Shortest paths and Hamilton-Jacobi equations on a network

joint work with Dirk Schieborn (Eberhard-Karls University, Tübingen)

11w5086 Advancing numerical methods for viscosity solutions and applications
The single target shortest path problem

Problem

In a weighted graph, finding the distance of the vertices from a prescribed target vertex and detect the shortest path (Dijkstra’s algorithm)
The multiple targets shortest path problem

Problem

In a weighted graph, finding the distance of the vertices from a prescribed target set and detect the shortest path.
The multiples targets shortest path problem with continuous running cost

Problem
Finding the distance of **any point in the graph** from a prescribed target set when the cost varies in a continuous way along the edges
In $\mathbb{R}^N$, finding the \textit{weighted distance} from a given target set is equivalent to solve the Eikonal equation $|Du(x)| = f(x)$ with $u = 0$ on the target.

To solve the target problem with continuous running cost, introduce Eikonal equations of the form $H(x, Du) = 0$ on a graph.
Motivation

Target problem in $\mathbb{R}^N$

In $\mathbb{R}^N$, finding the *weighted distance* from a given target set is equivalent to solve the Eikonal equation $|Du(x)| = f(x)$ with $u = 0$ on the target.

To solve the target problem with continuous running cost, introduce Eikonal equations of the form $H(x, Du) = 0$ on a graph.
Literature about differential equations on networks

- Lumer, Nicaise, von Below: linear and semilinear problem on networks (maximum principle, spectral theory, etc.)
- Lagnese-Leugering: applications to wave equations (networks of vibrating strings)
- Garavello-Piccoli: hyperbolic problems, traffic flow on a network
- Engel-Nagel and coauthors: semigroup theory and asymptotic behavior of linear system on networks
- Achdou-C.-Cutri-Tchou: control problem on a network (a controlled dynamic in \( \mathbb{R}^2 \) constrained to a network)
Literature about differential equations on networks

- Lumer, Nicaise, von Below: linear and semilinear problem on networks (maximum principle, spectral theory, etc.)
- Lagnese-Leugering: applications to wave equations (networks of vibrating strings)
- Garavello-Piccoli: hyperbolic problems, traffic flow on a network
- Engel-Nagel and coauthors: semigroup theory and asymptotic behavior of linear system on networks
- Achdou-C.-Cutrì-Tchou: control problem on a network (a controlled dynamic in $\mathbb{R}^2$ constrained to a network)
Literature about differential equations on networks

- Lumer, Nicaise, von Below: linear and semilinear problem on networks (maximum principle, spectral theory, etc.)
- Lagnese-Leugering: applications to wave equations (networks of vibrating strings)
- Garavello-Piccoli: hyperbolic problems, traffic flow on a network
- Engel-Nagel and coauthors: semigroup theory and asymptotic behavior of linear system on networks
- Achdou-C.-Cutri-Tchou: control problem on a network (a controlled dynamic in $\mathbb{R}^2$ constrained to a network)
Literature about differential equations on networks

- Lumer, Nicaise, von Below: linear and semilinear problem on networks (maximum principle, spectral theory, etc.)
- Lagnese-Leugering: applications to wave equations (networks of vibrating strings)
- Garavello-Piccoli: hyperbolic problems, traffic flow on a network
- Engel-Nagel and coauthors: semigroup theory and asymptotic behavior of linear system on networks
- Achdou-C.-Cutrì-Tchou: control problem on a network (a controlled dynamic in $\mathbb{R}^2$ constrained to a network)
Literature about differential equations on networks

- Lumer, Nicaise, von Below: linear and semilinear problem on networks (maximum principle, spectral theory, etc.)
- Lagnese-Leugering: applications to wave equations (networks of vibrating strings)
- Garavello-Piccoli: hyperbolic problems, traffic flow on a network
- Engel-Nagel and coauthors: semigroup theory and asymptotic behavior of linear system on networks
- Achdou-C.-Cutri-Tchou: control problem on a network (a controlled dynamic in $\mathbb{R}^2$ constrained to a network)
Aim

Introduce a concept of viscosity solution which preserves the main features of viscosity theory: uniqueness, existence, and stability; sufficiently “weak” to yield existence, while sufficiently “selective” to ensure uniqueness and stability with respect to uniform convergence.

Difficulties

1. How to modelize the differential structure of the network (which is not a regular manifold).

2. Which condition to impose at the vertices (transition condition). For second order linear equation, transition conditions are the key point to obtain the Maximum Principle.
Aim

Introduce a concept of viscosity solution which preserves the main features of viscosity theory: uniqueness, existence, and stability; sufficiently “weak” to yield existence, while sufficiently “selective” to ensure uniqueness and stability with respect to uniform convergence.

Difficulties

1. How to modelize the differential structure of the network (which is not a regular manifold).

2. Which condition to impose at the vertices (transition condition). For second order linear equation, transition conditions are the key point to obtain the Maximum Principle.
Aim

Introduce a concept of viscosity solution which preserves the main features of viscosity theory: uniqueness, existence, and stability; sufficiently "weak" to yield existence, while sufficiently "selective" to ensure uniqueness and stability with respect to uniform convergence.

Difficulties

1. How to modelize the differential structure of the network (which is not a regular manifold).

2. Which condition to impose at the vertices (transition condition). For second order linear equation, transition conditions are the key point to obtain the Maximum Principle.
A network $\Gamma$ is couple $(V, E)$ where

- $V := \{v_i, \ i \in I\}$ is a finite collection of pairwise different points in $\mathbb{R}^N$;
- $E := \{e_j : \ j \in J\}$ is a finite collection of differentiable curves in $\mathbb{R}^N$ given by $e_j := \pi_j((0, l_j))$ with $\pi_j : [0, l_j] \subset \mathbb{R} \to \mathbb{R}^N, j \in J$. Furthermore
  
  i) $\pi_j(0), \pi_j(l_j) \in V$ for all $j \in J$ and $#(\bar{e}_j \cap V) = 2$ for all $j \in J$

  ii) $\bar{e}_j \cap \bar{e}_k \subset V$, and $#(\bar{e}_j \cap \bar{e}_k) \leq 1$ for all $j, k \in J, j \neq k$.

  iii) For all $v, w \in V$ there is a path with end points $v$ and $w$ (i.e. a sequence of edges $\{e_j\}_{j=1}^N$ such that $#(\bar{e}_j \cap \bar{e}_{j+1}) = 1$ and $v \in \bar{e}_1$, $w \in \bar{e}_N$) (the graph is connected).
Some definitions

- \( \text{Inc}_i := \{ j \in J : e_j \text{ incident } v_i \} \) is the set of arcs incident the vertex \( v_i \).

- The parametrization of the arcs \( e_j \) induces an orientation on the edges, expressed by the signed incidence matrix \( A = \{ a_{ij} \}_{i,j \in J} \)

\[
   a_{ij} := \begin{cases} 
   1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\
   -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\
   0 & v_i \not\in \bar{e}_j.
   \end{cases}
\]

- Given a nonempty set \( I_B \subset I \), we define \( \partial \Gamma := \{ v_i, i \in I_B \} \) to be the set of boundaries vertices, while for \( I_T := I \setminus I_B \) is the set of transition vertices.
Some definitions

- \( \text{Inc}_i := \{ j \in J : e_j \text{ incident } v_i \} \) is the set of arcs incident the vertex \( v_i \).

- The parametrization of the arcs \( e_j \) induces an orientation on the edges, expressed by the signed incidence matrix \( A = \{ a_{ij} \}_{i,j \in J} \)

\[
\begin{align*}
a_{ij} &:= \begin{cases} 
1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\
-1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\
0 & v_i \notin \bar{e}_j.
\end{cases}
\end{align*}
\]

- Given a nonempty set \( I_B \subset I \), we define \( \partial \Gamma := \{ v_i, i \in I_B \} \) to be the set of boundaries vertices, while for \( I_T := I \setminus I_B \) is the set of transition vertices.
Some definitions

- $\text{Inc}_i := \{j \in J : e_j \text{ incident } v_i\}$ is the set of arcs incident the vertex $v_i$.
- The parametrization of the arcs $e_j$ induces an orientation on the edges, expressed by the signed incidence matrix $A = \{a_{ij}\}_{i,j \in J}$

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & v_i \not\in \bar{e}_j. \end{cases}$$

- Given a nonempty set $I_B \subset I$, we define $\partial \Gamma := \{v_i, i \in I_B\}$ to be the set of boundaries vertices, while for $I_T := I \setminus I_B$ is the set of transition vertices.
Continuity

Given $u : \bar{\Gamma} \to \mathbb{R}$, $u^j$ the restriction of $u$ to $\bar{e}_j$, i.e.

$$u^j := u \circ \pi_j : [0, l_j] \to \mathbb{R}.$$  

$u$ is continuous in $\bar{\Gamma}$ if $u^j \in C([0, l_j]$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{for any } i \in I, j, k \in Inc_i.$$  

Differentiation

We define differentiation along an edge $e_j$ by

$$\partial_j u(x) := \partial_j u^j(\pi_j^{-1}(x)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(x)), \quad \text{for all } x \in e_j,$$

and at a vertex $v_i$ by

$$\partial_j u(v_i) := \partial_j u^j(\pi_j^{-1}(v_i)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(v_i)) \quad \text{for } j \in Inc_i.$$
**Continuity**

Given $u : \bar{\Gamma} \to \mathbb{R}$, $u^j$ the restriction of $u$ to $\bar{e}_j$, i.e.

$$u^j := u \circ \pi_j : [0, l_j] \to \mathbb{R}.$$ 

$u$ is continuous in $\bar{\Gamma}$ if $u^j \in C([0, l_j]$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{for any } i \in I, j, k \in \text{Inc}_i.$$

**Differentiation**

We define differentiation along an edge $e_j$ by

$$\partial_j u(x) := \partial_j u^j(\pi_j^{-1}(x)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(x)), \quad \text{for all } x \in e_j,$$

and at a vertex $v_i$ by

$$\partial_j u(v_i) := \partial_j u^j(\pi_j^{-1}(v_i)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(v_i)) \quad \text{for } j \in \text{Inc}_i.$$
Figure: Differentiation along the edge
Hamiltonian

A Hamiltonian $H : \bar{\Gamma} \times \mathbb{R} \rightarrow \mathbb{R}$ of eikonal type is given by $H(x, p) = H^j(\pi^{-1}_j(x), p)$ for $x \in e_j$ where $(H^j)_{j \in J}$ with $H^j : [0, l_j] \times \mathbb{R} \rightarrow \mathbb{R}$

$H^j \in C^0([0, l_j] \times \mathbb{R}), \quad j \in J, \quad (1)$

$H^j(x, p)$ is convex in $p \in \mathbb{R}$ for any $x \in [0, l_j], j \in J, \quad (2)$

$H^j(x, p) \rightarrow +\infty \quad \text{as} \quad |p| \rightarrow \infty \quad \text{for any} \quad x \in [0, l_j], j \in J, \quad (3)$

$H^j(\pi^{-1}_j(v_i), p) = H^k(\pi^{-1}_k(v_i), p) \quad \text{for any} \quad p \in \mathbb{R}, i \in I, j, k \in Inc_i, \quad (4)$

$H^j(\pi^{-1}_j(v_i), p) = H^i(\pi^{-1}_j(v_i), -p) \quad \text{for any} \quad p \in \mathbb{R}, i \in I, j \in Inc_i. \quad (5)$

(1)–(3) are standard conditions. Assumptions (4)–(5) are compatibility conditions of $H$ at the vertices of $\bar{\Gamma}$, i.e. continuity at the vertices and independence of the orientation of the incident arc (the network is not oriented). For example, $H^j(x, p) := |p|^2 - f^j(x), j \in J,$ where $f^j \in C^0([0, l_j]), f^j(x) \geq 0,$ $f^j(\pi^{-1}_j(v_i)) = f^k(\pi^{-1}_k(v_i))$ for any $i \in I, j, k \in Inc_i.$
A Hamiltonian $H : \bar{\Gamma} \times \mathbb{R} \rightarrow \mathbb{R}$ of eikonal type is given by

$$H(x, p) = H_j^i(\pi_j^{-1}(x), p) \text{ for } x \in e_j \text{ where } (H^j_j)_{j \in J} \text{ with }$$

$$H^j_j : [0, l_j] \times \mathbb{R} \rightarrow \mathbb{R}$$

$$H^j_j \in C^0([0, l_j] \times \mathbb{R}), \quad j \in J, \quad (1)$$

$H^j_j(x, p)$ is convex in $p \in \mathbb{R}$ for any $x \in [0, l_j], j \in J, \quad (2)$

$H^j_j(x, p) \rightarrow +\infty$ as $|p| \rightarrow \infty$ for any $x \in [0, l_j], j \in J, \quad (3)$

$H^j_j(\pi_j^{-1}(v_i), p) = H^k_k(\pi_k^{-1}(v_i), p)$ for any $p \in \mathbb{R}, i \in I, j, k \in Inc_i, \quad (4)$

$H^j_j(\pi_j^{-1}(v_i), p) = H^j_j(\pi_j^{-1}(v_i), -p)$ for any $p \in \mathbb{R}, i \in I, j \in Inc_i. \quad (5)$

(1)–(3) are standard conditions. Assumptions (4)–(5) are compatibility conditions of $H$ at the vertices of $\bar{\Gamma}$, i.e. continuity at the vertices and independence of the orientation of the incident arc (the network is not oriented). For example, $H^j_j(x, p) := |p|^2 - f^j_j(x), j \in J$, where

$f^j_j \in C^0([0, l_j]), f^j_j(x) \geq 0, f^j_j(\pi_j^{-1}(v_i)) = f^k_k(\pi_k^{-1}(v_i)) \text{ for any } i \in I, j, k \in Inc_i.$
A Hamiltonian $H : \bar{\Gamma} \times \mathbb{R} \to \mathbb{R}$ of eikonal type is given by $H(x, p) = H^j(\pi_j^{-1}(x), p)$ for $x \in e_j$ where $(H^j)_{j \in J}$ with $H^j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$

\begin{align*}
H^j &\in C^0([0, l_j] \times \mathbb{R}), \quad j \in J, \quad (1) \\
H^j(x, p) &\text{ is convex in } p \in \mathbb{R} \text{ for any } x \in [0, l_j], j \in J, \quad (2) \\
H^j(x, p) &\to +\infty \quad \text{as } |p| \to \infty \text{ for any } x \in [0, l_j], j \in J, \quad (3) \\
H^j(\pi_j^{-1}(v_i), p) &= H^k(\pi_k^{-1}(v_i), p) \quad \text{for any } p \in \mathbb{R}, i \in I, j, k \in Inc_i, \quad (4) \\
H^j(\pi_j^{-1}(v_i), p) &= H^j(\pi_j^{-1}(v_i), -p) \quad \text{for any } p \in \mathbb{R}, i \in I, j \in Inc_i. \quad (5)
\end{align*}

(1)–(3) are standard conditions. Assumptions (4)–(5) are compatibility conditions of $H$ at the vertices of $\bar{\Gamma}$, i.e. continuity at the vertices and independence of the orientation of the incident arc (the network is not oriented). For example, $H^j(x, p) := |p|^2 - f^j(x), j \in J$, where $f^j \in C^0([0, l_j]), f^j(x) \geq 0, f^j(\pi_j^{-1}(v_i)) = f^k(\pi_k^{-1}(v_i))$ for any $i \in I, j, k \in Inc_i$. 
Test Functions

Definition

i) \( \varphi \) is differentiable at \( x \in e_j \), if \( \varphi^j := \varphi \circ \pi_j : [0, l_j] \to \mathbb{R} \) is differentiable at \( t = \pi_j^{-1}(x) \).

ii) Let \( x = v_i, i \in I_T, j, k \in Inc_i, j \neq k \). \( \varphi \) is \((j, k)\)-differentiable at \( x \) if

\[
 a_{ij} \partial_j \varphi_j(\pi_j^{-1}(x)) + a_{ik} \partial_k \varphi_k(\pi_k^{-1}(x)) = 0, \tag{6}
\]

where \((a_{ij})\) as is the incidence matrix.

Remark

Condition (6) demands that the derivatives in the direction of the incident edges \( j \) and \( k \) at the vertex \( v_i \) coincide, taking into account the orientation of the edges.
Test Functions

Definition

i) $\varphi$ is differentiable at $x \in e_j$, if $\varphi^j := \varphi \circ \pi_j : [0, l_j] \to \mathbb{R}$ is differentiable at $t = \pi_j^{-1}(x)$.

ii) Let $x = v_i$, $i \in I_T$, $j, k \in Inc_i$, $j \neq k$. $\varphi$ is $(j, k)$-differentiable at $x$ if

$$a_{ij} \partial_j \varphi_j(\pi_j^{-1}(x)) + a_{ik} \partial_k \varphi_k(\pi_k^{-1}(x)) = 0, \quad (6)$$

where $(a_{ij})$ as is the incidence matrix.

Remark

Condition (6) demands that the derivatives in the direction of the incident edges $j$ and $k$ at the vertex $v_i$ coincide, taking into account the orientation of the edges.
A function $u$ is called a **viscosity subsolution** if

i) If $x \in e_j$, $j \in J$, and for any $\varphi \in C(\Gamma)$ which is differentiable at $x$ and for which $u - \varphi$ attains a local maximum at $x$

$$H^i(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \leq 0.$$ 

ii) If $x = v_i$, $i \in I_T$, for any $j, k \in Inc_i$, $\varphi$ which is $(j, k)$-differentiable at $x$ and for which $u - \varphi$ attains a local maximum at $x$

$$H^i(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \leq 0.$$ 

A function $u$ is called a **viscosity supersolution** if:

i) If $x \in e_j$, $j \in J$, and for any $\varphi \in C(\Gamma)$ which is differentiable at $x$ and for which $u - \varphi$ attains a local minimum at $x$

$$H^i(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \geq 0.$$ 

ii) If $x = v_i$, $i \in I_T$, $j \in Inc_i$, there exists $k \in Inc_i$, $k \neq j$, such that for any $\varphi \in C(\Gamma)$ which is $(j, k)$-differentiable at $x$ and for which $u - \varphi$ attains a local maximum at $x$

$$H^i(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \geq 0.$$
i) For $x = v_i$, since $H^j(\pi_j^{-1}(v_i), p) = H^k(\pi_k^{-1}(v_i), p)$ it is indifferent to require the sub and supersolution conditions for $j$ or for $k$.

ii) If supersolutions would be defined similarly to subsolutions, the distance function from the boundary would not be a supersolution (but there is always a shortest path from a transition vertex to the boundary).
i) For $x = v_i$, since $H^j(\pi_j^{-1}(v_i), p) = H^k(\pi_k^{-1}(v_i), p)$ it is indifferent to require the sub and supersolution conditions for $j$ or for $k$.

$ii$) If supersolutions would be defined similarly to subsolutions, the distance function from the boundary would not be a supersolution (but there is always a shortest path from a transition vertex to the boundary).
The distance function

\[
S(y, x) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds : \gamma \in B_{y,x}^t \right\}, \quad x, y \in \Gamma
\]

where

- \(B_{y,x}^t\) is the set of paths \(\gamma : [0, t] \to \Gamma\) connecting \(y\) to \(x\) and piecewise differentiable (i.e. \(t_0 := 0 < t_1 < \cdots < t_{n+1} := t\) s.t. for any \(m = 0, \ldots, n\), we have \(\gamma([t_m, t_{m+1}]) \subset \bar{e}_{j_m}\) for some \(j_m \in J\), \(\pi^{-1}_{j_m} \circ \gamma \in \mathcal{C}^1(t_m, t_{m+1})\), and \(\dot{\gamma}(s) = \frac{d}{ds} (\pi_{j_m}^{-1} \circ \gamma)(s)\)).

- The Lagrangian \(L(x, q)\) is defined by

\[
L(x, q) = \sup_{p \in \mathbb{R}} \{ p \, q - H^j(\pi_{j_m}^{-1}(x), p) \} \quad x \in \bar{e}_j
\]

The path distance \(d(y, x)\) on the network coincides with \(S(y, x)\) for \(H(x, p) = |p|^2 - 1\).
The distance function

\[
S(y, x) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds : \gamma \in B_{y,x}^t \right\}, \quad x, y \in \Gamma
\]

where

- \( B_{y,x}^t \) is the set of paths \( \gamma : [0, t] \rightarrow \Gamma \) connecting \( y \) to \( x \) and piecewise differentiable (i.e. \( t_0 := 0 < t_1 < \cdots < t_{n+1} := t \) s.t. for any \( m = 0, \ldots, n \), we have \( \gamma([t_m, t_{m+1}]) \subset \bar{e}_{j_m} \) for some \( j_m \in J \), \( \pi_{j_m}^{-1} \circ \gamma \in C^1(t_m, t_{m+1}) \), and \( \dot{\gamma}(s) = \frac{d}{ds}(\pi_{j_m}^{-1} \circ \gamma)(s)) \).
- The Lagrangian \( L(x, q) \) is defined by

\[
L(x, q) = \sup_{p \in \mathbb{R}} \{ p q - H^i(\pi_j^{-1}(x), p) \} \quad x \in \bar{e}_j
\]

The path distance \( d(y, x) \) on the network coincides with \( S(y, x) \) for \( H(x, p) = |p|^2 - 1 \).
The distance function

\[
S(y, x) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds : \gamma \in B^t_{y,x} \right\}, \quad x, y \in \Gamma
\]

where

- \(B^t_{y,x}\) is the set of paths \(\gamma : [0, t] \to \Gamma\) connecting \(y\) to \(x\) and piecewise differentiable (i.e. \(t_0 := 0 < t_1 < \cdots < t_{n+1} := t\) s.t. for any \(m = 0, \ldots, n\), we have \(\gamma([t_m, t_{m+1}]) \subset \bar{e}_{j_m}\) for some \(j_m \in J\), \(\pi_{j_m}^{-1} \circ \gamma \in C^1(t_m, t_{m+1})\), and \(\dot{\gamma}(s) = \frac{d}{ds} (\pi_{j_m}^{-1} \circ \gamma)(s)\)).

- The Lagrangian \(L(x, q)\) is defined by
  \[
  L(x, q) = \sup_{p \in \mathbb{R}} \{ p \, q - H^i(\pi_{j}^{-1}(x), p) \}, \quad x \in \bar{e}_j
  \]

The path distance \(d(y, x)\) on the network coincides with \(S(y, x)\) for \(H(x, p) = |p|^2 - 1\).
Existence and uniqueness

**Theorem**

Assume that there exists a differentiable function $\psi$ such that $H(x, D\psi) < 0$ in $\Gamma$. Let $g : \partial \Gamma \to \mathbb{R}$ be a continuous function satisfying

$$g(x) - g(y) \leq S(y, x) \quad \text{for any } x, y \in \partial \Gamma = I_B,$$

Then

$$u(x) := \min\{g(y) + S(y, x) : y \in \partial \Gamma\}$$

is the **unique viscosity solution** of

$$\begin{cases} H(x, Du) = 0, & x \in \Gamma; \\ u(x) = g(x), & x \in \partial \Gamma. \end{cases}$$
Sketch of the proof

Uniqueness:
Classical doubling argument

\[ \Phi_\varepsilon(x, y) := u(x) - v(y) - \varepsilon^{-1}d(x, y)^2 \]

for Maximum Principle (\(d^2\) is an admissible test function) + Ishii’s trick.

Existence:
The function \(S(y, \cdot)\) is a subsolution in \(\Gamma\) and a supersolution in \(\Gamma \setminus \{y\}\). Moreover

\[ S(y, x) = \max\{u(x) : u \text{ is a subsolution s.t. } u(y) = 0\}. \]
Sketch of the proof

**Uniqueness:**
Classical doubling argument

\[ \Phi_\varepsilon(x, y) := u(x) - v(y) - \varepsilon^{-1} d(x, y)^2 \]

for Maximum Principle \((d^2\) is an admissible test function) + Ishii’s trick.

**Existence:**
The function \(S(y, \cdot)\) is a subsolution in \(\Gamma\) and a supersolution in \(\Gamma \setminus \{y\}\).
Moreover

\[ S(y, x) = \max\{u(x) : u \text{ is a subsolution s.t. } u(y) = 0\}. \]
Stability

**Theorem**

Assume $H_n(x, p) \to H(x, p)$ uniformly for $n \to \infty$ (i.e. $H'_n(\pi_j^{-1}(x), p) \to H'(\pi_j^{-1}(x), p)$ uniformly for $(x, p) \in \bar{e}_j \times \mathbb{R}$ for any $j \in J$). For any $n \in \mathbb{N}$ let $u_n$ be a solution of

$$H_n(x, Du) = 0, \quad x \in \Gamma,$$

and assume $u_n \to u$ uniformly in $\Gamma$ for $n \to \infty$. Then $u$ is a solution of

$$H(x, Du) = 0.$$
Classification of singularities

We consider the equation

\[
\begin{cases}
|Du|^2 - f(x) = 0, & x \in \Gamma; \\
u(x) = 0, & x \in l_B.
\end{cases}
\]

with \( f > 0 \) in \( \Gamma \). It is possible to prove that an edge contains at most one non-differentiability (singular) point. We define \( k^{\text{edge}} : E \to \{0, 1\} \) by

\[
k^{\text{edge}}(e_j) := \begin{cases} 
1, & \text{if } e_j \text{ contains a singular point;} \\
0, & \text{if } e_j \text{ does not contain a singular point.}
\end{cases}
\]

For a vertex \( v_i \), we

\[
k^{\text{vertex}}(v_i) := \#(Inc_i^-)
\]

where \( Inc_i^- \) are the edges entering “downhill” in \( v_i \) (the more incident edges lead “downhill”, the more \( v_i \) assumes the character of a local maximum and the higher it should be weighted when counting the singularities).

Fabio Camilli ("Sapienza" Univ. di Roma)
Classification of singularities

We consider the equation
\[
\begin{cases}
|Du|^2 - f(x) = 0, & x \in \Gamma; \\
\quad u(x) = 0, & x \in I_B.
\end{cases}
\]

with \( f > 0 \) in \( \Gamma \). It is possible to prove that an edge contains at most one non-differentiability (singular) point. We define \( k^{\text{edge}} : E \to \{0, 1\} \) by
\[
k^{\text{edge}}(e_j) := \begin{cases}
1, & \text{if } e_j \text{ contains a singular point}; \\
0, & \text{if } e_j \text{ does not contain a singular point}.
\end{cases}
\]

For a vertex \( v_i \), we
\[
k^{\text{vertex}}(v_i) := \#(\text{Inc}_i^-)
\]
where \( \text{Inc}_i^- \) are the edges entering “downhill” in \( v_i \) (the more incident edges lead “downhill”, the more \( v_i \) assumes the character of a local maximum and the higher it should be weighted when counting the singularities).
Figure: $\text{Inc}_1^- = 2$

**Theorem**

$$\sum_{i \in I} k^{\text{vertex}}(v_i) + \sum_{j \in J} k^{\text{edge}}(e_j) = \#(J)$$

i.e. the dimension of the singular set of the viscosity solution only depends on the number of edges of $\Gamma$. 
Figure: \( \text{Inc}_1^- = 2 \)

**Theorem**

\[
\sum_{i \in I} k^{\text{vertex}}(v_i) + \sum_{j \in J} k^{\text{edge}}(e_j) = \#(J)
\]

i.e. the dimension of the singular set of the viscosity solution only depends on the number of edges of \( \Gamma \).
A semi-Lagrangian approximation scheme

(see Falcone-Ferretti, J. Comput. Phys. 175 (2002))

Discretization in time

For $h > 0$, we define

i) An admissible trajectory $\gamma^h = \{\gamma^h_m\}_{m=0}^M \subset \Gamma$ is a finite number of points $\gamma^h_m = \pi_{j_m}(t_m) \in \Gamma$ such that for any $m = 0, \ldots, M$, the arc $\gamma^h_m\gamma^h_{m+1} \subset \bar{e}_{j_m}$ for some $j_m \in J$.

ii) $B^h_{x,y}$ is the set of all such paths with $\gamma^h_0 = x$, $\gamma^h_M = y$.

We set

$$u_h(x) = \inf \left\{ \sum_{m=0}^M hL(\gamma^h_m, q_m) + g(y) : \gamma^h \in B^h_{x,y}, y \in \partial \Gamma \right\} \quad x \in \Gamma$$
Set $x_{hq} := \pi_j(t - hq)$ (hence $d(x, x_{hq}) = h|q|$). Then $u_h$ is the unique Lipschitz-continuous solution of

- If $x = \pi_j(t) \in e_j$
  
  $$u^j_h(x) = \inf_{q \in \mathbb{R}: x_{hq} \in \bar{e}_j} \{ u_h(x_{hq}) + hL(x, q) \}$$
  $$= \inf_{q \in \mathbb{R}: x_{hq} \in \bar{e}_j} \{ u^j_h(t - hq) + hL^j(t, q) \}$$

- If $x = v_i \in l_T$
  
  $$u^j_h(v_i) = \inf_{k \in Inc_i} \left[ \inf_{q \in \mathbb{R}: x_{hq} \in \bar{e}_k} \{ u_h(x_{hq}) + hL(v_i, q) \} \right]$$
  $$= \inf_{k \in Inc_i} \left[ \inf_{q \in \mathbb{R}: x_{hq} \in \bar{e}_k} \{ u^k_h(t - hq) + hL^k(t, q) \} \right]$$

- If $x \in l_B$, $u^j_h(x) = g(x)$. 
A semi-Lagrangian approximation scheme

Discretization in space

For $j \in J$, consider a partition

$$P^j = \{ t_0^j = 0 < \cdots < t_m^j < \cdots < t_{M_j}^j = l_j \}$$

of $[0, l_j]$ such that $|P^j| = \max_{1, \ldots, M_j}(t_m^j - t_{m-1}^j) \leq k_j$. Set $x_m^j = \pi_j(t_m^j)$ and consider

$$W^j_{k_j} = \{ w \in C(\bar{e}_j) : \partial_j w(x) \text{ is constant in } (x_{m-1}^j, x_m^j), \ m = 1, \ldots, M_j \}.$$

Every element $w$ in $W^j_{k_j}$ can be expressed as

$$w(x) = \sum_{m=1}^{M_j} \bar{\beta}_m(x) w^j(x_m^j), \quad x \in e_j$$

for $\bar{\beta}_j(x) = \beta_j(\pi_j^{-1}(x))$ and $\beta_j$ tent functions for the partition $P_J$. 
Set $x_{hq}^{j,m} = \pi_j(t_m^j - hq)$ and $k = \max_{j \in J} k_j$ and consider:

Find $u_{hk} : \tilde{\Gamma} \rightarrow \mathbb{R}$ such that $u_{hk}^j \in W_{k_j}^j$ for $j \in J$ and

- If $x_m^j = \pi_j(t_m^j) \in e_j$

  $$u_{hk}^j(x_m^j) = \inf_{q \in \mathbb{R} : x_{hq}^j \in \bar{e}_j} \left\{ u_{hk}(x_{hq}^j) + hL(x_{hq}^j, q) \right\}$$

- If $x_m^j = v_i$

  $$u_{hk}^j(v_i) = \inf_{r \in Inc_i} \left[ \inf_{q \in \mathbb{R} : x_{hq}^r \in \bar{e}_r} \left\{ u_{hk}(x_{hq}^r) + hL(v_i, q) \right\} \right]$$

  $$= \inf_{r \in Inc_i} \left[ \inf_{q \in \mathbb{R} : x_{hq}^r \in \bar{e}_r} \left\{ u_{hk}(t_m^r - hq) + hL(t_m^r, q) \right\} \right]$$

- If $x_m^j = v_i \in I_B$, $u_h^j(x) = g(x)$. 
For \( j \in J \), we set \( U^j = \{ u^j_{hk}(t^j_m) \}_{m=1}^{M_j} \), \( B^j(q) = \{ \beta_l(t^j_m - hq) \}_{l,m=1}^{M_j} \) and \( L^j(q) = \{ L^j(t^j_m, q) \}_{m=1}^{M_j} \), and we rewrite the previous as the finite-dimensional system

\[
\begin{align*}
U^j_m &= \inf_{q \in \mathbb{R} : \pi_j(t^j_m -hq) \in \bar{e}_j} \{ B^j(q)U^j + hL^j_m(q) \} \\
U^i_m &= \inf_{r \in Inc_i} \inf_{q \in \mathbb{R} : \pi_r(t^i_m -hq) \in \bar{e}_r} \{ B^i(q)U^i + hL^i_m(q) \} \\
U^j_m &= g(x^j_m) \\
\end{align*}
\]

\( x^j_m = \pi_j(t^j_m) \in e_j \)

\( x^i_m = \pi_r(t^i_m) \in I_T \)

\( x^j_m = \pi_j(t^j_m) \in I_B \)
Theorem

Assume that $h \to 0$ and $\frac{k}{h} \to 0$. Then $u_{hk}$ converges uniformly to $u$ solution of

$$\begin{cases}
H(x, Du) = 0, & x \in \Gamma;
\quad \\quad \\
\quad u(x) = g(x), & x \in \partial \Gamma.
\end{cases}$$