Message Lengths for Noisy Network Coding

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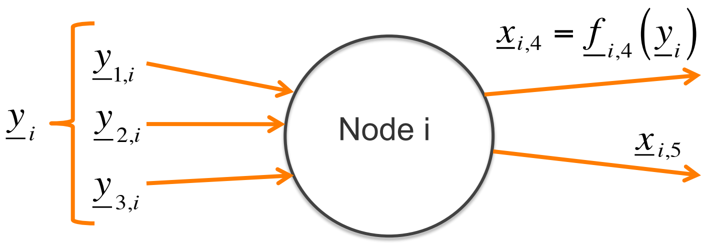
- **A serious problem with this talk**:
  No lattices! No finite field algebra!
- On the other hand: the methods directly link the fields of network coding (classic networks) and (classic) network information theory for general networks
- **Fun quote** on some topics of this workshop from David Slepian's Shannon Lecture "On Bandwidth" October 31,1974, Notre Dame University. Section "On Models and Reality": "Most of us would treat with great suspicion a model that predicts stable flight for an airplane if some parameter is irrational but predicts disaster if that parameter is a nearby rational number. Few of us would board a plane designed from such a model."

High-Level View of Network Coding

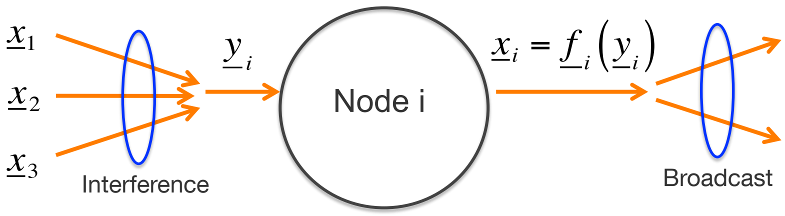
- Consider classic networks*
- For each edge (i,j), choose $\underline{f}_{i,j}(.)$ to "uniformly" map $\{\underline{y}_i\}$ to $\underline{x}_{i,j}$
- Linear <u>algebraic</u> or <u>random</u> coding is preferred, i.e., $\underline{f}_{i,j}(.)$ is a linear map ($\underline{x}_{i,j} = A_{i,j}\, \underline{y}_i$ where $A_{i,j}$ may be random) but these choices may not be best for non-multicast or wireless networks



* Ahlswede-Cai-Li-Yeung (2000)

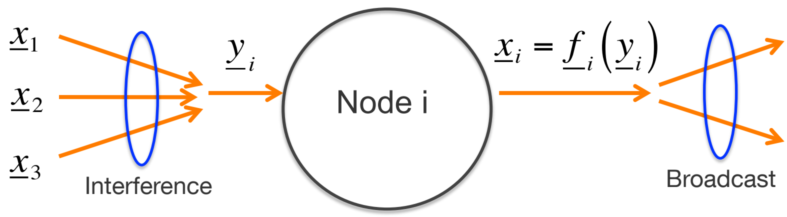
Network Coding for Wireless

- Nodes have interference and broadcast constraints (which may correspond to classic networks)
- For each node i, choose $\underline{f}_i(.)$ to "uniformly" map $\underline{y}_i$ to an $\underline{x}_i$
- Note: the symbols of $\underline{x}_i$ and $\underline{y}_i$ can be vectors. Non-linear $\underline{f}_i(.)$ needed in general (network & channel code). $\underline{x}_i$ are independent due to uniform map.



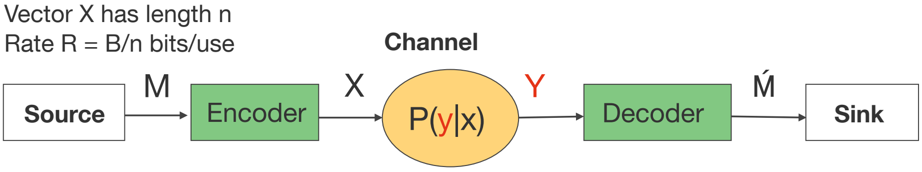
Open Problems

- Does linear network-coding (not channel coding) suffice for multicast? Likely yes (guess, speculation, conjecture?)
- How to co-ordinate network coding across nodes to achieve dependent $\underline{x}_i$? Can (dependent) algebraic/lattice structures play a role?



1) Channel and Source Coding: Notation

Message M has B bits
Vector X has length n
Rate R = B/n bits/use

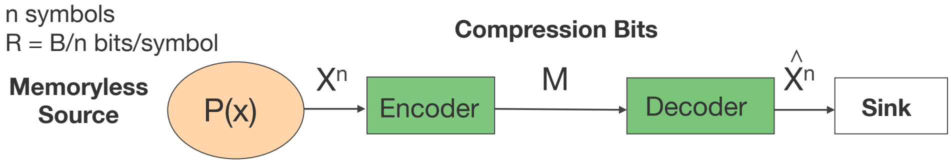


- Cost constraint: $\sum_i E[s(X_i, Y_i)] \leq nS$
- Problem: find the maximum R for reliable communications (small Pr[M≠Ḿ]) under cost constraint (e.g. energy $s(x,y)=x^2$)
- Shannon's Capacity-Cost Function:

$$C(S) = \max_{P(x)\,:\,E[s(X,Y)] \leq S} I(X;Y)$$

Source Coding

B compression bits
n symbols
R = B/n bits/symbol

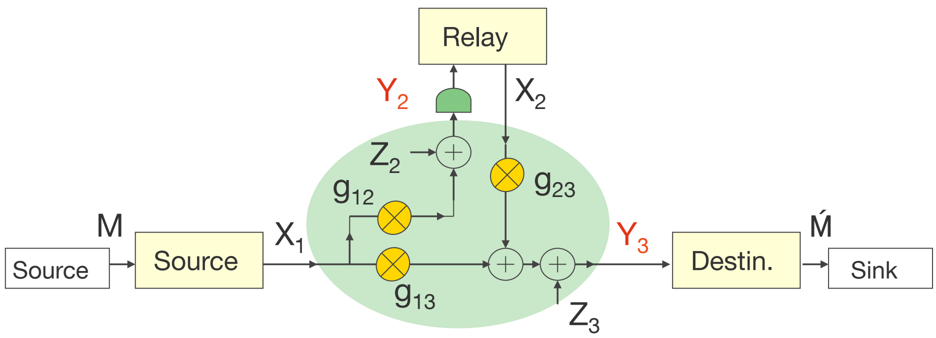


- Distortion constraint: $\sum_i E\left[d(X_i;\hat{X}_i)\right] \leq nD$
- Problem: find the minimum R under distortion constraint
- Shannon's Rate-Distortion Function:

$$R(D) = \min_{P_{\hat{X}|X}(.)\,:\,E[d(X,\hat{X})]\leq D} I(X;\hat{X})$$

2) Cooperative Relaying Strategies

- Single-Relay Channel (capacity an open problem)



- Gaussian noise Z_t (var. N), t=2,3
- Cost: $\Sigma_i |X_{ti}|^2/n \leq P_t$, t=1,2
- 2 Strategies: Amplify-Forward, Decode-Forward

Next: Review of Compression Strategies (all with i.i.d. Random Coding)

1) Classic Compress-Forward (CF), 1979
2) Quantize-Map-and-Forward (QF), 2007
3) Noisy Network Coding (NNC), 2010
4) Short-Message QF/NNC (SQF/SNNC), 2010
(also called "Cumulative encoding/block-by-block backward decoding")

1) Classic QF

	Block 1	Block 2
Source	$\underline{x}_{11}(m_1)$	$\underline{x}_{12}(m_2)$
Relay	$\underline{0}$	$\underline{x}_{22}(q)$
	$\hat{\underline{y}}_{21}(q)$	$\underline{0}$

- Relay quantizes $\underline{Y}_2$ to bits q representing $\hat{\underline{Y}}_2$ and transmits $\underline{x}_2(q)$
- Simple: use scalar quantization (good for high-rate quantization)
Better: use vector quantization with distortion D after canceling effect of X_2: $I(Y_2; \hat{Y}_2 \,|\, X_2) < R_Q(D)$
- Reliable transmission rate: $R_Q(D) < I(X_2; Y_3)$

Classic CF

	Block 1	Block 2
Source	$\underline{x}_{11}(m_1)$	$\underline{x}_{12}(m_2)$
Relay	$\underline{0}$	$\underline{x}_{22}(h(q))$
	$\underline{\hat{y}}_{21}(q)$	$\underline{0}$

- Improvement #1: relay hashes q (Wyner-Ziv coding)
  Quantization bound improves to: $I(Y_2; \hat{Y}_2|X_2Y_3) < R_Q(D)$
- Improvement #2: bursty transmission helps at low SNR, i.e., use high power for short time intervals. Formally take into account via a "time-sharing" random variable T.

CF Rate

	Block 1	Block 2
Source	$\underline{x}_{11}(m_1)$	$\underline{x}_{12}(m_2)$
Relay	$\underline{0}$	$\underline{x}_{22}(h(q))$
	$\underline{\hat{y}}_{21}(q)$	$\underline{0}$

- Final CF Rate*: $R < \max I(X_1; Y_3\hat{Y}_2|X_2T)$

  subject to $I(Y_2; \hat{Y}_2|X_2Y_3T) < I(X_2; Y_3|T)$
- Alternative expression** with a cut interpretation:

  $R < \max \min [\, I(X_1; \hat{Y}_2Y_3|X_2T),\ I(X_1X_2; Y_3|T) - I(Y_2; \hat{Y}_2|X_1\ X_2Y_3T)\,]$

*Cover-El Gamal (1979), **El Gamal-et al (2006)

2),3) QF/NNC*: Network Coding for Wireless and Beyond

	Block 1	Block 2
Source	$\underline{x}_{11}(m)$	$\underline{x}_{12}(m)$
Relay	$\underline{0}$	$\underline{x}_{22}(q)$
	$\underline{\hat{y}}_{21}(q)$	$\underline{0}$

- Source repetitively encodes a long message m
  Relay quantizes $\underline{Y}_{21}$ to bits q representing $\underline{\hat{Y}}_{21}$ and transmits $\underline{x}_{22}(q)$
  Destination decodes m and q jointly
- Advantage: extends to many relays and includes network coding
- Issues: long (en/de)coding delay, limited DF possibilities

* Avestimehr et al. (Allerton 2007), Lim et al. (ITW 2010)

4) SQF/SNNC*

	Block 1	Block 2
Source	$\underline{x}_{11}(m_1)$	$\underline{x}_{12}(m_2)$
Relay	$\underline{0}$	$\underline{x}_{22}(q)$
	$\underline{\hat{y}}_{21}(q)$	$\underline{0}$

- Recent result*: classic short messages m_1,m_2 and backward decoding achieves the same rates (see proof on next page)
- Minor generalization**: use long message m but hash m to short messages $h_1(m)$ and $h_2(m)$
- Advantage**: enable DF to improve reliability for slow fading

* Wu-Xie (2010), **Kramer-Hou (2011)

Proof of Equivalence for 1 Relay

- Consider a fixed coding distribution
- QF/NNC rate (long or short messages):

$$\max\{\, I(X_1; Y_3)^* ,\ \min [\, I(X_1; \hat{Y}_2 Y_3|X_2 T),\ I(X_1 X_2; Y_3|T) - I(Y_2; \hat{Y}_2|X_1\, X_2 Y_3 T) \,] \,\} \quad (1)$$

- Additionally for backward decoding if destination decodes X_2:

$$0 \le I(X_2; Y_3|X_1 T) - I(Y_2; \hat{Y}_2|X_1\, X_2 Y_3 T) \quad (2)$$

- Suppose (2) is violated. Subtract the negative of (2) (with strict inequality) from the 3rd expression in (1) to get

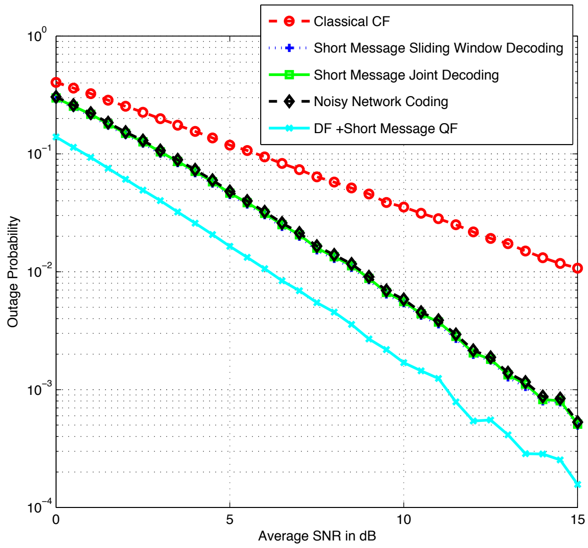
$$R < I(X_1; Y_3|T) \le I(X_1; Y_3)$$

- Proof method generalizes to many users and sources

* Destination treats X_2 as noise

Experiment

- Single-relay, ½ way between source and destination
- Attenuation exponent 3, slow Rayleigh fading, Gaussian noise
- Per-node power constraint
- Rate =1 bit/s/Hz
- SQF/SNNC gains 2 dB at outage prob. 10^{-3}
- Gains reduce for higher rates



Many Nodes, either Sources or Relays

	Block b	Block b+1
Source	$\underline{x}_{1b}(h_{1b}(m_1))$	$\underline{x}_{1(b+1)}(h_{1(b+1)}(m_1))$
Source/Relay k	$\underline{0}$	$\underline{x}_{k(b+1)}(h_{kb}(m_k),q_{kb})$
	$\hat{\underline{y}}_{kb}(q_{kb})$	$\underline{0}$

**Has a cut interpretation

- NNC properly extends Ahlswede-Cai-Li-Yeung network coding*
- Let node set S include the source but not the destination d. Let Ŝ be the complement of S. Then (S,Ŝ) is a cut. Achieve**

  $$R_S < \min_{(S,\hat{S})} I(X_S; \hat{Y}_{\hat{S}} Y_d | X_{\hat{S}} T) - I(Y_S; \hat{Y}_S | X_S X_{\hat{S}} \hat{Y}_{\hat{S}} Y_d T)$$

- SQF/SNNC achieves same rates (Wu-Xie 2010, Kramer-Hou 2011) and facilitates DF (Kramer-Hou 2011)

* Lim-Kim-El Gamal-Chung (ITW 2010)

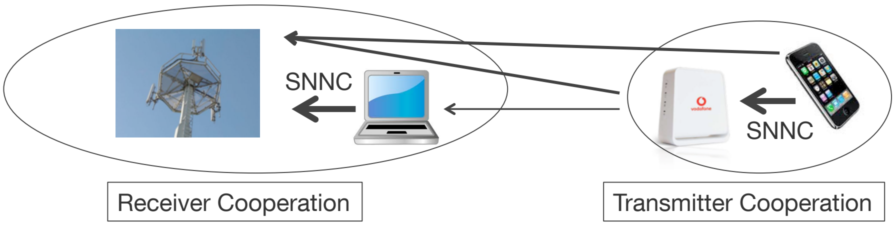
Discussion

- $R < \max \min_{(S,\hat{S})} I(X_S; \hat{Y}_{\hat{S}} Y_d | X_{\hat{S}} T) - I(Y_S; \hat{Y}_S | X_S X_{\hat{S}} \hat{Y}_{\hat{S}} Y_d T)$
- Deterministic (e.g. classic) networks: choose $\hat{Y}_i = Y_i$ and achieve cut-set bound with <u>independent</u> inputs
- Gaussian networks: choose $\hat{Y}_k = Y_k + \hat{Z}_k$, $\hat{Z}_k \sim CN(0,N_k)$, to get within 0.63|V| bits of the cut-set bound (here a true upper bound with <u>dependent</u> inputs)
- Problems inherent to long messages:
  - Encoding and decoding delays are large (latter problem also for joint or backward decoding)
  - Must hash w for reasonable modulation set sizes
  - Inflexible: relays cannot use multihop or DF

Application Question

Does QF/NNC have a practical future?

- relays can operate in a distributed and autonomous fashion
- achieves the "multi-output" gains of MIMO
- SQF/SNNC with DF achieves "multi-input" gains of MIMO
- method applies to more than radio, e.g., classic & optical networks
- Difficulty and Research: how to design practical codes and decoders?



Open Problems

- It's time to attack and solve the following problem (whose solution may or may not be difficult!):
  Find the capacity of deterministic relay networks with multicast
- Note: the source node controls everything, and the destination node can experiment with all (finite number of) possibilities
- Which techniques for classic networks extend to wireless? (Algebra, Grassmanians, etc?)
  First and practical step: separate channel and network coding