Schubert Calculus and its Relation to Network Coding

Algebraic Structure in Network Information Theory

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Grassmann variety and subspace codes

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Question: Why is $\mathrm{Grass}(k, V)$ a variety?

Plücker Embedding

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$$\begin{aligned} \varphi \ : \qquad \mathrm{Grass}(k, V) &\longrightarrow \mathbb{P}(\wedge^k V) \\ \mathrm{span}(v_1, \ldots, v_k) &\longmapsto \mathbb{F} v_1 \wedge \cdots \wedge v_k . \end{aligned} \tag{1}$$

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Assume

$$v_i = \sum_{j=1}^{n} a_{ij} e_j, \; i = 1, \dots, k.$$

Let A be the $k \times n$ matrix $(a_{i,j})$. The Plücker embedding writes:

$$\begin{array}{rrcl} \varphi \; : & Mat_{k\times n} & \longrightarrow & \mathbb{P}(\wedge^k V) \\ & \mathrm{rowspace}(A) & \longmapsto & \displaystyle\sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1,\dots,i_k} \cdot e_{i_1} \wedge \ldots \wedge e_{i_k}. \end{array} \quad (2)$$

The coordinates $x_{\underline{i}} := x_{i_1,\dots,i_k}$ are called the Plücker coordinates of $\mathrm{rowspace}(A)$.

Shuffle Relations

Theorem

$$\sum_{\lambda=1}^{k+1} (-1)^{\lambda} \cdot x_{i_1,\ldots,i_{k-1},j_\lambda} \cdot x_{j_1,\ldots,\hat{j_\lambda},\ldots,j_{k+1}} = 0 \tag{3}$$

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Example

$\mathrm{Grass}(2,\mathbb{F}^4)$ is embedded in $\mathbb{P}^5$ and $\varphi(Grass(2,4))$ is described by a single relation

$$x_{12}x_{34}-x_{13}x_{24}+x_{14}x_{23}=0 \qquad (4)$$

Shuffle Relations

Example

Grass$(2, \mathbb{F}^5)$ is embedded in $\mathbb{P}^9$ and the defining relations are:

$$
\begin{aligned}
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} &= 0 \\
x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} &= 0 \\
x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{14} &= 0 \\
x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} &= 0 \\
x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} &= 0
\end{aligned}
$$

Importance in Network Coding

Metric on Grassmannian: If $U, W \in \mathrm{Grass}(k, V)$ are two subspaces one defines its distance as:

$$d(U, W) := \dim(U + W) - \dim(U \cap W).$$

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Remark

The ball of radius t around the subspace W defines a so called Schubert variety:

$$\{U \in \mathrm{Grass}(k, V) \mid d(U, W) \leq t\}$$

Geometric Questions of Schubert

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Answer Schubert: By Poncelet's principle of conservation of numbers we can assume lines 1 and 2 intersect and lines 3 and 4 intersect. So there are 2 solutions in general.

Geometric Questions of Schubert

Theorem (Schubert [Sch79])

Given $N := k(n-k)$ linear subspace $U_i, i = 1, \ldots, N$ in V having dimension k each. If the base field $\mathbb{F}$ is algebraically closed and the subspaces are in general position then there exist exactly

$$\frac{1!2!\cdots(k-1)!(N)!}{(n-k)!(n-k+1)!\cdots(n-1)!} \tag{5}$$

subspaces W of dimension $(n-k)$ intersecting each of the subspaces U_i nontrivially.

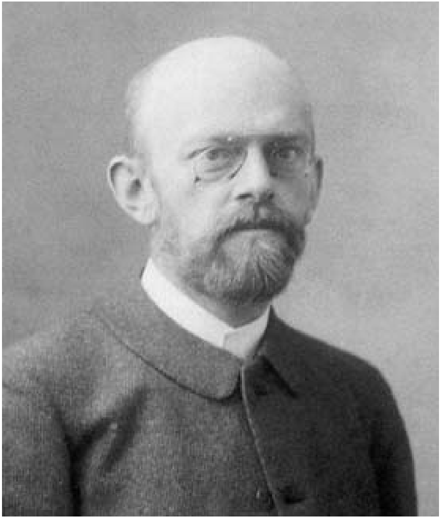
Hermann Cäsar Hannibal Schubert (1848-1911)

Hilbert Problem Number 15, Paris 1900

Rigorous foundation of Schubert's enumerative calculus

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.

Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.



David Hilbert (1862-1943)

Schubert Varieties

Definition

A flag $\mathscr{F}$ is a sequence of nested subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_n = V \tag{6}$$

where we assume that $\dim V_i = i$ for $i = 1, \ldots, n$.

Let $\underline{i} = (i_1, \ldots, i_k)$ denote a sequence of numbers having the property that

$$1 \leq i_1 < \ldots < i_k \leq n. \tag{7}$$

Definition

For each flag $\mathscr{F}$ and each multiindex $\underline{i}$

$$C(\underline{i}; \mathscr{F}) := \{W \in \mathrm{Grass}(k, V) \mid \dim(W \bigcap V_{i_s}) = s\}$$

is called a Schubert cell.

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Remark

The closure of the cell $C(\underline{i};\mathscr{F})$ is the Schubert variety $S(\underline{i};\mathscr{F})$. The equations describing the variety $S(\underline{i};\mathscr{F})$ consists of the quadratic equations describing the Grassmann variety and some additional linear equations.

Schubert Varieties

Remark

If $\{e_1, \ldots, e_n\}$ is a basis of V and $\mathscr{F}$ is the standard flag with respect to this basis then $C(\underline{i}; \mathscr{F})$ consists of all subspaces having a certain row reduced echelon form:

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots & * & \cdots & * & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Schubert Calculus

Theorem

For every fixed flag $\mathscr{F}$ the Schubert cells $C(\underline{i};\mathscr{F})$ decompose the Grassmann variety $\mathrm{Grass}(k,\mathbb{C}^n)$ into a finite cellular CW–complex. The integral homology $H_{2m}(\mathrm{Grass}(k,\mathbb{C}^n),\mathbb{Z})$ has no torsion and is freely generated by the fundamental classes of the Schubert varieties $S(\underline{i};\mathscr{F})$ of real dimension $2m$.

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The Poincaré-dual of the class $(i_1,\ldots,i_k)$ will be denoted by

$$\{\mu_1,\ldots,\mu_k\} := \{n-k-i_1+1, n-k-i_2+2,\ldots,n-i_k\}. \qquad (8)$$

viewed as an element of the cohomolgy ring
$H^*(\mathrm{Grass}(k,\mathbb{C}^n),\mathbb{Z})$.

Schubert Calculus

The cohomology ring

$$H^*(\mathrm{Grass}(k,\mathbb{C}^n),\mathbb{Z}) := \bigoplus_{m=0}^{k(n-k)} H^{2m}(k,\mathbb{C}^n),\mathscr{Z}) \tag{9}$$

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$$\sigma_j \;:=\; \{j,0,\ldots,\ldots,0\} \quad j=1,\ldots,n-k. \tag{10}$$

denotes the *j*th Chern class.

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Pieri's formula:

$$\{\mu_1, \ldots, \mu_k\} \cdot \sigma_j = \sum_{\substack{\mu_{i-1} \geq \nu_i \geq \mu_i \\ \sum_{i=1}^{k} \nu_i = (\sum_{i=1}^{k} \mu_i) + j}} \{\nu_1, \ldots, \nu_k\}$$

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Giambelli's formula:

$$\{\mu_1, \dots, \mu_k\} \quad = \quad = \det \begin{pmatrix} \sigma_{\mu_1} & \sigma_{\mu_1+1} & \cdots & \sigma_{\mu_1+k-1} \\ \sigma_{\mu_2-1} & \sigma_{\mu_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{\mu_k-k+1} & & \cdots & \sigma_{\mu_k} \end{pmatrix}$$

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Algebraic Problem: One has the equation of $\mathrm{Grass}(2,\mathbb{F}^4)$:

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together with 4 linear equations describing the 4 Schubert varieties. $\longmapsto$ 2 Solutions.

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Cohomology ring:

$$\{1,0\}\{1,0\}\{1,0\}\{1,0\} = 2\{2,2\}.$$

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