Structure in Some Multiterminal Problems

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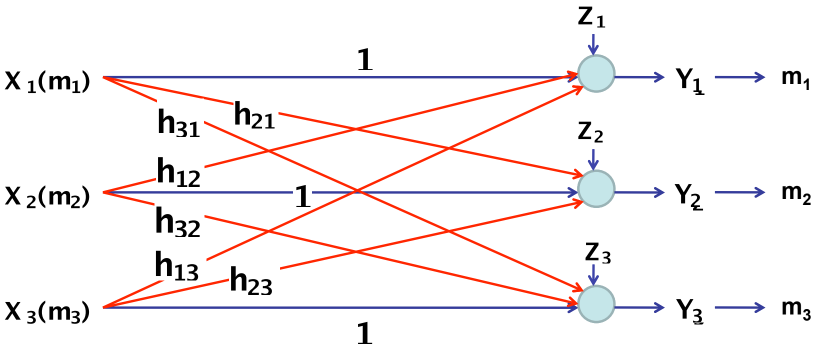
- Lattice foundations: Sloane, Conway, Rimoldi, Urbanke, Loeliger, Erez, Zamir, Shamai, Pradhan and many more...
- Interference foundations: Ahlswede, Han, Kobayashi, El Gamal, Cover, Sason, Sato, Kramer, Tse, Etkin, Avestimehr, Motani, Jafar, Khandani, Maddah-Ali, Cadambe, Niesen, Devroye, Shamai, Goldsmith, Viswanath and many many others.....
- Lattice application foundations: Nazer, Gastpar, Zamir, Erez, Pradhan, Shamai, Bresler, Narayanan, Yener, He, Chung, El Gamal, Caire and growing.....

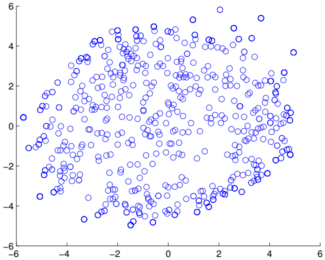
Channel Coding: Interference



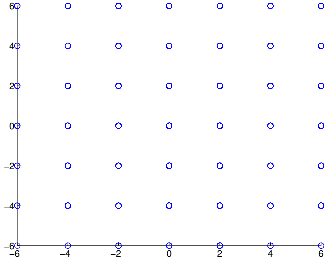
Three User Interference channel – What is the capacity region?

Lattice codes vs Single-letterized Uniform/Gaussian Random codes

- sum of signals/lattices
- Fewer distinct sums of lattice codewords than random points in n-D space



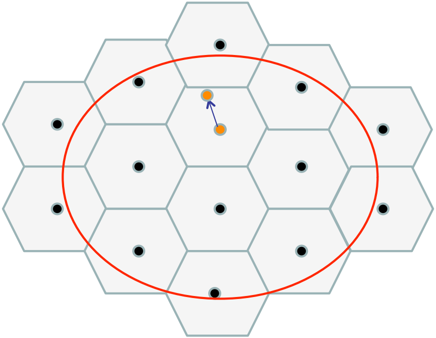
(a) Random - 108 points



(b) Lattice - 49 points

Figure: Sum of two sets of vectors in 2-D

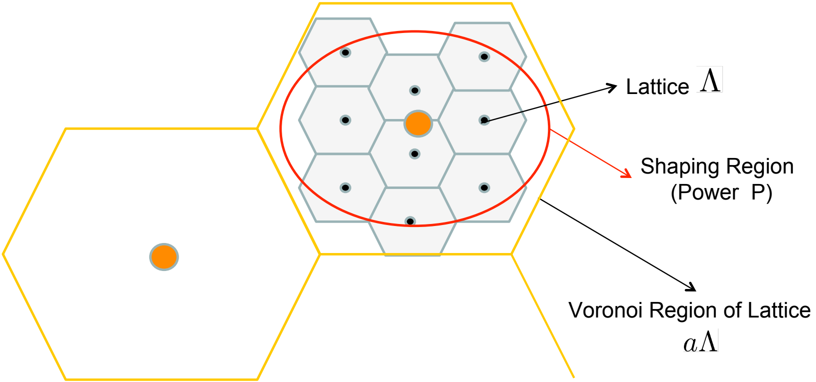
Lattice Preliminaries



Additive noise translates lattice point

- $\Lambda = \{x = z\mathbf{G} : z \in \mathbb{Z}^n, \mathbf{G} \in \mathbb{R}^{n \times n}\}$
- Quantization - $Q_\Lambda(x) = \text{argmin}_{r \in \Lambda} \|x - r\|$
- Fundamental Voronoi region - $\mathcal{V}_0 = \{x \in \mathbb{R}^n : Q - \Lambda(x) = 0\}$
- $x \bmod \Lambda = x - Q_\Lambda(x)$

Very Stong Interference: A 2-level Matryoshka Doll



Works only for all cross gains $h_{ij} = \Omega(\sqrt{P})$ and rationally related

More general: Multilevel Matryoshkas!

If all cross-channel gains are "a", then

Theorem

If $a^2 > 2$, then sum rate $\geq 3\log\frac{a^2-1}{2a^4-a^2}\log SNR$

Theorem

If $a^2 < 1/3$, then sum rate $\geq 3\frac{\log(1-a^2)-\log(2a^2)}{log(1+a^2)-log(2a^4)}\log SNR$

Dead End?

- Any improvements possible?
- Yes, using a transformation to a noiseless n-dimenstion channel

Theorem

The **central dogma** *of lattices:*

- *Let* X_1, X_2, X_3 *all be lattice points*
- $Y_1 = X_1 + h_{21}X_2 + h_{31}X_3 + N_1$ *is a corrupted lattice point.*
- *Recover* $Z_1 = X_1 + h_{21}X_2 + h_{31}X_3 + N_1$. *Now,* Y_1 *to* Z_1 *is a discrete memoryless noiseless channel.*
- *Use Algebra and geometry to find codebooks to maximize rate*

- One dimensional noisy to n-dimensional noiseless

..For emphasis... repeat!

- If $\mathcal{L}$ is the subset of lattice points ("input alphabet", not codebook)
- Then output alphabet at Receiver 1 is $\mathcal{L} + h_{21}\mathcal{L} + h_{31}\mathcal{L}$
- Real multiplication and Minkowski-sum
- Similar alphabets at other receivers
- Use Algebraic (geometric) coding to design maximally separable codebook, vectors of lattice points

Is this Useful?

Theorem

All cross-gains are a for simplicity, then symmetric rate of

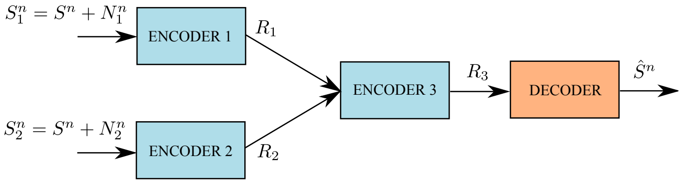
$$\frac{1}{2}\left(1 - \frac{\log(K-1)}{\log a}\right)\log SNR$$

is achievable, where K is users in system.

- Use a code, not necessarily linear, over the lattice
- For example, symbol alignment can be superposed on lattices
- Rate $=$ (rational DoF) *$\log SNR$

Multiterminal Source Coding

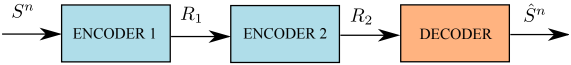
Multiterminal Source Coding Through a Relay



What are efficient coding schemes?

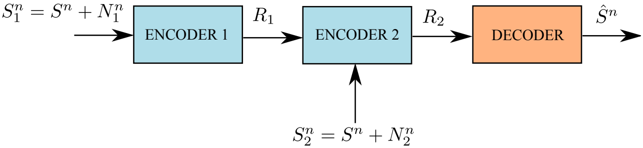
Simple Relay Model

Relay only needs to forward message from Encoder 1



Relays with Side Information

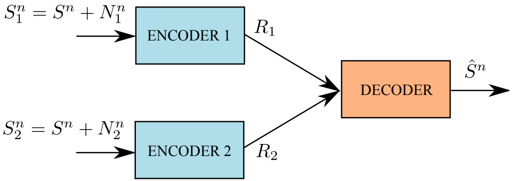
Tension between whether relay should forward message from Encoder 1 or decode and compress desired function



Multiterminal Source Coding

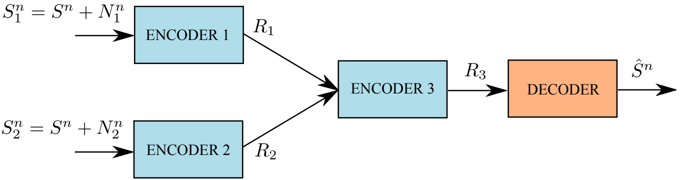
Quantize and bin architecture based on random codes is optimal for

- Quadratic Gaussian CEO Problem - [Oohama], [PrabhakaranTseRamchandran]
- Gaussian Two Terminal Source Coding - [WagnerTavildarViswanath]



Are random codes optimal for multiterminal source coding through relays?

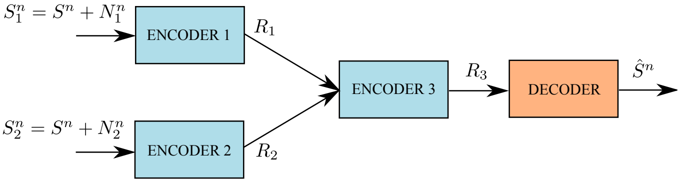
System Model



- $S \sim \mathcal{N}(0, \sigma_S^2)$, $N_1 \sim \mathcal{N}(0, \sigma_{N_1}^2)$, $N_2 \sim \mathcal{N}(0, \sigma_{N_2}^2)$
- Distortion constraint - $\sum_{i=1}^{n} \mathbb{E}\left[(S_i - \hat{S}_i)^2\right] \leq D$
- Find the set of all achievable tuples (R_1, R_2, R_3, D)

Key Questions

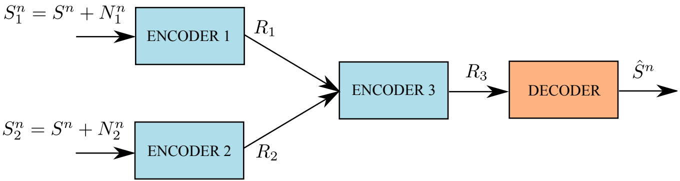
How to integrate incoming messages?



- Forward messages
- Reconstructing linear function requires further compression
- Compute and forward - Directly compute linear function of codewords and forward
- Compress and forward - Estimate desired linear function and compress

Main Results

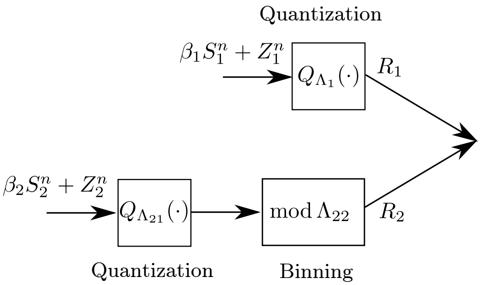
Given a distortion constraint D



- for symmetric noise variances, compute and forward achieves optimum $R_1 + R_2$ and within $1/2$ bit of optimum R_3
- can further reduce R_3 at at cost of higher $R_1 + R_2$ using compress and forward

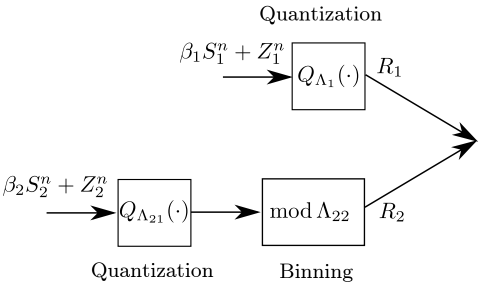
Compute and Forward - Lattice codes

- $\mathbb{E}[S|S_1, S_2] = \beta_1 S_1 + \beta_2 S_2$ is the linear function to be compressed
- Λ_1, Λ_{21}, Λ_{22} are ‘good’ lattices with suitable nesting structure



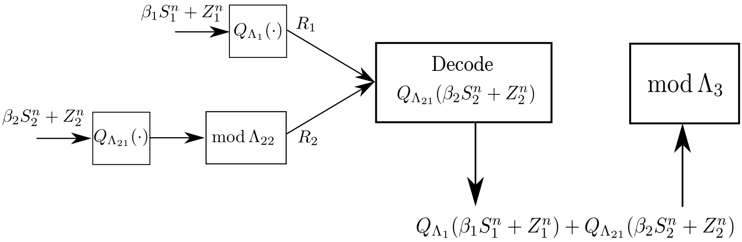
Compute and Forward - Lattice codes

- $Z_1^n, Z_2^n \sim \mathrm{Unif}(\mathcal{V}_0)$ - dithers for quantization
- Λ_1, Λ_{21} - Quantization at Encoder 1 and 2, $R_1 = I(S_1; U_1)$
- Λ_{22} - Binning to achieve $R_2 = I(S_2; U_2|U_1)$



Compute and Forward - Lattice codes

- Λ_3 - 'good' channel coding lattice
- Λ_3 - useful for analysis, not used for quantization
- Lattice points summed up at relay to compute function



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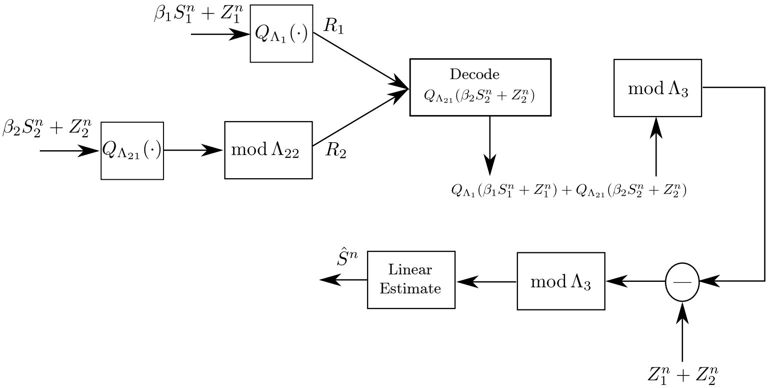
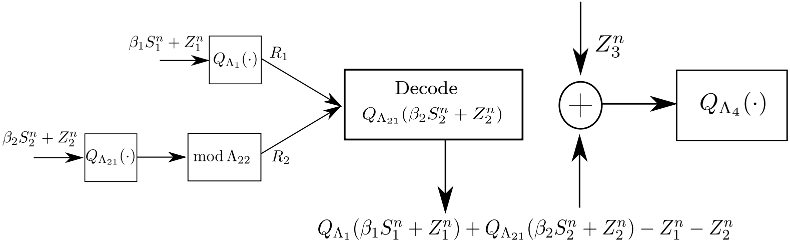


Compress and Forward - Lattice codes

- Λ_4 - used for dithered quantization at Encoder 3
- Random or lattice codes achieve same performance here
- Compute the sum estimate and then compress



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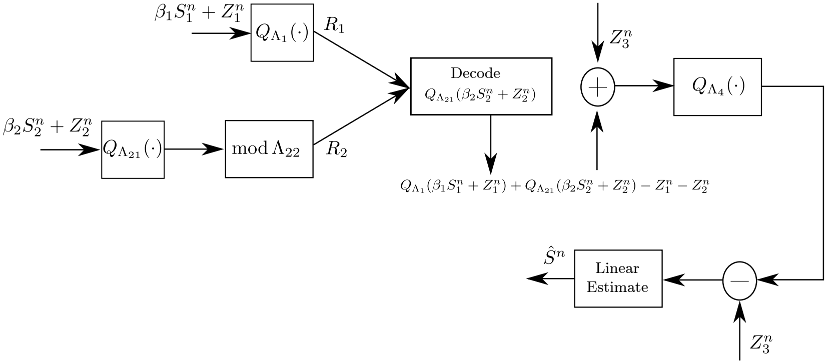
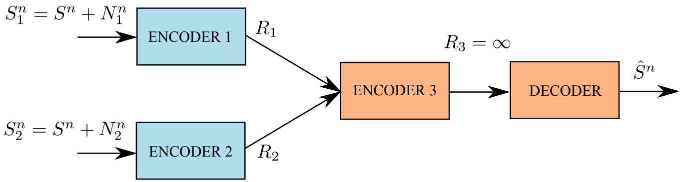


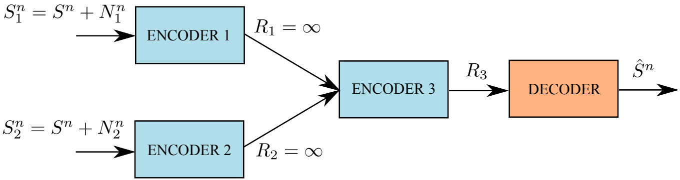
Outer Bound on (R_1, R_2)

- Allow Encoder 3 and Decoder to cooperate
- (R_1, R_2) bounded by rate region of the CEO problem



Lower Bound on R_3

- Allow Encoder 1, 2 and 3 to cooperate
- R_3 bounded by rate distortion function of remote source



Results

Compute and Forward

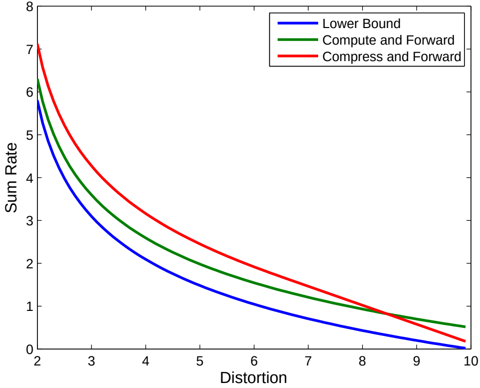
- (R_1, R_2) achieve sum rate of CEO problem
- R_3 is within $1/2$ bit of optimum for symmteric case $\sigma^2_{N_1} = \sigma^2_{N_2} = \sigma^2_N$

Compress and Forward

- (R_1, R_2) achieve higher than sum rate of CEO problem
- R_3 is lower than the rate achievable by compute and forward

Numerical Results - Sum Rate $R_1 + R_2 + R_3$

$\sigma_S^2 = 10$, $\sigma_{N_1}^2 = \sigma_{N_2}^2 = 4$



- Compute and forward is within $1/2$ bit of total sum rate
- Compress and forward achieves a smaller sum rate for higher distortions

Joint (Source/Channel) Settings

Setting one: Gaussian broadcast channels with Gaussian sources

Setting two: Linear functions over Gaussian MACs

Broadcast with Correlated Sources

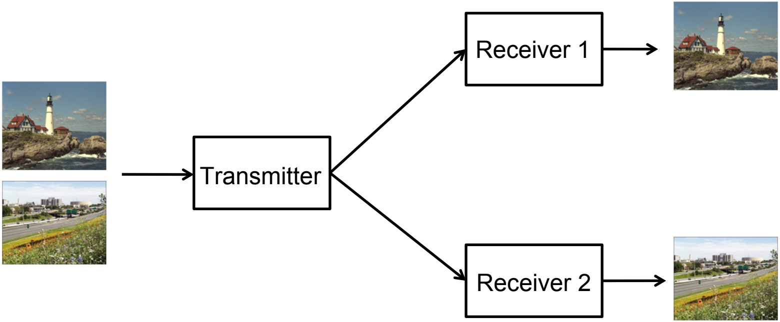


Figure: Correlated data over a broadcast channel with minimum distortion

Mathematical Setup

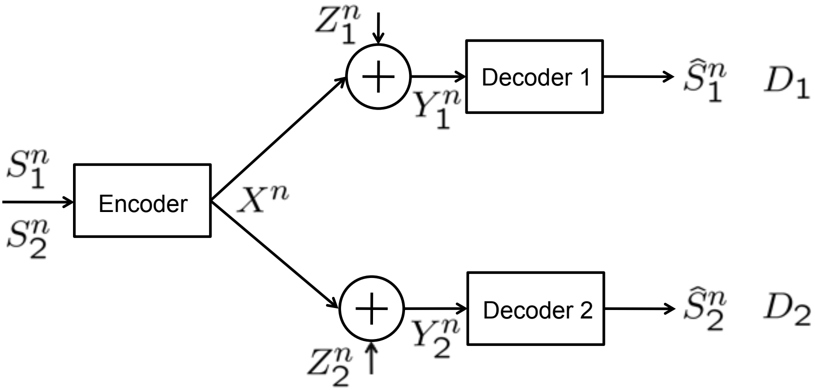
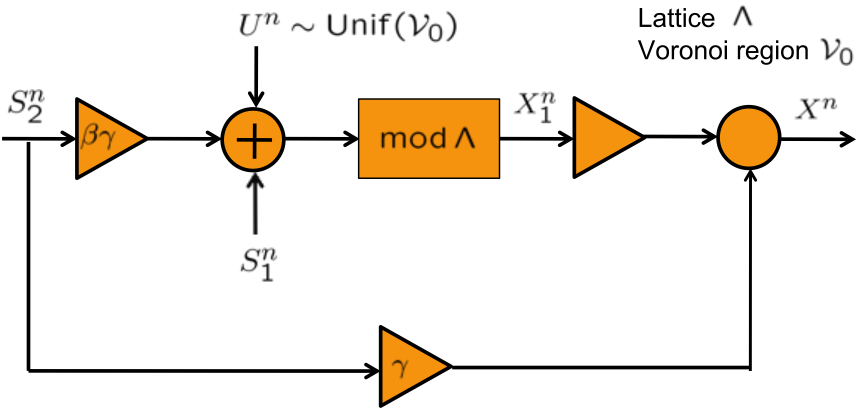


Figure: Correlated Gaussian sources over a Gaussian channel

Main Results

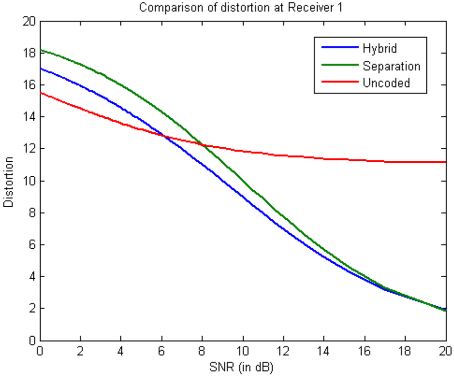
A **hybrid** coding scheme = "analog" dirty paper coding.
Hybrid = part lattice + part analog for one source

- independent sources = optimal
- correlated sources
  - uniformly better than separation
  - Better than all analog beyond a threshold SNR



$$X^n = \eta[S_1^n + \beta\gamma S_2^n + U^n] \bmod \Lambda + \gamma S_2^n$$

Numerical Comparison

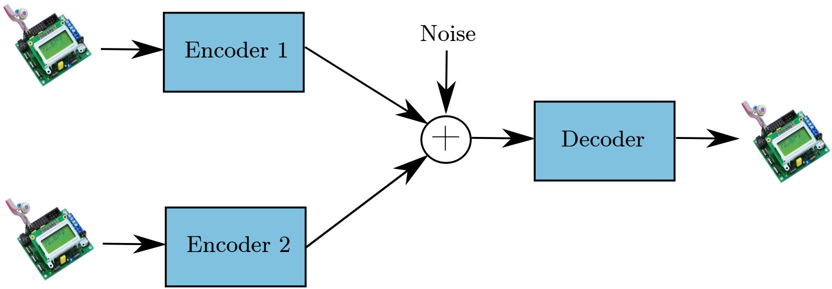


$$\sigma^2 = 20$$
$$P = 10$$
$$\alpha = 0.1$$
$$\rho = 0.4$$

All schemes achieve the same distortion at Receiver 2

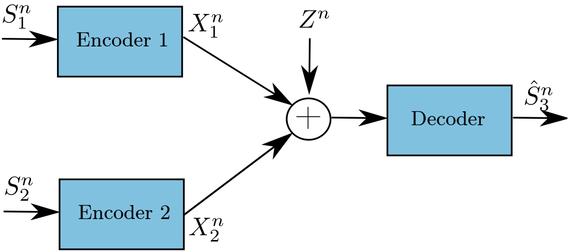
Figure: Hybrid can do uniformly better, and is optimal if independent sources

Setting two: Linear Functions over a MAC



- Characterize optimal distortion in linear functions
- Challenge - Source channel separation not optimal

System Model



- $(S_1, S_2) \sim \mathcal{N}(0, \mathbf{\Sigma})$ where $\mathbf{\Sigma} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$, $\rho \geq 0$
- Power constraint P at Encoder 1 and 2, $Z \sim \mathcal{N}(0, N)$
- Linear function $S_3 = S_1 + cS_2$
- Squared error distortion in function - $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[(S_{3i} - \hat{S}_{3i})^2\right]$

What is the smallest distortion in the function that can be achieved?

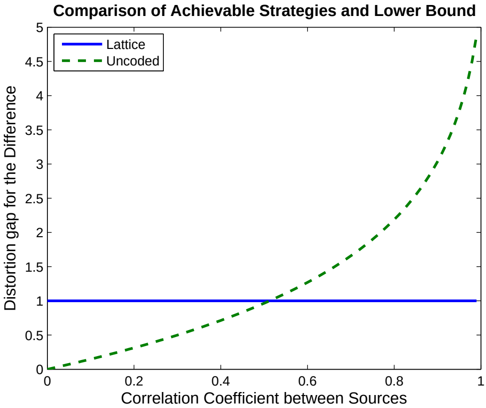
Main Result

Theorem

If SNR $>$ -3 dB, then a lattice coding scheme achieves a distortion

1. *within 1 bit of the optimal distortion when* $c \in [-1, -\rho]$
2. *within 2 bits of the of the optimal distortion when* $c \in \mathbb{R} \setminus [-1, -\rho]$

Performance Comparison



σ_S^2=20, P=10, N=0.5

Conclusion

- Structure plays a role across domains
- Interference: Signals naturally mix, and lattices curtail the cardinality growth of interference
- Source Coding: Lattices enable more efficient representations of functions of source
- Joint: Both advantages mix