A Generalized Fast Marching Method on Unstructured Grids

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Outline

- The model problem
- The Generalized Fast Marching Method (GFMM)
- GFMM on Unstructured grids
- Properties and Numerical simulations

Propagation of front: level set approach

The curve

$$\Gamma_t = \{(x, y) \in \mathbb{R}^2, v(x, y, t) = 0\}$$

moves with normal velocity c, if the function v solves the PDE

$$\begin{cases} v_t = c(x, y, t)|Dv| & \mathbb{R}^2 \times (0, T) \\ v(x, y, 0) = dist(x, y, \Gamma_0). \end{cases}$$

in the class of continuous viscosity solutions.
Ref. Crandall, Lions, Evans, Ishii, etc...

Some references

- $c(x, y) > 0$
  Fast Marching Method
  (Tsitsiklis 95, Sethian 96)
- $c(x, y) \geq 0$
  Semi-Lagrangian Fast Marching Methods
  (Falcone, Cristiani 05)
- $c(x, y, t) > 0$
  Ordered Upwind Method
  (Sethian,Vladimirsky 01)
- non-signed $c(x)$
  Bidirectional Fast Marching Method
  (Chopp 09)
- non-signed $c(x, y, t)$
  Generalized Fast Marching Method
  (C., Falcone, Forcadel, Monneau 08)

A Generalized Fast Marching Method (GFMM)

AIM: to extend the FMM to the case $c(x, y, t)$ non signed.

ADVANTAGE :

1. no need of techniques of reinitialization, in case of small gradient of the solution
2. no need of extension of the speed on all the numerical domain
3. complexity $O(NlogN)$ in case of smooth speed c

TOOL : an auxiliary discontinuous function $\theta(x, y, t)$ to track the front.

Non monotone evolution

If the speed function is NOT always positive then the crossing time $u(x,y)$ is NOT single-valued function.
Then we decide to use a discontinuous function to follow the position of the front

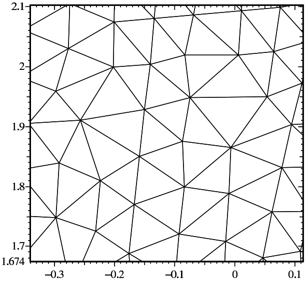
$$\theta(x,y,t) = \begin{cases} 1 & \text{if} \quad x,y \in \Omega_t, \\ -1 & \text{if} \quad x,y \notin \Omega_t. \end{cases}$$

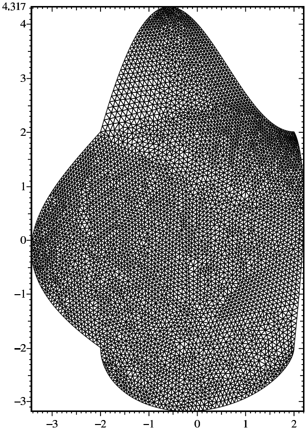
and to solve locally in time the stationary equation for the time evolution

$$\begin{cases} |c(x,y,t_n)||Du(x,y)| = 1 & NB_n \\ u(x,y) = \widehat{u}(x,y) & \partial NB_n \end{cases}$$

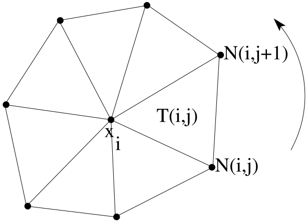
GFMM on UNSTRUCTURED meshes: local solver

Acute final mesh





GFMM on UNSTRUCTURED meshes: local solver

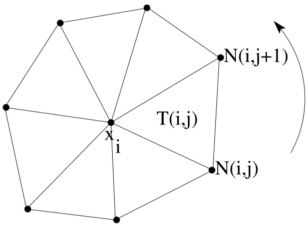


The neighborhood of the node i, is the set of nodes defined

$$V(i) = \{N(i,l), l \in \mathcal{V}(i)\}$$

$N(i,j)$ is the global index of j-th neighboring vertex with
$j \in \mathcal{V}(i) = \{1, \ldots, \mathcal{N}_v(i)\ \}$
$\mathcal{N}_v(i)$ is the number of neighboring vertexes of the node i.

GFMM on UNSTRUCTURED meshes: local solver



We suppose there exists a $\gamma_0 > 0$ s.t. for any mesh

$$\gamma_0 \leq \frac{h_{min}}{h_{max}} \leq 1$$

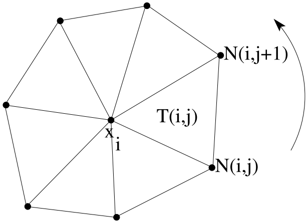
where $h_{max} := \max\{|l_{ij}|,\ i,j \in \{1,\ldots,\mathcal{N}_v\}\}$,
$h_{min} := \min\{|l_{ij}|,\ i,j \in \{1,\ldots,\mathcal{N}_v\}\}$
and l_{ij} is the edge connecting vertex i to vertex j.

GFMM on UNSTRUCTURED mesh

Local problem

$$|Du(x)| = \frac{1}{|c(x_i, t_n)|} \quad \text{in} \quad D_i$$

where D_i is:



General local solver

$$Q\left(x_i, u_i, \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j\in\mathcal{V}(i)}\}\right) = \frac{1}{|c(x_i, t_n)|} \quad i \in \{1, \ldots, \mathcal{N}_v\}.$$

Properties Local Solver: Consistency

(H1)
For any $\psi \in C^2(\mathbb{R}^2)$, let us denote by $\psi_i := \psi(x_i)$ for any $i \in \{1 \dots \mathcal{N}_v\}$ and consider true the following assumptions:

$$\lim_{m\to\infty} Q\left(x_{i_m}, \psi_{i_m}, \{\psi_{N(i_m,j_m)}, \psi_{N(i_m,j_m+1)}\}_{j_m\in\mathcal{V}(i_m)}\right) = |D\psi(x)|$$

where m is an index of refinement for a family of grids $\{\mathcal{M}^T_m\}_{m\geq 0}$ and $(x_{i_m}) \in \mathcal{M}^T_m$ is a sequence of nodes such that for $m \to \infty$

$$(h_{max})_m \to 0 \quad \text{and} \quad x_{i_m} \to x.$$

Properties Local Solver: Monotonicity

(H2)

Let us suppose $u_i \leq \psi_i$ and define

$$\mathcal{C}(i) := \{j \in \mathcal{V}(i),\ \text{s. t. } u_{N(i,j)} \geq \psi_{N(i,j)},\ \ u_{N(i,j+1)} \geq \psi_{N(i,j+1)}\}$$

then

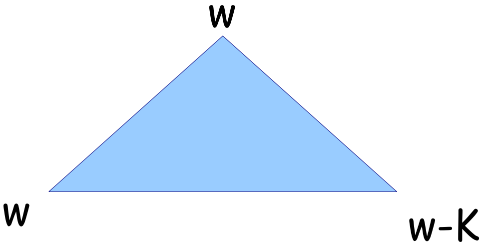
$$Q(x_i, u_i, \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j \in \mathcal{C}(i)}) \leq$$
$$Q(x_i, \psi_i, \{\psi_{N(i,j)}, \psi_{N(i,j+1)}\}_{j \in \mathcal{C}(i)}).$$

Properties Local Solver

(H3)

$$\frac{K}{h_{max}} \leq Q(x_i, w, \{w, w-K\}) \leq \frac{K}{h_{min}}$$

for any positive constant K, for any $w \in \mathbb{R}$.



Properties Local Solver

(H4) Let $\mathcal{I}(i)$,$\mathcal{J}(i)$ two set of indices, s.t.

$$\mathcal{I}(i) \subset \mathcal{J}(i),$$

then

$$Q(x_i, u_i, \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j \in \mathcal{I}(i)}) \leq$$
$$Q(x_i, u_i, \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j \in \mathcal{J}(i)}).$$

Example of Local Solver

1. Local problem

$$\begin{cases} |Du(x)| = \frac{1}{|c(x_i,t_n)|} & x \in D_i \\ u(x) = u_h(x) & x \in \partial D_i \end{cases}$$

   with u_h linear function, affine when restricted to a simplex.

2. The Hopf-Lax formula :

$$u(x_i) = \min_{y \in \partial D_i} (u_h(y) + \frac{|x_i - y|}{|c(x_i, t_n)|})$$

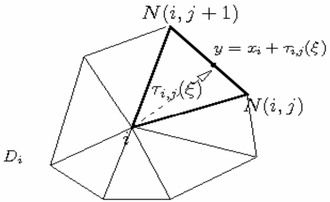
Example of Local Solver: Semi-Lagrangian

From the Hopf-Lax formula

$$\max_{y \in \partial D_i} \left(\frac{u(x_i) - u_h(y)}{|x_i - y|} \right) = \frac{1}{|c(x_i, t_n)|},$$

and since u_h is affine on each simplex:

$$Q(x_i, u(x_i), \{u_{N(i,j)}, u_{N(i,j+1)}\}_{j \in \mathcal{V}(i)}) = \max_{j \in \mathcal{V}(i)} \max_{0 \le \xi \le 1} \left(\frac{u_i - (1-\xi) u_{N(i,j+1)} - \xi u_{N(i,j)}}{|\tau_{i,j}(\xi)|} \right)$$



Ref. Sethian Vladimirsky(2006)

Example of Local Solver: Bornemann-Rash

Since u_h is an affine function on each simplex:

$$u(x_i) = \min_{j \in \mathcal{V}(i)} \min_{y \in [y_i, z_i]} \left(u_h(y) + \frac{|x_i - y|}{|c(x_i, t_n)|} \right) = \min_{j \in \mathcal{V}(i)} (u_j^*)$$

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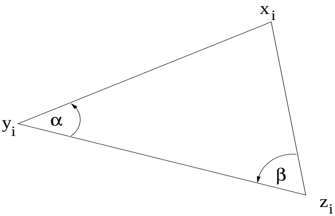

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and defining $\Delta = \frac{(u_h(z_i) - u_h(y_i))}{|z_i - y_i|}$,

$$u_h(y) = u_h(y_i) + \Delta |y - y_i| = u_h(z_i) - \Delta |y - z_i|$$

$$u_j^* = u_h(y_i) + \min_{y \in [y_i, z_i]} \left(\Delta |y - y_i| + \frac{|x_i - y|}{|c(x_i, t_n)|} \right)$$

Example of Local Solver: Bornemann-Rash

By geometric argument, the min can be explicIty evaluated

$$u_j^* = u_h(y_i) + \min_{y \in [y_i, z_i]} \left(\Delta |y - y_i| + \frac{|x_i - y|}{|c(x_i, t_n)|} \right)$$

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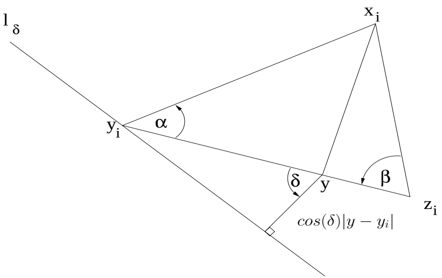

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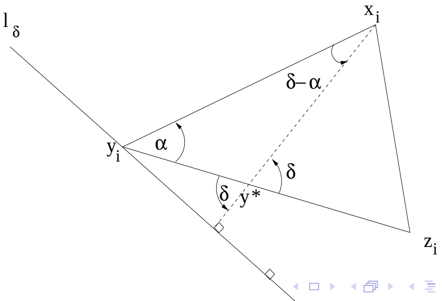

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Defining $\cos(\delta) = \Delta$, if $|\Delta| \leq 1$, we get

$$u_j^* = \begin{cases} u_h(y_i) + \frac{|y_i - x_i|}{|c(x_i,t_n)|}, & \cos(\alpha) < \Delta, \\ u_h(y_i) + \cos(\delta - \alpha)\frac{|y_i - x_i|}{|c(x_i,t_n)|}, & -\cos(\beta) \leq \Delta \leq \cos(\alpha), \\ u_h(z_i) + \frac{|z_i - x_i|}{|c(x_i,t_n)|}, & \Delta < -\cos(\beta). \end{cases}$$

Ref. Kimmel and Sethian (1998), Bornemann-Rash(2005)

GFMM on UNSTRUCTURED meshes

We introduce an auxiliary discrete function

$$\theta_i^n = \begin{cases} 1 & \text{if } x_i \in \Omega_n \\ -1 & \text{otherwise.} \end{cases}$$

We give a slightly different definition, of the two phases:

Definition

$$\Theta_\pm^n \equiv \{i : \theta_i^n = \pm 1 \text{ and } \exists\, j \in V(i) \text{ such that } \theta_j^n = \pm 1\},$$

Note: *isolated nodes*:

$$IN_\pm^n \equiv \{i : \ \theta_i^n = \pm 1 \text{ and } \theta_j^n = \mp 1 \text{ for all } j \in V(i)\}$$

An *isolated node* can only change its phase *but* it can not contribute to change the phase of its neighboring.

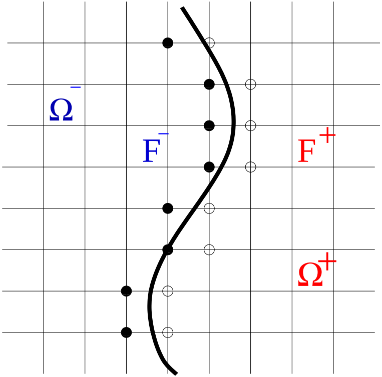
GFMM on UNSTRUCTURED meshes

GFMM on UNSTRUCTURED meshes

We define

- the fronts $F^n_\pm$

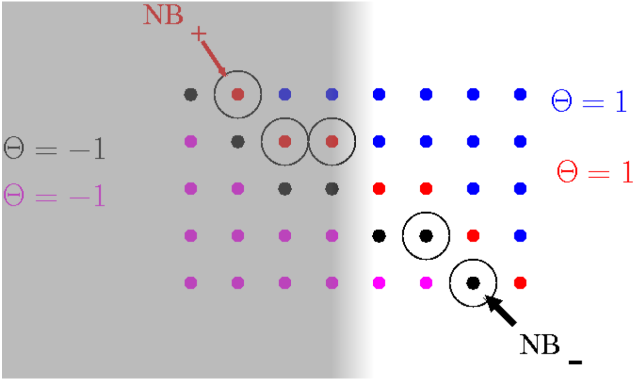
$$F^n_+ \equiv V(\Theta^n_-)\backslash\Theta^n_-, \quad F^n_- \equiv V(\Theta^n_+)\backslash\Theta^n_+$$



GFMM on UNSTRUCTURED meshes

- the Narrow Bands $NB^n_{\pm}$

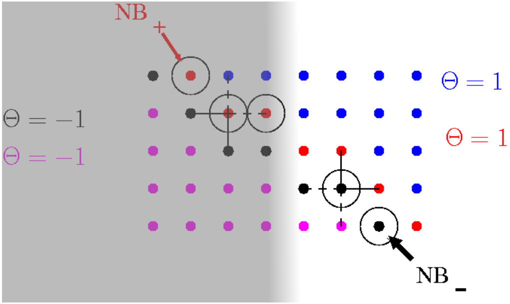
$$NB^n_+ = F^n_+ \cap \{i, \hat{c}^n_i < 0\}, \quad NB^n_- = F^n_- \cap \{i, \hat{c}^n_i > 0\}.$$



GFMM on UNSTRUCTURED meshes

- the Useful nodes for $i \in NB_{\pm}^n$

$$\mathcal{U}^n(i) = \{j \in V(i),\ j \in \Theta_{\mp}^n\}, \quad \mathcal{U}^n = \bigcup_{i \in NB^n} \mathcal{U}^n(i).$$



GFMM on Unstructured Meshes

Initialization

- *Initialization of the matrix* θ^0
  $$\theta_i^0 = \begin{cases} 1 & x_i \in \Omega_0 \\ -1 & x_i \notin \Omega_0 \end{cases}$$
- *Initialization of the time on the front*
  $u_i^0 = 0$ for all $i \in \mathcal{U}^0$
- $n = 1$

GFMM on Unstructured Mesh

Main Cycle

1 Compute the time $\tilde{u}_i^{n-1}$ in the NB_+^{n-1} and NB_-^{n-1} using a local solver

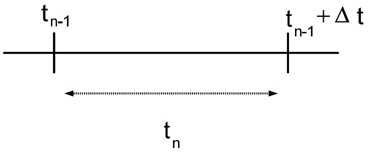
$$Q(x_i, \tilde{u}_i^{n-1}, \{u_{N(i,j)}^{n-1}, u_{N(i,j+1)}^{n-1}\}_{j\in V(i)}) = \frac{1}{|c(x_i, t_n)|}$$

using respectively the values u^{n-1} defined on $\mathcal{U}^{n-1} \cap F_-^{n-1}$ or $\mathcal{U}^{n-1} \cap F_+^{n-1}$.

2 Compute the minimal time $\widetilde{t}_n = \min\{\tilde{u}^{n-1},\ i \in NB_\pm^{n-1}\}$

3 $t_n = \max\{t_{n-1}, \min\{\widetilde{t}_n, t_{n-1} + \Delta t\}$

4 if $t_n < \widetilde{t}_n$ go to 1



GFMM on Unstructured Mesh

Main Cycle

5 Initialize the new accepted points
$NA^n_\pm = \{i \in NB^{n-1}_\pm \; u^n_i = \widetilde{t}_n\}$,

6 *Update* θ^n

$$\theta^n_i = \begin{cases} -\theta^{n-1}_i & \text{for } i \in NA^n \\ \theta^{n-1}_i & \text{elsewhere} \end{cases}$$

7 Update $F^n_\pm$ and $NB^n_\pm$

8 If $i \in \mathcal{U}^n$ then

- if $i \notin \mathcal{U}^{n-1}$ or $i \in NA^n$, then $u^n_i = t_n$.
- if $i \in \mathcal{U}^{n-1} \backslash NA^n$, then $u^n_i = u^{n-1}_i$.

9 *Remove isolated points*
If $i \in IN^n$ and $i \in IN^{n-1}$ then $\theta^n_i = -\theta^{n-1}_i$

10 $n := n + 1$ and go to 1

Non constant time step!

The time step $\Delta t_n = t_{n+1} - t_n$ is not constant and we can actually have:

1. $\Delta t_n >> 1$ too large time step
2. $\Delta t_n < 0$ not increasing time

To avoid case 1. we choose

$$\widehat{t}_n \equiv t_n + \Delta t$$

and to avoid case 2.

$$t_n = t_{n-1}.$$

Then one always gets

$$0 \leq \Delta t_n < \Delta t$$

If case 1) occurs: do not advance the front!

GFMM on UNSTRUCTURED MESH: Definition of $\theta^\epsilon(x,t)$

$\{t_{k_n}, n \in \mathbb{N}\}$ is a strictly increasing subsequence of $(t_n)_n$ such that

$$t_{k_{n-1}} < t_{k_n} < t_{k_{n+1}}.$$

Extension of $(\theta_i^n)_{n,i}$ on the continuous time interval $[0,T]$

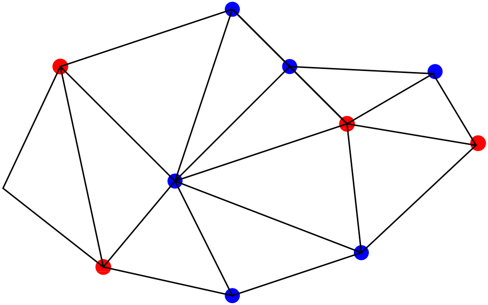
$$\theta(x_i,t) = \theta_i^{k_{n+1}-1} \quad \text{if } (x_i,t) \in \{x_i\} \times [t_{k_n}, t_{k_{n+1}}[$$

(Same extension on structured grids.)

GFMM on UNSTRUCTURED MESH: Definition of $\theta^\epsilon(x,t)$

Let $\epsilon = (h_{max}, \Delta t)$ and $\theta^\epsilon(x,t)$ be an extension of $(\theta(x_i,t_n))_i$ on a continuous domain Ω of $\mathbb{R}^2$

• $\theta = 1$, • $\theta = -1$



(Different than structured grids!)

GFMM on UNSTRUCTURED MESH: Definition of $\theta^\epsilon(x,t)$

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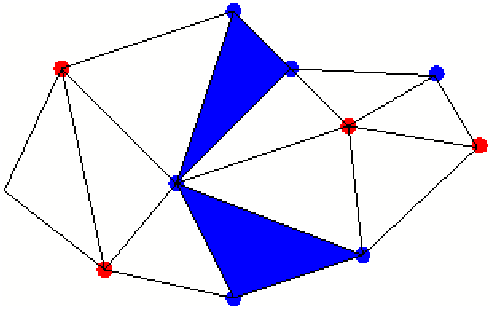

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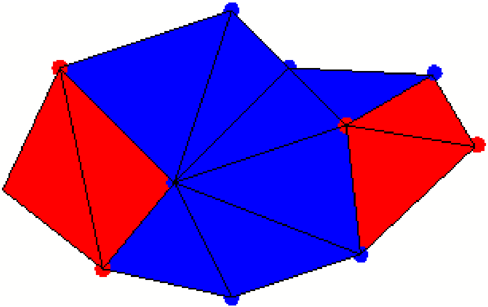

Convergence result

Theorem (C., Falcone, Hoch)
Let $c(x,t)$ be globally Lipschitz continuous in space and time, the initial set Ω_0 be with piece wise smooth boundary then

$$\overline{\theta}^0(x,t) = \limsup_{\epsilon\to 0, z\to x, s\to t} \theta^\epsilon(z,s)$$

(resp. $\underline{\theta}^0(x,t) = \liminf_{\epsilon\to 0, z\to x, s\to t} \theta^\epsilon(z,s)$)
is a viscosity sub-solution (resp. super-solution) of the problem

$$\begin{cases} \theta_t = c(x,y,t)|D\theta| & \mathbb{R}^2 \times (0,T) \\ \theta = 1_{\Omega_0} - 1_{\Omega_0^c} & \mathbb{R}^2. \end{cases}$$

Skip Proof

Idea of the proof

By contradiction, assume that there are (x_0, t_0) and $\varphi \in C^2$ such that $(\overline{\theta}^0) - \varphi$ reaches a strict maximum (x_0, t_0) with $(\overline{\theta}^0)(x_0, t_0) = \varphi(x_0, t_0) = 1$ and

$$\varphi_t(x_0, t_0) > c(x_0, t_0)|D\varphi(x_0, t_0)|,$$

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$$\varphi_t(x_0, t_0) > c(x_0, t_0)|D\varphi(x_0, t_0)|,$$

If $|D\varphi(x_0, t_0)| \neq 0$, there exists $\alpha > 0$ s.t.

$$\varphi_t(x_0, t_0) = \alpha + c(x_0, t_0)|D\varphi(x_0, t_0)| = \bar{c}|D\varphi(x_0, t_0)|$$

with $\bar{c} > c(x_0, t_0)$

Idea of the proof

By contradiction, assume that there are (x_0, t_0) and $\varphi \in C^2$ such that $(\overline{\theta}^0) - \varphi$ reaches a strict maximum (x_0, t_0) with $(\overline{\theta}^0)(x_0, t_0) = \varphi(x_0, t_0) = 1$ and

$$\varphi_t(x_0, t_0) > c(x_0, t_0)|D\varphi(x_0, t_0)|,$$

By classical argument, $\exists\ (x_\epsilon, t_\epsilon) \to (x_0, t_0)$ as $\epsilon \to 0$ s.t.

$$\max((\theta^\epsilon)^* - \varphi)) = ((\theta^\epsilon)^* - \varphi))(x_\epsilon, t_\epsilon) = 0,$$

where

$$(\theta^\epsilon)^*(x, t) = \limsup_{z \to x, s \to t} \theta^\epsilon(z, s)$$

Idea of the proof

- $c(x_0, t_0) > 0$

Since $\varphi_t(x_0, t_0) > 0$ (by the property of φ and the $(\theta^\epsilon)^*$)$\Rightarrow$

$$\theta_i^{n-1} = -1, \quad \theta_i^n = 1$$

where $(x_i, t_n) \in B_r(x_0, t_0)$

Idea of the proof

- $c(x_0, t_0) > 0$

Since $\varphi_t(x_0, t_0) > 0$ (by the property of φ and the $(\theta^\epsilon)^*$)$\Rightarrow$

$$\theta_i^{n-1} = -1, \quad \theta_i^n = 1$$

where $(x_i, t_n) \in B_r(x_0, t_0)$
and (by the Implicit Function theorem) there exists a function Ψ s.t.

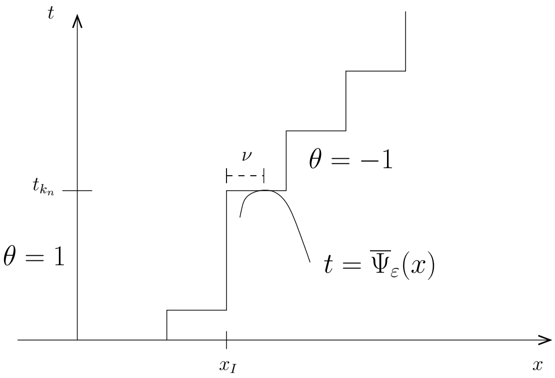
$$\{\varphi(x,t) \geq 1\} = \{t \geq \Psi(x)\}$$

then, since $(\theta^\epsilon)^*(x,t) \leq \varphi(x,t)$

$$\{(\theta^\epsilon)^*(x,t) = 1\} \subset \{t \geq \Psi(x)\}$$

for any $(x,t) \in B_r(x_0, , t_0)$

Idea of the proof



Then applying the local solver on the test function Ψ and numerical solution u, we obtain an absurd

$$\bar{c}(x_0, t_0) \leq c(x_0, t_0)$$

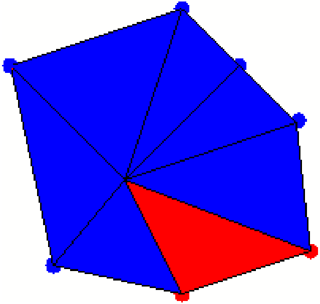
.

Idea of the proof: difficulty with unstructured grids

Let us suppose φ is a test function s.t. $\varphi \geq (\theta^\epsilon)^*$ and $\varphi_t(x_\epsilon, t_\epsilon) > 0$. Then

$$\theta_i^{n-1} = -1, \quad \theta_i^n = 1$$

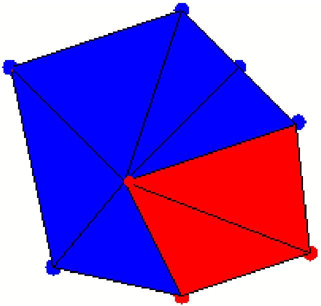
Back to proof



Idea of the proof: difficulty with unstructured grids

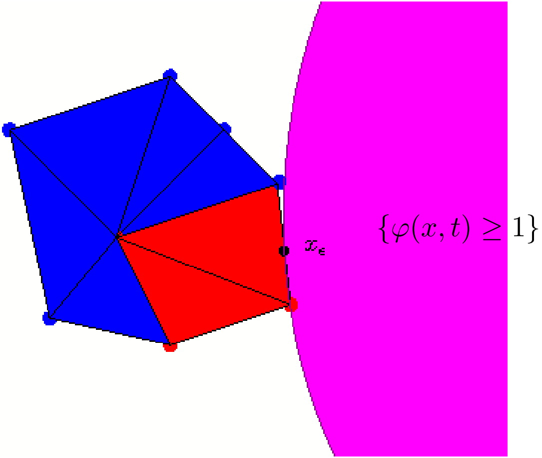
Let us suppose φ is a test function s.t. $\varphi \geq (\theta^{\epsilon})^*$ and $\varphi_t(x_\epsilon, t_\epsilon) > 0$. Then

$$\theta_i^{n-1} = -1, \quad \theta_i^n = 1$$



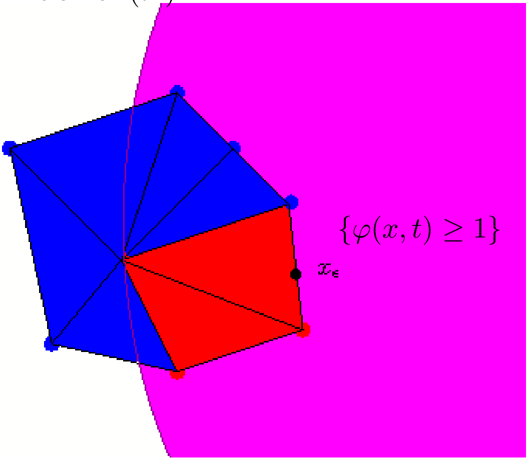
Idea of the proof: difficulty with unstructured grids

Then $\{(\theta^\epsilon)^*(x,t)=1\} \subset \{\varphi(x,t)\geq 1\}$



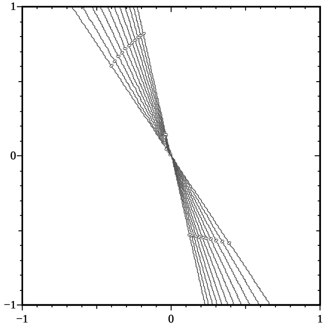
Idea of the proof: difficulty with unstructured grids

We would like to define a φ_ϵ such that $\varphi_\epsilon(x_i) = \varphi(x_\epsilon)$.
But translations of φ on unstructured grids do not generally maintain the same definition of $(\theta^\epsilon)^*$!



Numerical tests: rotating line

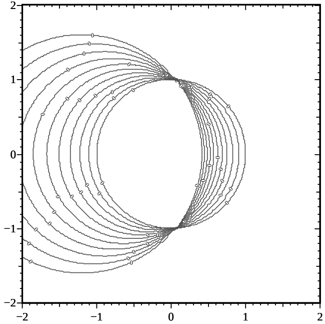
Speed $c(x, y, t) = x$



Hausdorff Error	
h_{max}	$H(C^{ex}, C^{ap})$
.04	0.0350906
.02	0.0169257
.01	0.00886822
.005	0.00436559

Numerical tests: evolution of one circles

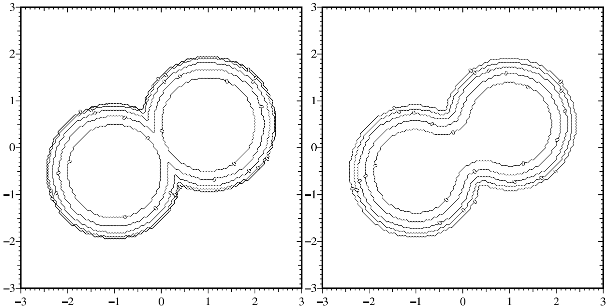
Speed $c(x, y, t) = 0.1t - x$



Hausdorff Error	
h_{max}	$H(C^{ex}, C^{ap})$
0.08	0.0745711
0.04	0.0319709
0.02	0.0189972
0.01	0.0133406

Numerical tests: evolution of two circles

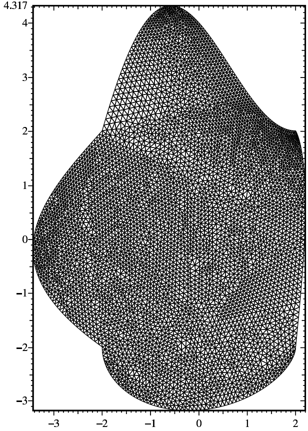
Speed $c(x, y, t) = 1 - t$

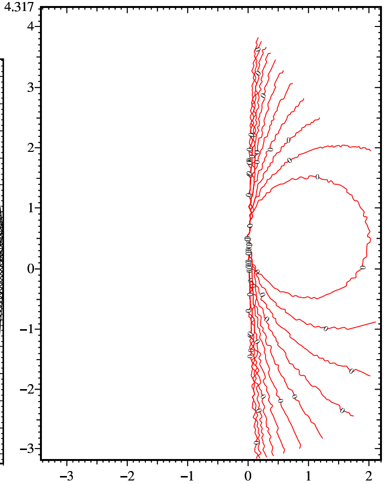


Increasing (left) and decreasing (right) evolution of two circles

Numerical tests: general domain

Speed $c(x, y, t) = x$





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