Equivalence for Rank-Metric and Matrix Codes with Applications to Network Coding

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Koetter and Kschischang show subspace codes are valuable for error correction of network coding.

- A **subspace code** is a non-empty collection $C$ of subspaces of $\mathbb{F}_q^n$.
- **Constant-dimension subspace codes**: all the codewords (subspaces) have fixed dimension $l$.
- The **subspace distance** between $U$ and $V$ is

$$d_S(U, V) = \dim(U + V) - \dim(U \cap V)$$
Subspace Code Construction

- **Matrix code:** A subset $T \subseteq \mathbb{F}_q^{l \times m}$.

- **Lifted matrix code:** A constant-dimension subspace code where all the RREF matrices corresponding to each codeword have the same pivot locations, and the non-pivot locations are filled by the entries of a matrix from a matrix code.
  
  E.g. $C = \{\text{rowspan}[I|A] : A \in T\}$ for some code $T \subseteq \mathbb{F}_q^{l \times m}$.

- Silva, Kschischang, and Koetter show that the subspace distance between $U = \text{rowspan}[I|A]$ and $V = \text{rowspan}[I|B]$ is
  
  $$d_S(U, V) = 2 \text{rank}(A - B)$$
Subspace Code Construction Cont’d

- **Rank-metric code:** a block code over $\mathbb{F}_{q^m}$, where each codeword $\mathbf{x}$ is associated with a matrix $\epsilon_B(\mathbf{x})$; row $i$ of $\epsilon_B(\mathbf{x})$ is the expansion of $x_i$ w.r.t. a fixed basis $B$ for $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.

- **Lifted rank-metric code:** lifting of the matrix expansion of a rank-metric code.

- The **rank-metric distance** between two vectors $\mathbf{x}$ and $\mathbf{y}$ is
  $$d_R(\mathbf{x}, \mathbf{y}) = \dim\langle \mathbf{x} - \mathbf{y} \rangle_{\mathbb{F}_q} = \text{rank}(\epsilon_B(\mathbf{x}) - \epsilon_B(\mathbf{y})).$$
Equivalence of Rank-Metric Codes

Any invertible $\mathbb{F}_{q^m}$-linear map $f : \mathbb{F}_{q^m}^n \to \mathbb{F}_{q^m}^n$ that preserves rank weight is called a rank-metric equivalence map.

**Theorem (Berger)**

The set of rank-metric equivalence maps $G_{RM}(\mathbb{F}_{q^m}^n)$ is generated by the non-zero $\mathbb{F}_{q^m}$-scalar multiplications and the linear group $GL_n(\mathbb{F}_q)$. The group is isomorphic to the product $(\mathbb{F}_{q^m}^*/\mathbb{F}_q^*) \times GL_n(\mathbb{F}_q)$.

**Note:** For $f \in G_{RM}(\mathbb{F}_{q^m}^n)$, we represent $f$ by an ordered pair $(\alpha, A)$ for some $\alpha \in \mathbb{F}_{q^m}^*$, $A \in GL_n(\mathbb{F}_q)$.

The rank-metric automorphism group $Aut_{RM}(C)$ of a code $C \subseteq \mathbb{F}_{q^m}^n$ is the set of rank-metric equivalence maps $f \in G_{RM}(\mathbb{F}_{q^m}^n)$ satisfying $f(C) = C$. 
The $[n, k, n-k+1]_{q^m}$ rank-metric code $C_{k,g,q^m}$ with generator matrix

$$G = \begin{bmatrix} g_1, & g_2, & \cdots, & g_n \\ g_1^{q^1}, & g_2^{q^1}, & \cdots, & g_n^{q^1} \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{q(k-1)}, & g_2^{q(k-1)}, & \cdots, & g_n^{q(k-1)} \end{bmatrix},$$

where the entries of $g = [g_1, \ldots, g_n] \in \mathbb{F}_{q^m}^n$ are linearly independent over $\mathbb{F}_q$, is called a Gabidulin code.

Gabidulin codes are $q^m$-ary analogues of Reed-Solomon codes that are optimal for the rank metric.

Used in the first subspace code construction by Koetter and Kschischang; also used in the GPT public-key cryptosystem.
Theorem

Let $k \leq n \leq m$. Let $g = [g_1, \ldots, g_n] \in \mathbb{F}_{q^m}^n$ have entries that are linearly independent over $\mathbb{F}_q$, and let $C_{k,g,q^m}$ be the Gabidulin code of dimension $k$ generated by $g$. Let $d$ be the largest integer such that $\langle g_1, \ldots, g_n \rangle_{\mathbb{F}_q}$ is a vector space over $\mathbb{F}_{q^d} \subseteq \mathbb{F}_{q^m}$. Then

1. $d$ divides $\gcd(n, m)$.
2. $\text{Aut}_{RM}(C_{k,g,q^m}) = \left\{ \left( \alpha, \epsilon_g ([\beta g_1, \ldots, \beta g_n])^\top \right) : \alpha \in \mathbb{F}_{q^m}^*, \beta \in \mathbb{F}_{q^d}^* \right\}$. 
A matrix-equivalence map is an invertible $\mathbb{F}_q$-linear map $f : \mathbb{F}^{n \times m}_q \to \mathbb{F}^{n \times m}_q$ that preserves rank weight.

**Theorem**

Let $f \in G_{\text{Mat}}(\mathbb{F}^{n \times m}_q)$ be a matrix-equivalence map. If $n \neq m$, then there exist $A \in \text{GL}_n(\mathbb{F}_q)$, $B \in \text{GL}_m(\mathbb{F}_q)$ such that

- $f(M) = AMB$ for all $M \in \mathbb{F}^{n \times m}_q$.

If $n = m$, then there exist $A, B \in \text{GL}_n(\mathbb{F}_q)$ such that either

- $f(M) = AMB$ for all $M \in \mathbb{F}^{n \times m}_q$, or
- $f(M) = AM^\top B$ for all $M \in \mathbb{F}^{n \times m}_q$.

**Note:** When $n \neq m$,

$$G_{\text{Mat}}(\mathbb{F}^{n \times m}_q) \cong \text{GL}_n(\mathbb{F}_q) \times \text{PGL}_m(\mathbb{F}_q),$$

and so we can choose a representative for $f \in G_{\text{Mat}}(\mathbb{F}^{n \times m}_q)$ of the form $(A, B)$ where $A \in \text{GL}_n(\mathbb{F}_q)$ and $B \in \text{GL}_m(\mathbb{F}_q)$. 
Matrix-Automorphism Group of Gabidulin Codes

The matrix-automorphism group $\text{Aut}_{\text{Mat}}(C)$ of a code $C \subseteq \mathbb{F}_q^{n \times m}$ is the set of matrix-equivalence maps that fix $C$.

**Theorem**

Let $k \leq n < m$ and $\mathcal{B} = \{b_1, \ldots, b_m\}$ be a basis for $\mathbb{F}_q^m$ over $\mathbb{F}_q$. Let $g = [g_1, \ldots, g_n] \in \mathbb{F}_q^n$ have entries that are linearly independent over $\mathbb{F}_q$, and let $\epsilon_{\mathcal{B}}(C_k, g, q^m)$ be the matrix expansion of the Gabidulin code of dimension $k$ generated by $g$. Let $d$ be maximal such that $\langle g_1, \ldots, g_n \rangle_{\mathbb{F}_q}$ is a vector space over $\mathbb{F}_{q^d} \subseteq \mathbb{F}_q^m$. Then

1. $d$ divides $\gcd(n, m)$.
2. $\text{Aut}_{\text{Mat}}(\epsilon_{\mathcal{B}}(C_k, g, q^m)) \supseteq \left\{ (\epsilon_g([\alpha g_1, \ldots, \alpha g_n]), \epsilon_{\mathcal{B}}([\beta b_1, \ldots, \beta b_m])) : \alpha \in \mathbb{F}^{*d}_q, \beta \in \mathbb{F}^{*m}_{q^d} \right\}$.
Determine if either the matrix equivalence maps provide better protection against cryptanalysis than the permutation equivalence map currently used in the GPT public-key cryptosystem.

Use these notions of equivalence to enumerate all inequivalent self-dual matrix codes.

Extend the notion of equivalence to subspace codes and determine the automorphism groups of various families of subspace codes.