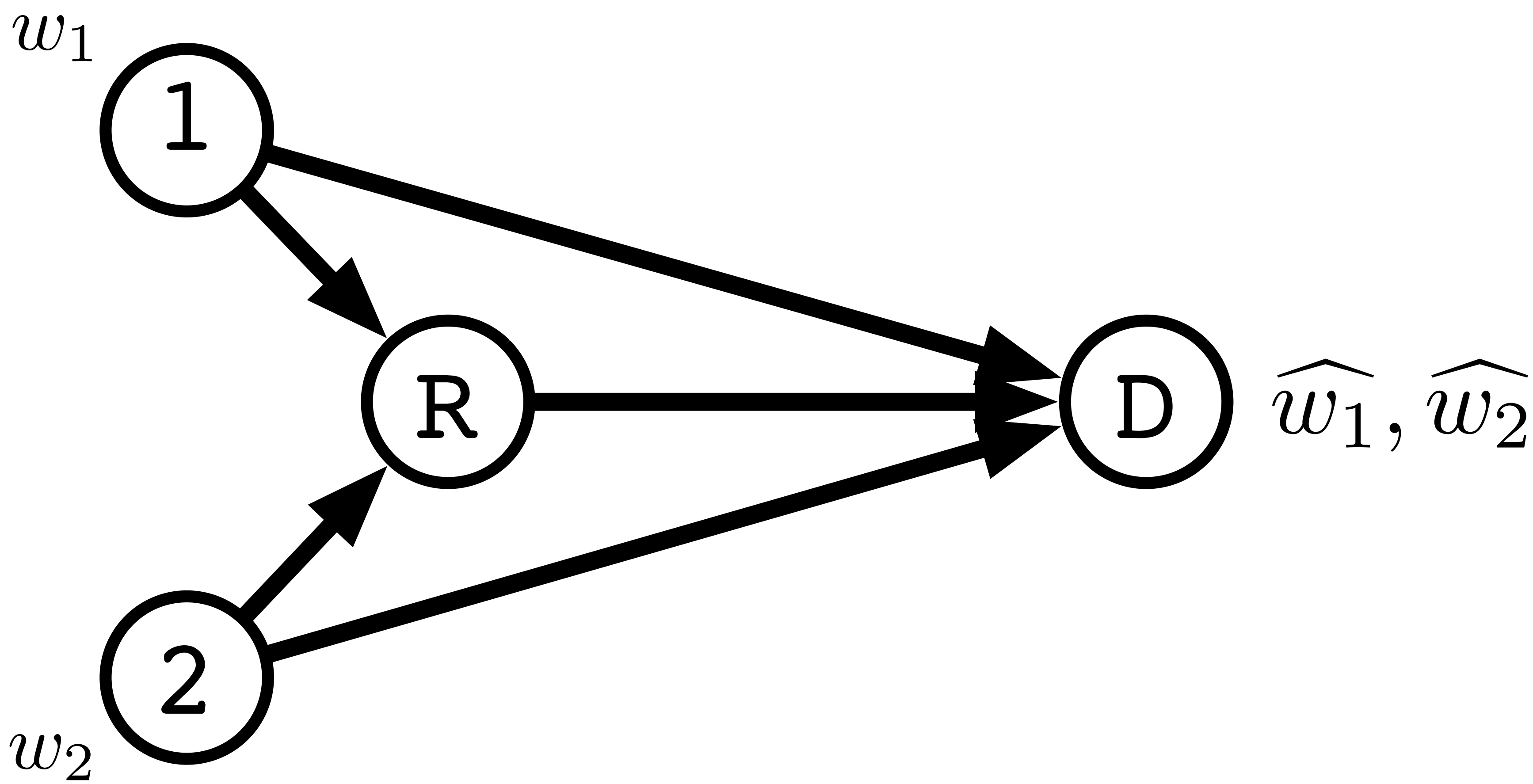
- Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay

  } **Same rates as random**

- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel

- Lattices achieve CF rate for AWGN relay

Lattices for the multiple-access relay channel



$$Y_R = X_1 + X_2 + Z_R, \quad Z_R \sim \mathcal{N}(0, N_R)$$

$$Y_D = X_1 + X_2 + X_R + Z_D, \quad Z_D \sim \mathcal{N}(0, N_D)$$

Key idea: decode+forward sum at the relay

Lattices for the multiple-access relay channel

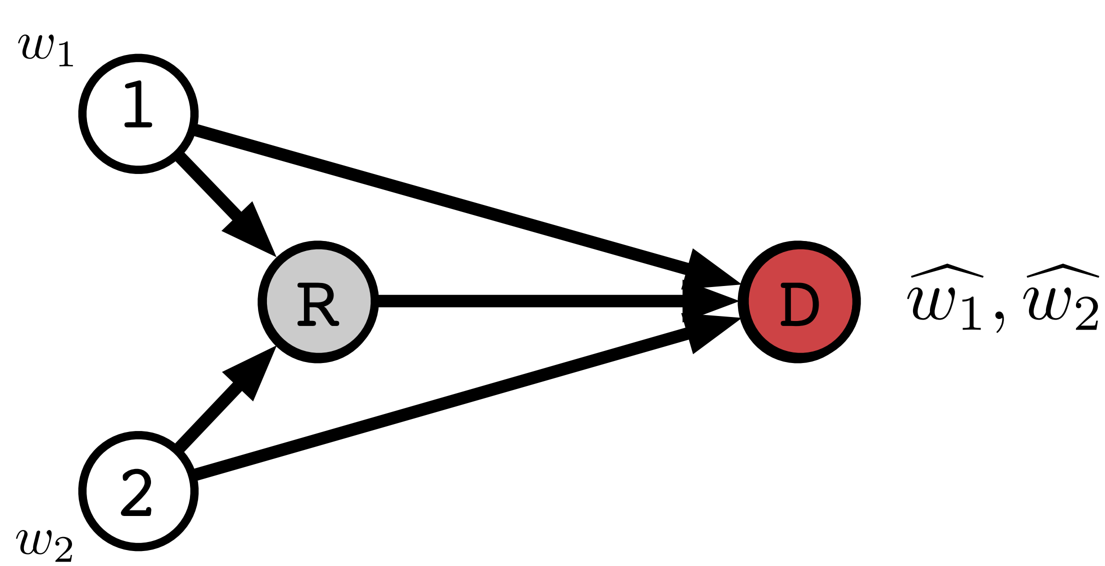
Theorem: The following rates are achievable for the AWGN multiple access relay channel:

$$R_1 < \min\left(\left[\frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+\frac{P_1}{N_R}\right)\right]^+, \frac{1}{2}\log\left(1+\frac{P_1+P_R}{N_D}\right)\right)$$

$$R_2 < \min\left(\left[\frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+\frac{P_2}{N_R}\right)\right]^+, \frac{1}{2}\log\left(1+\frac{P_2+P_R}{N_D}\right)\right)$$

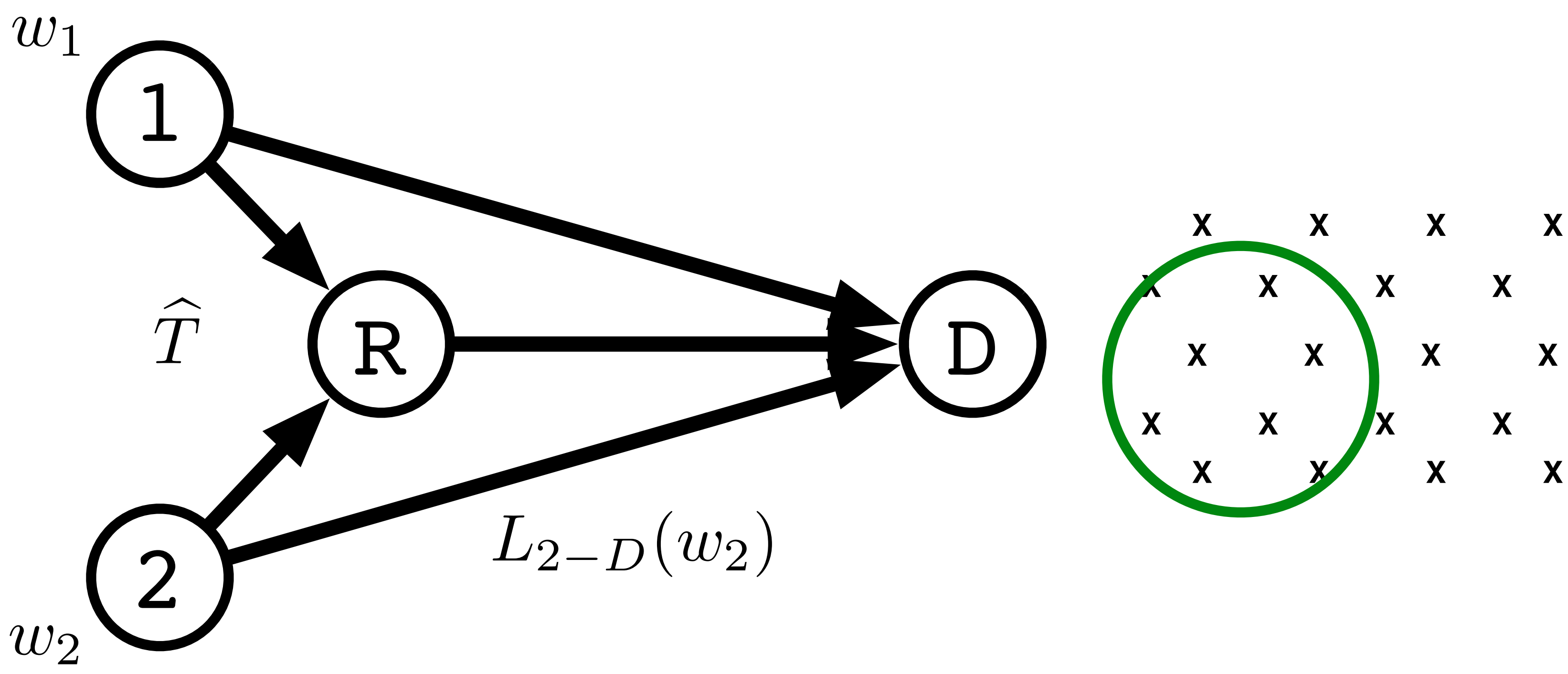
$$R_1 + R_2 < \frac{1}{2}\log\left(1+\frac{P_1+P_2+P_R}{N_D}\right).$$

missing sum-rate constraint at relay!

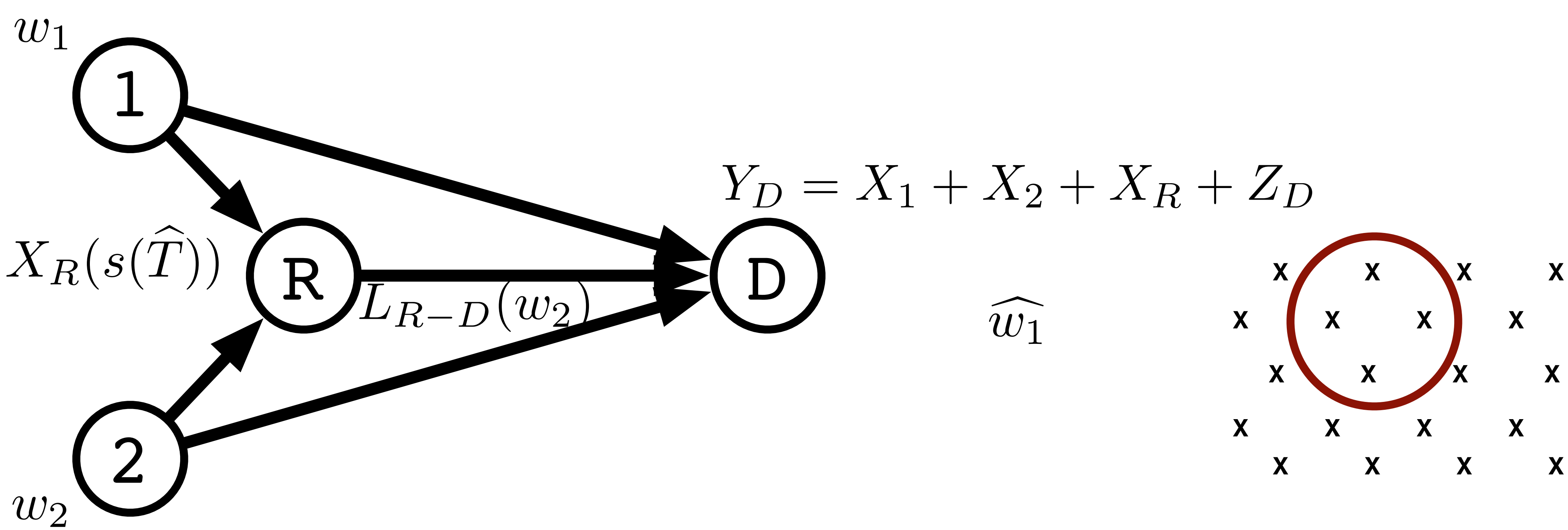


Key idea: decode+forward sum at the relay

Decoding, order 1 then 2



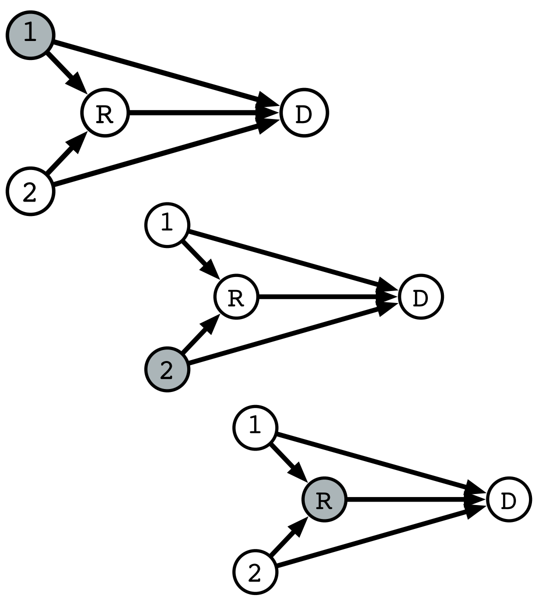
intersect 2 lists for *w*$_2$



switch order, time share

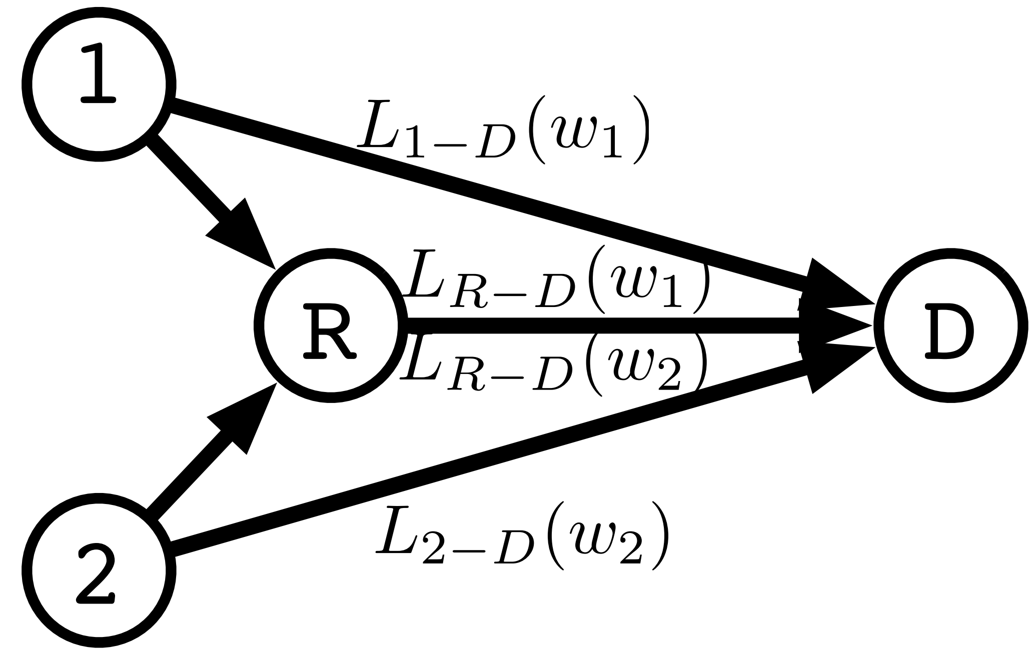
$L_{R-D}(w_{1(b-1)}) := \{w_{1(b-1)} : (x_R(s(T(b-1))), X_2(w_{2b}), Y_D(b)) \in A_\epsilon^N\}$
$L_{R-D}(w_{2(b-1)}) := \{w_{2(b-1)} : (x_R(s(T(b-1))), X_1(w_{1b}), Y_D(b)) \in A_\epsilon^N\}$

Encoding

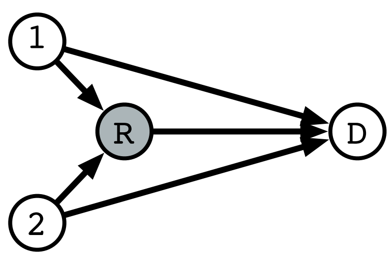


Block b

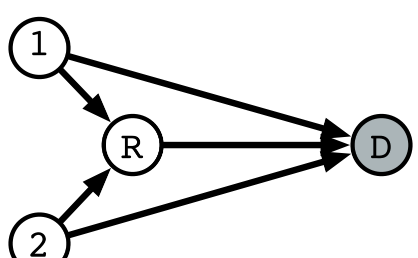
$\mathbf{X_1}(w_{1b}) = (\mathbf{t_1}(w_{1b}) - \mathbf{U_1}) \mod \Lambda_1$
$\mathbf{X_2}(w_{2b}) = (\mathbf{t_2}(w_{2b}) - \mathbf{U_2}) \mod \Lambda_2$
$\mathbf{X_R}(s(\hat{\mathbf{T}}(b-1)))$



Decoding, order 1 then 2



$Y_R = X_1 + X_2 + Z_R$



$Y_D = X_1 + X_2 + X_R + Z_D$

Block b-1	Block b
$\hat{\mathbf{T}}(b-1) = (\mathbf{t_1}(w_{1(b-1)}) + \mathbf{t_2}(w_{2(b-1)}) - Q_2(\mathbf{t_2}(w_{2(b-1)})) + \mathbf{U_2})) \mod \Lambda_1$	$\hat{\mathbf{T}}(b) = (\mathbf{t_1}(w_{1(b)}) + \mathbf{t_2}(w_{2(b)}) - Q_2(\mathbf{t_2}(w_{2(b)})) + \mathbf{U_2})) \mod \Lambda_1$
decode w_{1b-1} from $Y_D = X_1 + X_2 + X_R(s(\hat{T})) + Z_D$ $L_{2-D}(w_{2(b-1)})$	decode w_{1b} from $Y_D = X_1 + X_2 + X_R(s(\hat{T})) + Z_D$ $L_{R-D}(w_{2(b-1)})$ knows $w_{1(b-1)}$ and $w_{2(b-1)}$ $L_{2-D}(w_{2b})$

Intersect

Outline of achievability scheme

- Relay node: $Y_R = X_1 + X_2 + Z_R$, decodes $\hat{T}$:

$$R_1 \leq \frac{1}{2}\log\left(\frac{P_1}{P_1+P_2} + \frac{P_1}{N_R}\right)$$
$$R_2 \leq \frac{1}{2}\log\left(\frac{P_2}{P_1+P_2} + \frac{P_2}{N_R}\right)$$

- Node D decode w_{1b} if $R_1 < \frac{1}{2}\log\left(1 + \frac{P_1}{P_2+P_R+N_D}\right)$
- Node D decode $w_{2(b-1)}$ if

$$R_2 < I(X_R : Y_2|X_1) + C\left(\frac{P_2}{N_D}\right)$$
$$= \frac{1}{2}\log\left(1+\frac{P_R}{P_2+N_D}\right) + \frac{1}{2}\log\left(1+\frac{P_2}{N_D}\right)$$
$$= \frac{1}{2}\log\left(1+\frac{P_2+P_R}{N_D}\right)$$

} Order w_{1b} then w_{2b}

Reverse order and time-share

- subtract $w_{1b}, w_{1(b-1)}, w_{2(b-1)}$ and decode list $L_{2-D}(w_{2b})$ of size $2^{n(R_2 - C(P_2/N_D))}$

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(as/than known random)

- Lattice notation
- Lattice list decoder
- Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay

} **Same rates as random**

- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel

} **Sometimes better rates than random**

- Lattices achieve CF rate for AWGN relay

Why sometimes better?

Structured

$$R_1 \leq \frac{1}{2}\log\left(\frac{P_1}{P_1+P_2}+\frac{P_1}{N_R}\right)$$
$$R_2 \leq \frac{1}{2}\log\left(\frac{P_2}{P_1+P_2}+\frac{P_2}{N_R}\right)$$

- no coherent gain (yet) when decode sum

(Tx needs to know exact relay message, does not if decode sum)

Random

$$R_1 \leq \frac{1}{2}\log\left(1+\frac{P_1}{N_R}\right)$$
$$R_2 \leq \frac{1}{2}\log\left(1+\frac{P_2}{N_R}\right)$$
$$R_1+R_2 \leq \frac{1}{2}\log\left(1+\frac{P_1+P_2}{N_R}\right)$$

- coherent gains

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Outline - enabling *cooperation* via lattices

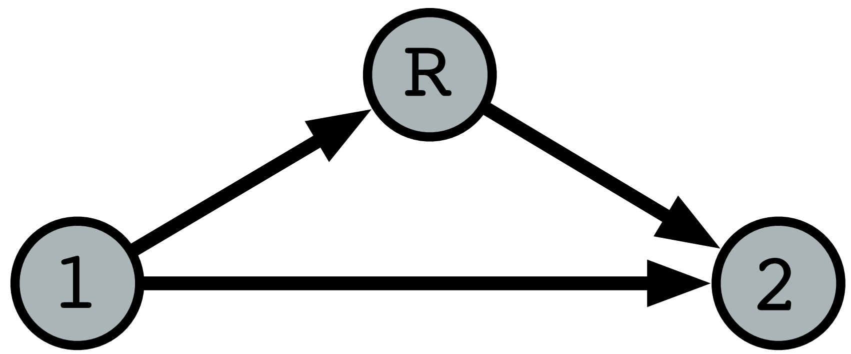
(as/than known random)

- Lattice notation
- Lattice list decoder
- Single source DF applications:
  - Lattices achieve DF rate for AWGN relay
  - Lattices achieve DF rate for AWGN multi-relay

  } **Same rates as random**
- Multi-source DF applications:
  - Lattices for two-way relay channel with direct links
  - Lattices for multiple-access relay channel

  } **Sometimes better rates than random**
- Lattices achieve CF rate for AWGN relay

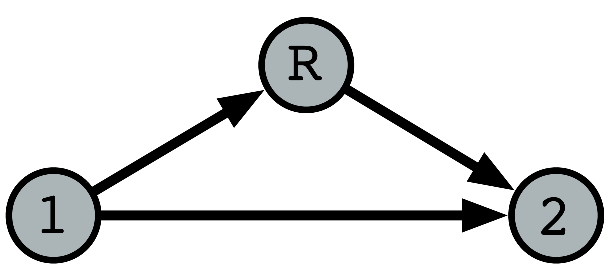
Compress and forward (CF)



- DF limited by need to decode at relay

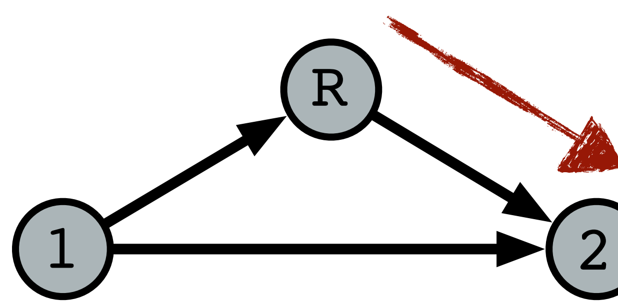
- CF is NOT limited in this fashion

$Y_R \rightarrow \widehat{Y_R}$

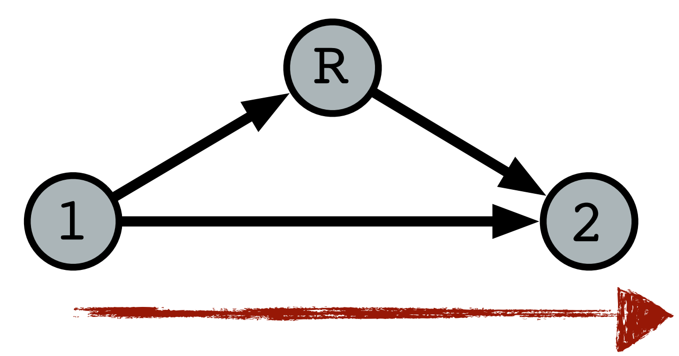


direct link side-information

$\widehat{Y_R} \rightarrow X_R$ Wyner-Ziv

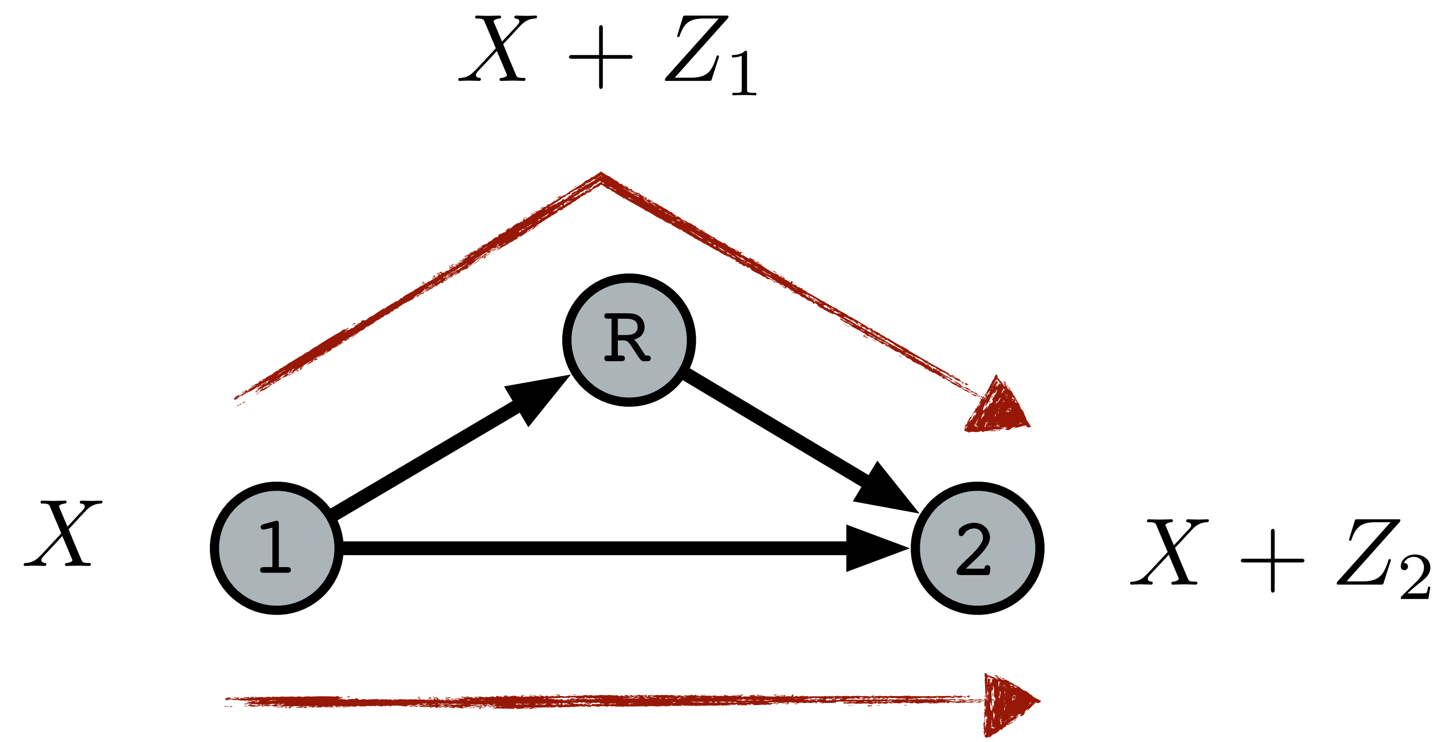


$\widehat{Y_R}$



combine with direct link

The *(X+Z1, X+Z2)* Wyner-Ziv problem



- Gaussian Wyner-Ziv $(X + Z_1, X + Z_2)$

- in *[Zamir,Shamai,Erez, 2002]* demonstrated a lattice scheme for Gaussian $(X + Z, X)$ Wyner-Ziv which is fully general

- demonstrate $(X + Z_1, X + Z_2)$ for completeness

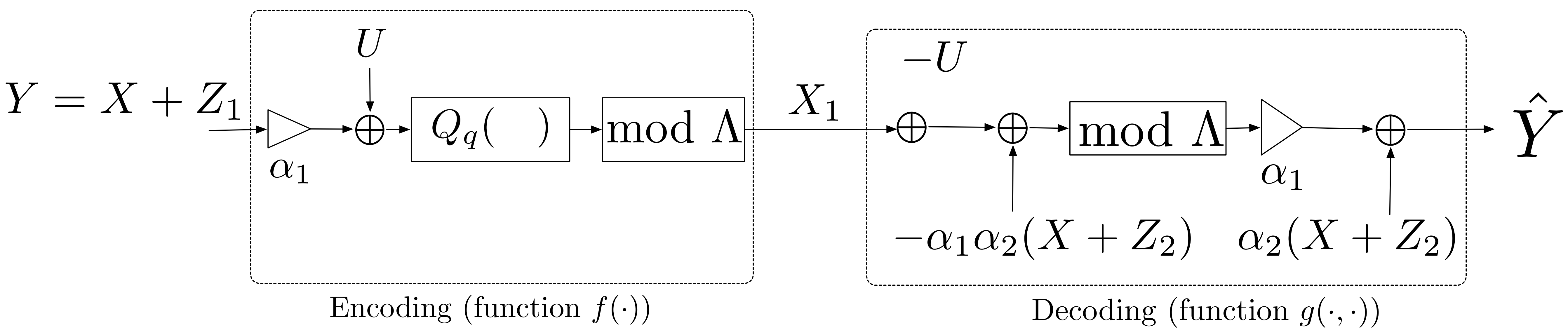
The (X+Z1, X+Z2) Wyner-Ziv problem

Theorem. The following rate-distortion function for the lossy compression of the source $X + Z_1$ subject to the reconstruction side-information $X + Z_2$ and squared error distortion metric may be achieved using lattice codes:

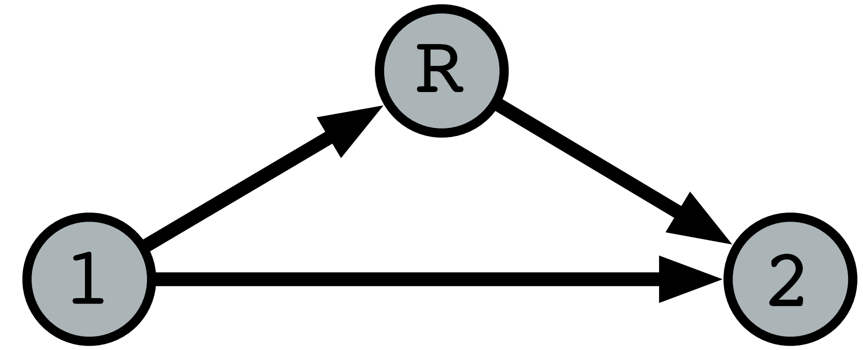
$$R(D) = \frac{1}{2}\log\left(\frac{\sigma^2_{X+Z_1|X+Z_2}}{D}\right), \qquad 0 \leq D \leq \sigma^2_{X+Z_1|X+Z_2}$$

$$= \frac{1}{2}\log\left(\frac{N_1 + \frac{PN_2}{P+N_2}}{D}\right), \qquad 0 \leq D \leq N_1 + \frac{PN_2}{P+N_2},$$

and 0 otherwise.



A Lattice CF scheme

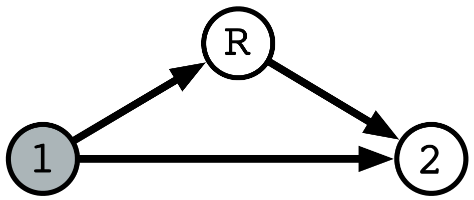


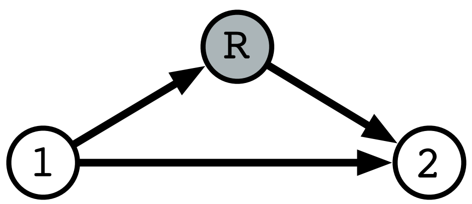
Theorem. For the three user Gaussian relay channel described by the input/output equations $Y_R = X_1 + N_R$ at the relay's receiver and $Y_2 = X_1 + X_R + N_2$ at the destination, with corresponding input and noise powers P_1, P_R, N_R, N_2, the following rate may be achieved using lattice codes in a lattice Compress-and-Forward fashion:

$$R < \frac{1}{2}\log\left(1 + \frac{P_1}{N_2} + \frac{P_1 P_R}{P_1 N_R + P_1 N_2 + P_R N_R + N_R N_2}\right).$$

same as that achieved by Gaussian codes in the CF scheme of [Cover, El Gamal, 1979]

Encoding





Block b

$x_1(w_b)$
$x_R(w_{b-1})$, encoding of index i of compression of $Y_R(w_{b-1}) = X_1(w_{b-1}) + Z_2$

rate R

$w \leftrightarrow t_1,\ \Lambda_1 \subseteq \Lambda_{c1}, \sigma^2(\Lambda_1) = P_1$

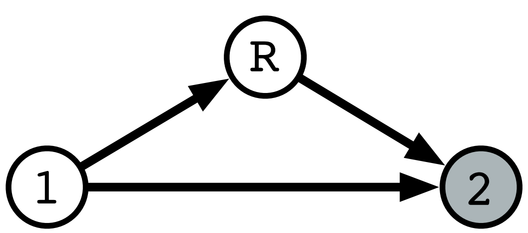
compression $\Lambda \subseteq \Lambda_q$

rate $\hat{R}$ $\quad \sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1 + N_3} + D$

$i \leftrightarrow t_R,\ \Lambda_R \subseteq \Lambda_{cR}$

rate R' $\quad \sigma^2(\Lambda_R) = P_R$

Decoding



Block b-1	*Block b*
• $Y_2'(w_{b-1}) = X_1(w_{b-1}) + Z_2$	• compression index i from $Y_2 = X_1 + X_r + Z_2$ • $Y_2'(w_b) = Y_2(w_b) - X_R(w_{b-1}) = X_1(w_b) + Z_2$ • $X_R(w_{b-1})$ to reconstruct $\hat{Y}_R(w_{b-1})$ • coherently combine $Y_2'(w_{b-1})$ and $\hat{Y}_R(w_{b-1})$ to decode w_{b-1}

Wyner-Ziv

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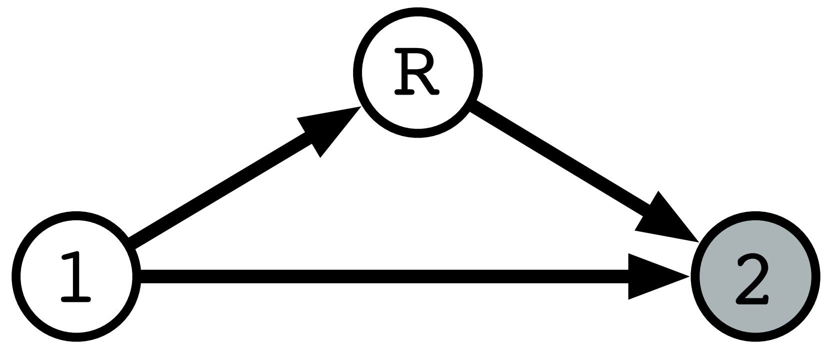
compression $\Lambda \subseteq \Lambda_q$ rate $\hat{R}$ $\sigma^2(\Lambda) = N_2 + \frac{P_1 N_3}{P_1+N_3} + D$

$i \leftrightarrow t_R,\ \Lambda_R \subseteq \Lambda_{cR}$ rate R' $\sigma^2(\Lambda_R) = P_R$

rate R $w \leftrightarrow t_1,\ \Lambda_1 \subseteq \Lambda_{c1}$

$\sigma^2(\Lambda_1) = P_1$



- decodes i from $Y_2 = X_1 + X_R + Z_2$ as long as $R' < \frac{1}{2}\log_2\left(1 + \frac{P_R}{P_1+N_2}\right)$

- source coding < channel coding rate: $\frac{1}{2}\log\left(1 + \frac{N_R + \frac{P_1 N_2}{P_1+N_2}}{D}\right) < \frac{1}{2}\log\left(1 + \frac{P_R}{P_1+N_2}\right)$

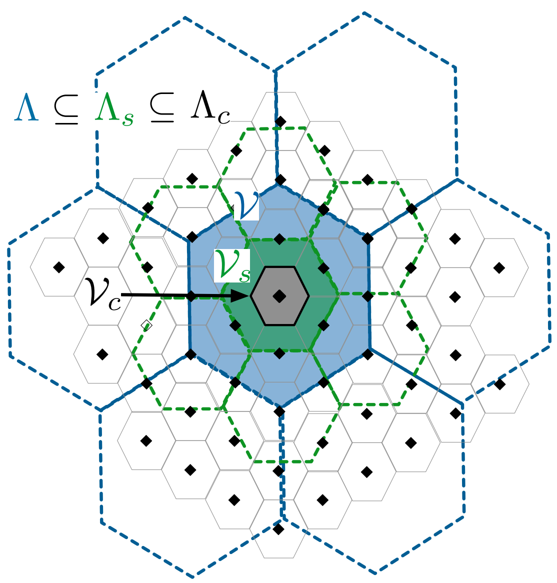
- use $Y_2' = Y_2 - X_R$ from previous block and X_R from current block to reconstruct $\hat{Y_R}$ as in $(X + Z_R, X + Z_2)$ Wyner-Ziv

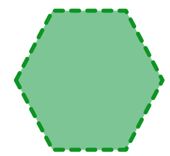
- coherently combine Y_2' and $\hat{Y_R}$ to decode w, as long as $R < \frac{1}{2}\log\left(1 + \frac{P_1}{N_2} + \frac{P_1}{N_R+D}\right)$

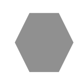
- note: pick $\alpha_1 = 1$ rather than source coding MMSE to render compression noise independent of all else

Enabling lattice "Cooperation"

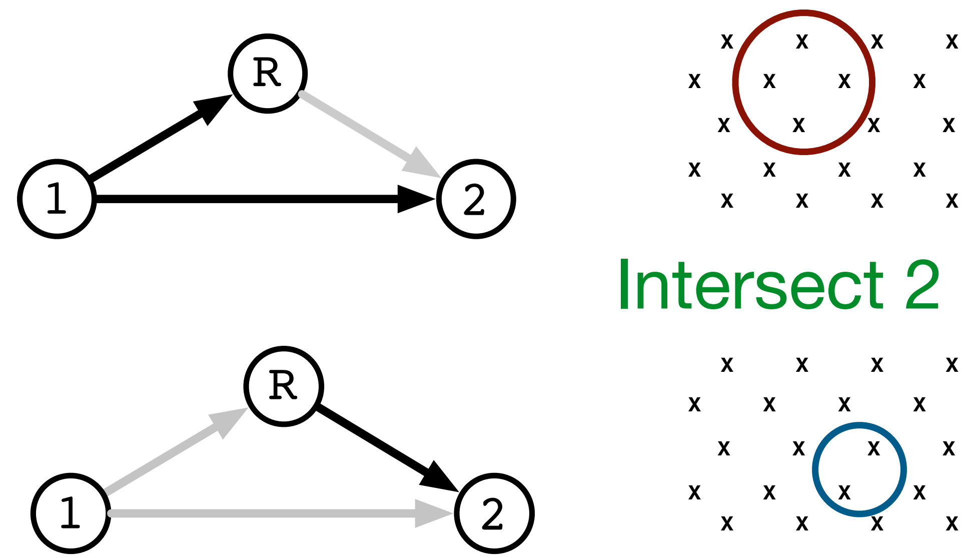
Lattice list decoder



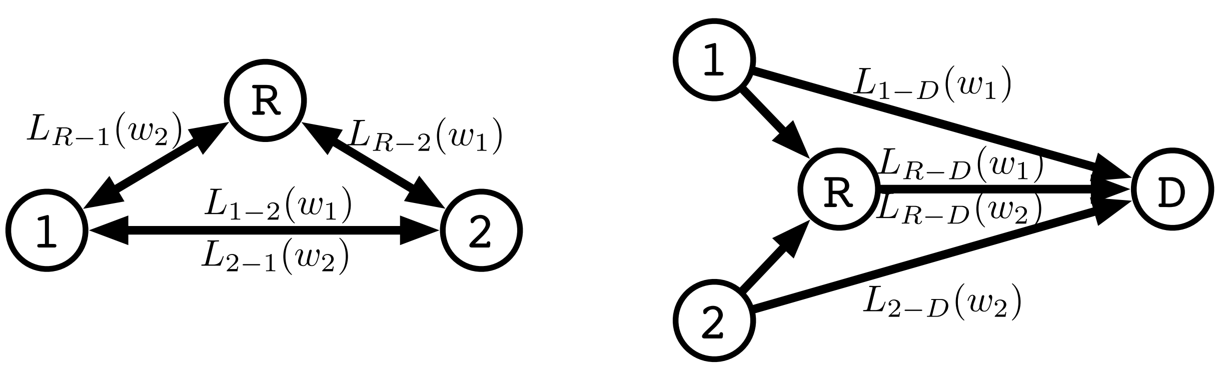
decode to 

rather than 

Lattices achieve DF rate

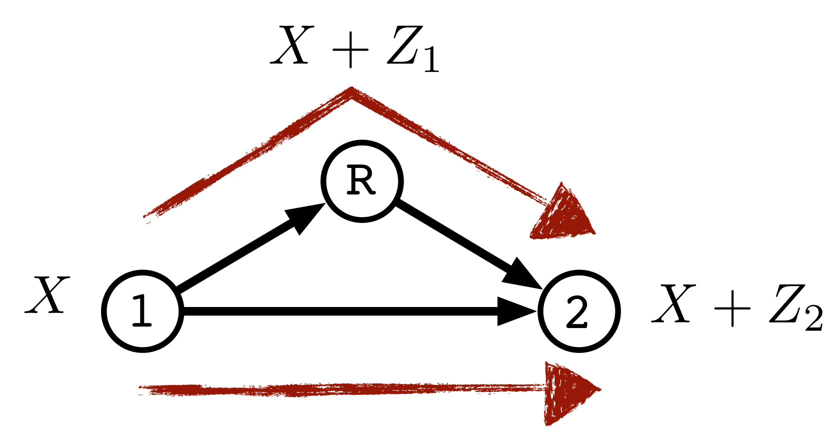


Lattices in multi-source networks



combine "decode sum"
and direct-link cooperation

Lattices achieve CF rate



lattices good source and channel codes, special Wyner-Ziv

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Conclusion

- can random codes be replaced by structured codes in Gaussian networks?
- capacity of relay channel? can structured codes help?
- list decoding - can help in explaining transmission above capacity?

Questions?

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