Schubert Calculus and its Relation to Network Coding
Algebraic Structure in Network Information Theory

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Grassmann variety and subspace codes

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**Definition**

The Grassmann variety $\text{Grass}(k, V)$ is the set of all $k$-dimensional subspaces $U \subset V$. 

**Remark**

A subset $C \subset \text{Grass}(k, V)$ can be viewed as a constant dimensional linear network code.
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A subset $\mathcal{C} \subset \text{Grass}(k, V)$ can be viewed as a constant dimensional linear network code.

**Question:** Why is $\text{Grass}(k, V)$ a variety?
Consider the vector space of alternating $k$–tensors $\wedge^k V$. Let $\mathbb{P}(\wedge^k V)$ be the projective space consisting of all lines in $\wedge^k V$. 

Plücker Embedding
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$$\varphi : \text{Grass}(k, V) \rightarrow \mathbb{P}(\wedge^k V)$$

$$\text{span}(v_1, \ldots, v_k) \mapsto Fv_1 \wedge \cdots \wedge v_k.$$
Plücker Coordinates

Let \( \{ e_1, \ldots, e_n \} \) be a basis of \( V \).
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Then

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\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n \}
\]
is a basis of \( \wedge^k V \).
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Assume
\[
v_i = \sum_{j=1}^{n} a_{ij} e_j, \quad i = 1, \ldots, k.
\]

Let \( A \) be the \( k \times n \) matrix \( (a_{i,j}) \). The Plücker embedding writes:

\[
\varphi : \quad \text{Mat}_{k \times n} \longrightarrow \mathbb{P}(\wedge^k V) \quad (2)
\]

\[
\text{rowspace}(A) \quad \mapsto \quad \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1, \ldots, i_k} \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}.
\]

The coordinates \( x_{i_1, \ldots, i_k} := x_{i_1, \ldots, i_k} \) are called the Plücker coordinates of \( \text{rowspace}(A) \).
Shuffle Relations

**Theorem**

\[
\frac{1}{k+1} \sum_{\lambda} (-1)^\lambda \cdot x_{i_1, \ldots, i_{k-1}, j_{\lambda}} \cdot x_{j_1, \ldots, \hat{j}_{\lambda}, \ldots, j_{k+1}} = 0
\]  

(3)

describes the image of the Grassmannian in the projective space \( \mathbb{P}(\wedge^k V) \)

Example

\( \text{Grass}(2, 4) \) is embedded in \( \mathbb{P}(5) \) and \( \phi (\text{Grass}(2, 4)) \) is described by a single relation

\[
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0
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\sum_{\lambda=1}^{k+1} (-1)^{\lambda} \cdot x_{i_1, \ldots, i_{k-1}} \cdot j_{\lambda} \cdot x_{j_1, \ldots, j_{k+1}} = 0
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**Example**

Grass(2, \( \mathbb{F}^4 \)) is embedded in \( \mathbb{P}^5 \) and \( \varphi(\text{Grass}(2, 4)) \) is described by a single relation

\[
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0
\]  

(4)
Shuffle Relations

Example

Grass$(2, \mathbb{F}^5)$ is embedded in $\mathbb{P}^9$ and the defining relations are:

\begin{align*}
  x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} &= 0 \\
  x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} &= 0 \\
  x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24} &= 0 \\
  x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} &= 0 \\
  x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} &= 0
\end{align*}
Importance in Network Coding

**Metric on Grassmannian:** If $U, W \in \text{Grass}(k, V)$ are two subspaces one defines its distance as:

$$d(U, W) := \dim(U + W) - \dim(U \cap W).$$
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**Question:** What is the algebraic structure of the balls of radius $t$ around an element $W \in \text{Grass}(k, V)$?
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**Answer:** $d(U, W) \leq t$ if and only if $\dim(U \cap W) \geq k - t/2 =: r$. 

**Remark** The ball of radius $t$ around the subspace $W$ defines a so-called Schubert variety:

$$\{ U \in \text{Grass}(k, V) \mid d(U, W) \leq t \}.$$
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Answer Schubert: By Poncelet’s principle of conservation of numbers we can assume lines 1 and 2 intersect and lines 3 and 4 intersect. So there are 2 solutions in general.
Theorem (Schubert [Sch79])

Given \( N := k(n - k) \) linear subspace \( U_i, i = 1, \ldots, N \) in \( V \) having dimension \( k \) each. If the base field \( \mathbb{F} \) is algebraically closed and the subspaces are in general position then there exist exactly

\[
\frac{1!2! \cdots (k-1)! (N)!}{(n-k)!(n-k+1)! \cdots (n-1)!}
\]

subspaces \( W \) of dimension \( (n-k) \) intersecting each of the subspaces \( U_i \) nontrivially.
Hermann Cäsar Hannibal Schubert (1848-1911)
The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him. Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.
Schubert Varieties

**Definition**

A flag $\mathcal{F}$ is a sequence of nested subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_n = V$$

where we assume that $\dim V_i = i$ for $i = 1,\ldots,n$.

Let $\underline{i} = (i_1,\ldots,i_k)$ denote a sequence of numbers having the property that

$$1 \leq i_1 < \ldots < i_k \leq n.$$  \hspace{1cm} (7)

**Definition**

For each flag $\mathcal{F}$ and each multiindex $\underline{i}$

$$C(\underline{i}; \mathcal{F}) := \{W \in \text{Grass}(k, V) \mid \dim(W \cap V_{i_s}) = s\}$$

is called a Schubert cell.
Schubert Varieties

**Definition**

For each flag $\mathcal{F}$ and each multiindex $i$

$$S(i; \mathcal{F}) := \{ W \in \text{Grass}(k, V) \mid \dim(W \cap V_i) \geq s \}$$

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**Remark**

The closure of the cell $C(i; \mathcal{F})$ is the Schubert variety $S(i; \mathcal{F})$. The equations describing the variety $S(i; \mathcal{F})$ consists of the quadratic equations describing the Grassmann variety and some additional linear equations.
Remark

If \( \{ e_1, \ldots, e_n \} \) is a basis of \( V \) and \( \mathcal{F} \) is the standard flag with respect to this basis then \( C(i; \mathcal{F}) \) consists of all subspaces having a certain row reduced echelon form:

\[
\begin{bmatrix}
\ast & \cdots & \ast & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\ast & \cdots & \ast & 0 & \ast & \cdots & \ast & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ast & \cdots & \ast & 0 & \ast & \cdots & \ast & 0 & \cdots & \ast & \cdots & \ast & 1 & 0 & \cdots & 0
\end{bmatrix}
\]
For every fixed flag $\mathcal{F}$ the Schubert cells $C(i; \mathcal{F})$ decompose the Grassmann variety $\text{Grass}(k, \mathbb{C}^n)$ into a finite cellular CW–complex. The integral homology $H_{2m}(\text{Grass}(k, \mathbb{C}^n), \mathbb{Z})$ has no torsion and is freely generated by the fundamental classes of the Schubert varieties $S(i; \mathcal{F})$ of real dimension $2m$. 

\[ \{\mu_1, \ldots, \mu_k\} := \{n - k - i_1 + 1, n - k - i_2 + 2, \ldots, n - i_k\}. \]
### Theorem

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The Poincaré-dual of the class $(i_1, \ldots, i_k)$ will be denoted by

$$\{\mu_1, \ldots, \mu_k\} := \{n - k - i_1 + 1, n - k - i_2 + 2, \ldots, n - i_k\}. \quad (8)$$

viewed as an element of the cohomology ring $H^*(\text{Grass}(k, \mathbb{C}^n), \mathbb{Z})$. 

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**Schubert Calculus**
The cohomology ring

\[ H^*(\text{Grass}(k, \mathbb{C}^n), \mathbb{Z}) := \bigoplus_{m=0}^{k(n-k)} H^{2m}(k, \mathbb{C}^n), \mathbb{Z} \]  

has in a natural way the structure of a graded ring.
The cohomology ring

\[ H^*(\text{Grass}(k, \mathbb{C}^n), \mathbb{Z}) := \bigoplus_{m=0}^{k(n-k)} H^{2m}(k, \mathbb{C}^n), \mathbb{Z} ) \] (9)

has in a natural way the structure of a graded ring.

\[ \sigma_j := \{j, 0, \ldots, 0\} \quad j = 1, \ldots, n-k. \] (10)

denotes the \( j \)th Chern class.
Schubert Calculus

Computations in the cohomology ring are done by the classical formulas of Pieri and Giambelli and by the Littlewood Richardson rule:

Pieri's formula:

\[ \{\mu_1, \ldots, \mu_k\} \cdot \sigma_j = \sum_{\mu_i - 1 \geq \nu_i \geq \mu_i} \sum_{k_i=1}^{\nu_i=\sum_{k_i=1}^{k-1}} \nu_i + j \{\nu_1, \ldots, \nu_k\} \]

Giambelli's formula:

\[ \{\mu_1, \ldots, \mu_k\} = \det \begin{bmatrix} \sigma_{\mu_1} & \sigma_{\mu_1+1} & \ldots & \sigma_{\mu_1+k-1} \\ \sigma_{\mu_2-1} & \sigma_{\mu_2} & \ldots & \sigma_{\mu_2+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\mu_k-k+1} & \sigma_{\mu_k-k+2} & \ldots & \sigma_{\mu_k} \end{bmatrix} \]
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Giambelli’s formula:

$$\begin{align*}
\{\mu_1, \ldots, \mu_k\} &= \det \begin{bmatrix}
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\sigma_{\mu_1+1} & \sigma_{\mu_1+2} & \cdots & \sigma_{\mu_k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{\mu_k} & \sigma_{\mu_k+1} & \cdots & \sigma_{\mu_k+k-1}
\end{bmatrix} \\
&= \det \begin{bmatrix}
\sigma_{\mu_1} & \sigma_{\mu_1+1} & \cdots & \sigma_{\mu_k} \\
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\{\mu_1, \ldots, \mu_k\} \cdot \sigma_j = \sum_{\mu_i - 1 \geq v_i \geq \mu_j} \sum_{i=1}^{k} v_i = (\sum_{i=1}^{k} \mu_i) + j \sum_{i=1}^{k} \sigma_{\mu_i} - \sigma_{\mu_i+1} \cdots \sigma_{\mu_i+k-1}
\]

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\sigma_{\mu_2-1} & \sigma_{\mu_2} & \cdots & \\
\vdots & \ddots & \ddots & \\
\sigma_{\mu_k-k+1} & \cdots & \sigma_{\mu_k}
\end{pmatrix}
\]
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Algebraic Problem: One has the equation of $\text{Grass}(2, \mathbb{F}^4)$:

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together with 4 linear equations describing the 4 Schubert varieties. $\longrightarrow$ 2 Solutions.
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**Cohomology ring:**

$$\{1, 0\}\{1, 0\}\{1, 0\}\{1, 0\} = 2\{2, 2\}.$$


