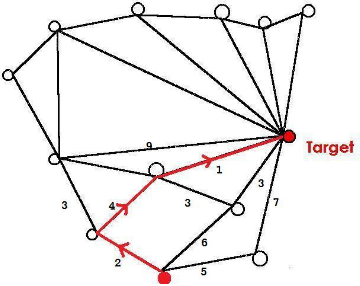
# Shortest paths and Hamilton-Jacobi equations on a network

joint work with Dirk Schieborn (Eberhard-Karls University, Tübingen)

11w5086 Advancing numerical methods for viscosity solutions and applications

# The single target shortest path problem




## Problem

In a weighted graph, finding the distance of the vertices from a prescribed target vertex and detect the shortest path (Dijkstra's algorithm)

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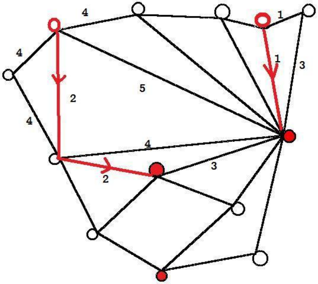
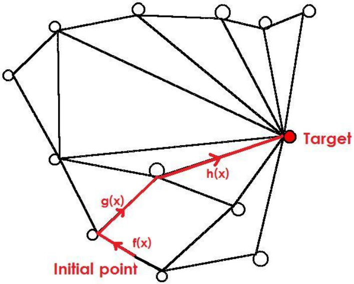



### Problem

In a weighted graph, finding the distance of the vertices from a prescribed target set and detect the shortest path.

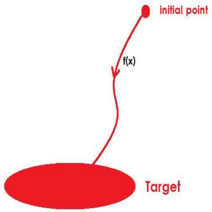
# The multiples targets shortest path problem with continuous running cost




## Problem

Finding the distance of **any point in the graph** from a prescribed target set when the cost varies in a continuous way along the edges

# Target problem in $\mathbb{R}^N$




In $\mathbb{R}^N$, finding the weighted distance from a given target set is equivalent to solve the Eikonal equation $|Du(x)| = f(x)$ with $u = 0$ on the target.

To solve the target problem with continuous running cost, introduce Eikonal equations of the form $H(x, Du) = 0$ on a graph.

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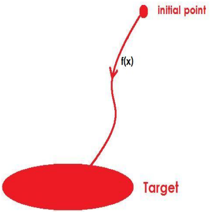


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# Literature about differential equations on networks

- Lumer, Nicaise, von Below: linear and semilinear problem on networks (maximum principle, spectral theory, etc.)
- Lagnese-Leugering: applications to wave equations (networks of vibrating strings)
- Garavello-Piccoli: hyperbolic problems, traffic flow on a network
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## Aim

Introduce a concept of viscosity solution which preserves the main features of viscosity theory: uniqueness, existence, and stability; sufficiently **“weak”** to yield existence, while sufficiently **“selective”** to ensure uniqueness and stability with respect to uniform convergence.

## Difficulties

1. How to modelize the differential structure of the network (which is not a regular manifold).
2. Which condition to impose at the vertices (**transition condition**).For second order linear equation, transition conditions are the key point to obtain the Maximum Principle.

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# The definition of Network

A **network** $\Gamma$ is couple $(V, E)$ where

• $V := \{v_i,\ i \in I\}$ is a finite collection of pairwise different points in $\mathbb{R}^N$;

• $E := \{e_j :\ j \in J\}$ is a finite collection of *differentiable curves* in $\mathbb{R}^N$ given by $e_j := \pi_j((0, l_j))$ with $\pi_j : [0, l_j] \subset \mathbb{R} \to \mathbb{R}^N$, $j \in J$. Furthermore

i) $\pi_j(0), \pi_j(l_j) \in V$ for all $j \in J$ and $\#(\bar{e}_j \cap V) = 2$ for all $j \in J$

ii) $\bar{e}_j \cap \bar{e}_k \subset V$, and $\#(\bar{e}_j \cap \bar{e}_k) \leq 1$ for all $j, k \in J$, $j \neq k$.

iii) For all $v, w \in V$ there is a path with end points $v$ and $w$ (i.e. a sequence of edges $\{e_j\}_{j=1}^N$ such that $\#(\bar{e}_j \cap \bar{e}_{j+1}) = 1$ and $v \in \bar{e}_1$, $w \in \bar{e}_N$) (the graph is *connected*).

# Some definitions

• $Inc_i := \{j \in J : \; e_j \text{ incident } v_i\}$ is the set of arcs incident the vertex $v_i$.

• The parametrization of the arcs $e_j$ induces an orientation on the edges, expressed by the signed incidence matrix $A = \{a_{ij}\}_{i,j \in J}$

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & v_i \notin \bar{e}_j. \end{cases}$$

• Given a nonempty set $I_B \subset I$, we define $\partial\Gamma := \{v_i, \; i \in I_B\}$ to be the set of boundaries vertices, while for $I_T := I \setminus I_B$ is the set of transition vertices.

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## Continuity

Given $u : \bar{\Gamma} \to \mathbb{R}$, $u^j$ the restriction of $u$ to $\bar{e}_j$, i.e.

$$u^j := u \circ \pi_j : [0, l_j] \to \mathbb{R}.$$

$u$ is continuous in $\bar{\Gamma}$ if $u^j \in C([0, l_j]$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \qquad \text{for any } i \in I,\ j,k \in Inc_i.$$

## Differentiation

We define differentiation along an edge $e_j$ by

$$\partial_j u(x) := \partial_j u^j(\pi_j^{-1}(x)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(x)), \qquad \text{for all } x \in e_j,$$

and at a vertex $v_i$ by

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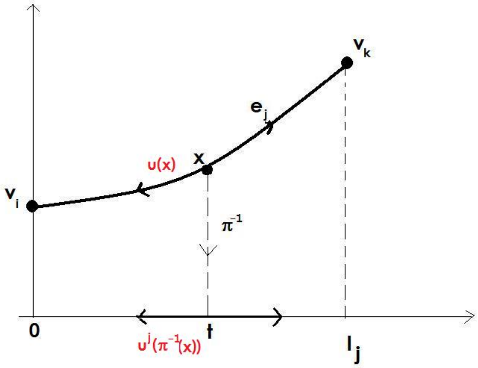


Figure: Differentiation along the edge

# Hamiltonian

A Hamiltonian $H : \bar{\Gamma} \times \mathbb{R} \to \mathbb{R}$ of **eikonal type** is given by $H(x,p) = H^j(\pi_j^{-1}(x), p)$ for $x \in e_j$ where $(H^j)_{j\in J}$ with $H^j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$

$$H^j \in C^0([0, l_j] \times \mathbb{R}), \quad j \in J, \tag{1}$$

$$H^j(x,p) \quad \text{is convex in } p \in \mathbb{R} \text{ for any } x \in [0, l_j],\ j \in J, \tag{2}$$

$$H^j(x,p) \to +\infty \quad \text{as } |p| \to \infty \text{ for any } x \in [0, l_j],\ j \in J, \tag{3}$$

$$H^j(\pi_j^{-1}(v_i), p) = H^k(\pi_k^{-1}(v_i), p) \quad \text{for any } p \in \mathbb{R},\ i \in I,\ j,k \in Inc_i, \tag{4}$$

$$H^j(\pi_j^{-1}(v_i), p) = H^j(\pi_j^{-1}(v_i), -p) \quad \text{for any } p \in \mathbb{R},\ i \in I,\ j \in Inc_i. \tag{5}$$

(1)–(3) are standard conditions. Assumptions (4)–(5) are compatibility conditions of $H$ at the vertices of $\bar{\Gamma}$, i.e. continuity at the vertices and independence of the orientation of the incident arc (the network is not oriented).For example, $H^j(x,p) := |p|^2 - f^j(x)$, $j \in J$, where $f^j \in C^0([0, l_j])$, $f^j(x) \geq 0$, $f^j(\pi_j^{-1}(v_i)) = f^k(\pi_k^{-1}(v_i))$ for any $i \in I$, $j, k \in Inc_i$.

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# Test Functions

## Definition

i) $\varphi$ is differentiable at $x \in e_j$, if $\varphi^j := \varphi \circ \pi_j : [0, l_j] \to \mathbb{R}$ is differentiable at $t = \pi_j^{-1}(x)$.

ii) Let $x = v_i$, $i \in I_T$, $j, k \in Inc_i$, $j \neq k$. $\varphi$ is $(j, k)$-differentiable at $x$ if

$$\boxed{a_{ij}\partial_j\varphi_j(\pi_j^{-1}(x)) + a_{ik}\partial_k\varphi_k(\pi_k^{-1}(x)) = 0,} \tag{6}$$

where $(a_{ij})$ as is the incidence matrix.

## Remark

Condition (6) demands that the derivatives in the direction of the incident edges $j$ and $k$ at the vertex $v_i$ coincide, taking into account the orientation of the edges.

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A function $u$ is called a viscosity subsolution if

i) If $x \in e_j$, $j \in J$, and for any $\varphi \in C(\Gamma)$ which is differentiable at $x$ and for which $u - \varphi$ attains a local maximum at $x$

$$H^j(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x)) \le 0.$$

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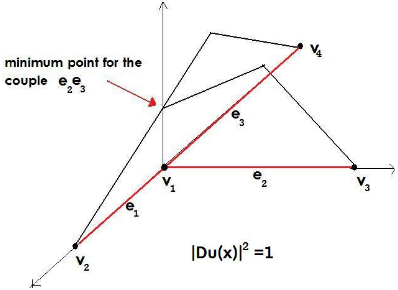
*i)*For $x = v_i$, since $H^j(\pi_j^{-1}(v_i), p) = H^k(\pi_k^{-1}(v_i), p)$ it is indifferent to require the sub and supersolution conditions for $j$ or for $k$.

*ii)* If supersolutions would be defined similarly to subsolutions, the distance function from the boundary would not be a supersolution (but there is always a shortest path from a transition vertex to the boundary).



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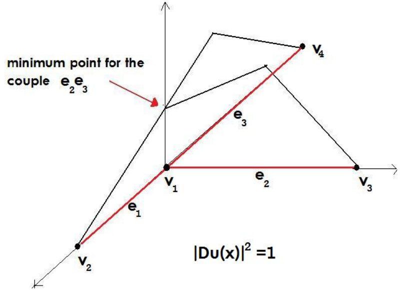

## The distance function

$$S(y,x) = \inf\left\{\int_0^t L(\gamma(s),\dot\gamma(s))ds:\ \gamma\in B^t_{y,x}\right\},\qquad x,y\in\Gamma$$

where

- $B^t_{y,x}$ is the set of paths $\gamma:[0,t]\to\Gamma$ connecting $y$ to $x$ and piecewise differentiable (i.e. $t_0 := 0 < t_1 < \cdots < t_{n+1} := t$ s.t. for any $m = 0,\dots,n$, we have $\gamma([t_m,t_{m+1}])\subset \bar e_{j_m}$ for some $j_m\in J$, $\pi^{-1}_{j_m}\circ\gamma\in C^1(t_m,t_{m+1})$, and $\dot\gamma(s) = \frac{d}{ds}(\pi^{-1}_{j_m}\circ\gamma)(s))$.
- The Lagrangian $L(x,q)$ is defined by

$$L(x,q) = \sup_{p\in\mathbb{R}}\{p\,q - H^j(\pi_j^{-1}(x),p)\}\qquad x\in\bar e_j$$

The **path distance** $d(y,x)$ on the network coincides with $S(y,x)$ for $H(x,p) = |p|^2 - 1$.

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# Existence and uniqueness

**Theorem**

Assume that there exists a differentiable function $\psi$ such that $H(x, D\psi) < 0$ in $\Gamma$. Let $g : \partial\Gamma \to \mathbb{R}$ be a continuous function satisfying

$$g(x) - g(y) \leq S(y, x) \qquad \text{for any } x, y \in \partial\Gamma = I_B,$$

Then

$$\boxed{u(x) := \min\{g(y) + S(y, x) : \ y \in \partial\Gamma\}}$$

is the **unique viscosity solution** of

$$\begin{cases} H(x, Du) = 0, & x \in \Gamma; \\ u(x) = g(x), & x \in \partial\Gamma. \end{cases}$$

# Sketch of the proof

### Uniqueness:

Classical doubling argument

$$\Phi_\varepsilon(x,y) := u(x) - v(y) - \varepsilon^{-1} d(x,y)^2$$

for Maximum Principle ($d^2$ is an admissible test function) + Ishii's trick.

### Existence:

The function $S(y,\cdot)$ is a subsolution in $\Gamma$ and a supersolution in $\Gamma \setminus \{y\}$. Moreover

$$S(y,x) = \max\{u(x) : u \text{ is a subsolution s.t. } u(y) = 0\}.$$

# Sketch of the proof

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# Stability

**Theorem**

Assume $H_n(x,p) \to H(x,p)$ uniformly for $n \to \infty$ (i.e. $H_n^j(\pi_j^{-1}(x),p) \to H^j(\pi_j^{-1}(x),p)$ uniformly for $(x,p) \in \bar{e}_j \times \mathbb{R}$ for any $j \in J$). For any $n \in \mathbb{N}$ let $u_n$ be a solution of

$$H_n(x,Du) = 0, \qquad x \in \Gamma,$$

and assume $u_n \to u$ uniformly in $\Gamma$ for $n \to \infty$. Then $u$ is a solution of

$$H(x,Du) = 0.$$

# Classification of singularities

We consider the equation

$$\begin{cases} |Du|^2 - f(x) = 0, & x \in \Gamma; \\ u(x) = 0, & x \in I_B. \end{cases}$$

with $f > 0$ in $\Gamma$. It is possible to prove that an edge contains at most one non-differentiability (singular) point. We define $k^{edge} : E \to \{0, 1\}$ by

$$k^{edge}(e_j) := \begin{cases} 1, & \text{if } e_j \text{ contains a singular point;} \\ 0, & \text{if } e_j \text{ does not contain a singular point.} \end{cases}$$

For a vertex $v_i$, we

$$k^{vertex}(v_i) := \#(Inc_i^-)$$

where $Inc_i^-$ are the edges entering “downhill” in $v_i$ (the more incident edges lead “downhill”, the more $v_i$ assumes the character of a local maximum and the higher it should be weighted when counting the singularities).

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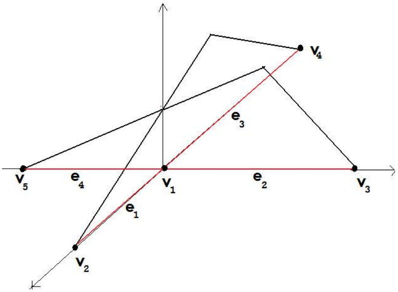
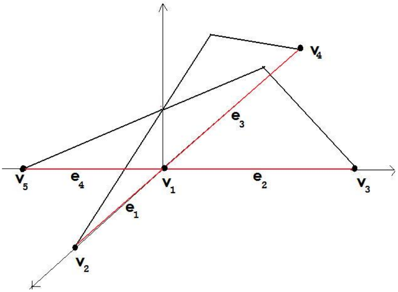



Figure: $Inc_1^- = 2$

**Theorem**

$$\sum_{i \in I} k^{vertex}(v_i) + \sum_{j \in J} k^{edge}(e_j) = \#(J)$$

i.e. the dimension of the singular set of the viscosity solution only depends on the number of edges of Γ.

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# A semi-Lagrangian approximation scheme

(see Falcone-Ferretti, J. Comput. Phys. 175 (2002))

## Discretization in time

For $h > 0$, we define

i) An admissible trajectory $\gamma^h = \{\gamma^h_m\}_{m=0}^M \subset \Gamma$ is a finite number of points $\gamma^h_m = \pi_{j_m}(t_m) \in \Gamma$ such that for any $m = 0, \ldots, M$, the arc $\widehat{\gamma^h_m \gamma^h_{m+1}} \subset \bar{e}_{j_m}$ for some $j_m \in J$.

ii) $B^h_{x,y}$ is the set of all such paths with $\gamma^h_0 = x$, $\gamma^h_M = y$.

We set

$$u_h(x) = \inf\{\sum_{m=0}^{M} hL(\gamma^h_m, q_m) + g(y) : \gamma^h \in B^h_{x,y},\, y \in \partial\Gamma\} \qquad x \in \Gamma$$

Set $x_{hq} := \pi_j(t - hq)$ (hence $d(x, x_{hq}) = h|q|$).Then $u_h$ is the unique Lipschitz-continuous solution of

- If $x = \pi_j(t) \in e_j$

$$u_h^j(x) = \inf_{q \in \mathbb{R}:\, x_{hq} \in \bar{e}_j} \{u_h(x_{hq}) + hL(x, q)\}$$
$$= \inf_{q \in \mathbb{R}:\, x_{hq} \in \bar{e}_j} \{u_h^j(t - hq) + hL^j(t, q)\}$$

- If $x = v_i \in I_T$

$$u_h^j(v_i) = \inf_{k \in Inc_i} \left[ \inf_{q \in \mathbb{R}:\, x_{hq} \in \bar{e}_k} \{u_h(x_{hq}) + hL(v_i, q)\} \right]$$
$$= \inf_{k \in Inc_i} \left[ \inf_{q \in \mathbb{R}:\, x_{hq} \in \bar{e}_k} \{u_h^k(t - hq) + hL^k(t, q)\} \right]$$

- If $x \in I_B$, $u_h^j(x) = g(x)$.

## Discretization in space

For $j \in J$,consider a partition

$$P^j = \{t_0^j = 0 < \cdots < t_m^j < \cdots < t_{M_j}^j = l_j\}$$

of $[0, l_j]$ such that $|P^j| = \max_{1,\dots,M_j}(t_m^j - t_{m-1}^j) \leq k_j$. Set $x_m^j = \pi_j(t_m^j)$ and consider

$$W_{k_j}^j = \{w \in C(\bar{e}_j) : \partial_j w(x) \text{ is constant in } (x_{m-1}^j, x_m^j),\, m = 1, \dots, M_j\}.$$

Every element $w$ in $W_{k_j}^j$ can be expressed as

$$w(x) = \sum_{m=1}^{M_j} \bar{\beta}_m^j(x) w^j(x_m^j), \qquad x \in e_j$$

for $\bar{\beta}_j(x) = \beta_j(\pi_j^{-1}(x))$ and $\beta_j$ tent functions for the partition $P_J$.

Set $x^{j,m}_{hq} = \pi_j(t^j_m - hq)$ and $k = \max_{j\in J} k_j$ and consider:
Find $u_{hk} : \bar{\Gamma} \to \mathbb{R}$ such that $u^j_{hk} \in W^j_{k_j}$ for $j \in J$ and

- If $x^j_m = \pi_j(t^j_m) \in e_j$

$$u^j_{hk}(x^j_m) = \inf_{q\in\mathbb{R}:\, x^{j,m}_{hq}\in\bar{e}_j} \{u_{hk}(x^{j,m}_{hq}) + hL(x^{j,m}, q)\}$$
$$= \inf_{q\in\mathbb{R}:\, x^{j,m}_{hq}\in\bar{e}_j} \{u^j_{hk}(t^j_m - hq) + hL^j(t^j_m, q)\}$$

- If $x^j_m = v_i$

$$u^j_{hk}(v_i) = \inf_{r\in Inc_i} \Big[ \inf_{q\in\mathbb{R}:\, x^{l,m}_{hq}\in\bar{e}_r} \{u_{hk}(x^{r,m}_{hq}) + hL(v_i, q)\} \Big]$$
$$= \inf_{r\in Inc_i} \Big[ \inf_{q\in\mathbb{R}:\, x^{r,m}_{hq}\in\bar{e}_r} \{u^r_{hk}(t^r_m - hq) + hL^r(t^r_m, q)\} \Big]$$

- If $x^j_m = v_i \in I_B$, $u^j_h(x) = g(x)$.

For $j \in J$, we set $U^j = \{u^j_{hk}(t^j_m)\}_{m=1}^{M_j}$, $B^j(q) = \{\beta_l(t^j_m - hq)\}_{l,m=1}^{M_j}$ and $\mathcal{L}^j(q) = \{L^j(t^j_m, q)\}_{m=1}^{M_j}$, and we rewrite the previous as the finite-dimensional system

$$
\begin{aligned}
U^j_m &= \inf_{q\in\mathbb{R}:\, \pi_j(t^j_m - hq)\in\bar{e}_j} \{B^j(q)U^j + h\mathcal{L}^j_m(q)\} && x^j_m = \pi_j(t^j_m) \in e_j \\
U^j_m &= \inf_{r\in Inc_i} \inf_{q\in\mathbb{R}:\, \pi_r(t^j_m - hq)\in\bar{e}_r} \{B^r(q)U^j + h\mathcal{L}^r_m(q)\} && v_i = \pi_r(t^j_m) \in I_T \\
U^j_m &= g(x^j_m) && x^j_m = \pi_j(t^j_m) \in I_B
\end{aligned}
$$

**Theorem**

Assume that $h \to 0$ and $\dfrac{k}{h} \to 0$. Then $u_{hk}$ converges uniformly to $u$ solution of

$$\begin{cases} H(x, Du) = 0, & x \in \Gamma; \\ u(x) = g(x), & x \in \partial\Gamma. \end{cases}$$