Split Bregman Method for Minimization of Region-Scalable Fitting Energy for Image Segmentation

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Outline

- Review of Region-based Active Contour Models
  - Mumford Shah Model
  - CV Piecewise Constant Model
  - VC Piecewise Smooth Model
- Images with intensity inhomogeneity
- Region-Scalable Fitting Energy Model
- Split Bregman Method for Minimization of Region-Scalable Fitting Energy
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Mumford Shah Model

Given an image \( I \), find a contour \( C \) in \( \Omega \), and a piecewise smooth image \( u \) approximating the original image \( I \) which minimize the energy functional

\[
F^{MS}(u, C) = \int_{\Omega} (u - I)^2 + \mu \int_{\Omega \setminus C} |\nabla u|^2 + \nu |C|
\]

where \( |C| \) is the length of contour \( C \).

-- data fidelity
-- smooth approximation
-- contour compactness
**Piecewise Constant Model**

**Assumption:** Intensity are piecewise constant inside and outside of the contour $C$

$$u(x) = \begin{cases} 
  c_1, & x \in \text{outside}(C) \\
  c_2, & x \in \text{inside}(C)
\end{cases}$$

The model they proposed is to minimize the following energy:

$$F_{CV}^C(C, c_1, c_2) = \lambda_1 \int_{\text{outside}(C)} |I(x) - c_1|^2 \, dx + \lambda_2 \int_{\text{inside}(C)} |I(x) - c_2|^2 \, dx + \nu |C|$$

Where $\lambda_1$, $\lambda_2$ and $\nu$ are positive constants, $\text{outside}(C)$ and $\text{inside}(C)$ represent the regions outside and inside the contour $C$, respectively.
Piecewise Constant Model

Chan & Vese: IEEE 2001:

Assumption: Intensity are piecewise constant inside and outside of the contour,

\[ u(x, y) = c_1 H(\phi(x, y)) + c_2 (1 - H(\phi(x, y))) \]

Remind: Mumford and Shah functional

\[ F^{MS}(u, \Gamma) = \alpha \int_\Omega (u - I)^2 \, dxdy + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dxdy + \mu |\Gamma| \]

Consider the following functional

\[ E^{CVC}(c_1, c_2, \phi) = \alpha_1 \int_\Omega |c_1 - I|^2 H(\phi) \, dxdy + \alpha_2 \int_\Omega |c_2 - I|^2 (1 - H(\phi)) \, dxdy \]

\[ + \mu \int_\Omega \delta(\phi) |\nabla \phi| \, dxdy \]
Minimization Procedure

How can we minimize $F^{C^V C}(c_1, c_2, \Gamma)$ ???

$E^{C^V C}(c_1, c_2, \phi) = \alpha_1 \int_{\Omega} |c_1 - I|^2 H(\phi)\,dxdy + \alpha_2 \int_{\Omega} \left|c_2 - I\right|^2 (1 - H(\phi))\,dxdy + \mu \int_{\Omega} \delta(\phi) \left|\nabla \phi\right|\,dxdy$

* start from an initial guess for $\Gamma$

* morph $\Gamma$ and update $c_1$ and $c_2$ in the descent direction of the functional until they reach the optimal solutions

Keep the contour fixed and minimize the energy:

$c_1(\phi) = \frac{\int_{\Omega} I(x, y)H(\phi)\,dxdy}{\int_{\Omega} H(\phi)\,dxdy} ; \quad c_2 = \frac{\int_{\Omega} I(x, y)(1 - H(\phi))\,dxdy}{\int_{\Omega} (1 - H(\phi))\,dxdy}$

Keep $c_1$ and $c_2$ fixed and minimize w.r.t. $\phi$

$$\frac{\partial \phi}{\partial t} = \delta(\phi) \left[ \mu \nabla \cdot \left( \frac{\nabla \phi}{\left|\nabla \phi\right|} \right) - \mu_2 - \lambda_1 (I - c_1)^2 + \lambda_2 (I - c_2)^2 \right]$$
Images with intensity inhomogeneity

Intensity Images:  
- gray scale images \( I : \Omega \rightarrow \mathbb{R} \)
- color images \( I : \Omega \rightarrow \mathbb{R}^3 \)
Numerical Results: Difficulty for images with inhomogeneity
Instead of considering piecewise constant inside and outside of the contour $C$, introduce two functions $u^+$ and $u^-$ such that:

$$u(x) = \begin{cases} u^+(x), & x \in \text{outside}(C) \\ u^-(x), & x \in \text{inside}(C) \end{cases}$$

Then the energy functional becomes:

$$F^{VCS}(u^+, u^-, C) = \int_{\text{outside}(C)} (u^+ - I)^2 + \int_{\text{inside}(C)} (u^- - I)^2 +$$

$$\mu \int_{\text{outside}(C)} |\nabla u^+|^2 + \mu \int_{\text{inside}(C)} |\nabla u^-|^2 + \nu |C|$$
Instead of considering piecewise constant inside and outside of the contour, introduce two functions $u^+$ and $u^-$ such that

$$u(x, y) = u^+(x, y)H(\phi(x, y)) + u^-(x, y)(1 - H(\phi(x, y)))$$

Remind: Mumford and Shah functional

$$F^{MS}(u, \Gamma) = \alpha \int_{\Omega} (u - I)^2 \, dx \, dy + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx \, dy + \mu |\Gamma|$$

Consider the following functional

$$E^{VCS}(u^+, u^-, \phi) = \alpha \int_{\Omega} |u^+ - I|^2 H(\phi) \, dx \, dy + \alpha \int_{\Omega} |u^- - I|^2 (1 - H(\phi)) \, dx \, dy$$

$$+ \beta \int_{\Omega} |\nabla u^+|^2 H(\phi) \, dx \, dy + \beta \int_{\Omega} |\nabla u^-|^2 (1 - H(\phi)) \, dx \, dy + \mu \int_{\Omega} \delta(\phi) |\nabla \phi| \, dx \, dy$$
Piecewise Smooth Model

\[ E^{VCS}(u^+, u^-, \phi) = \alpha \int_{\Omega} |u^+ - I|^2 H(\phi) \, dx \, dy + \alpha \int_{\Omega} |u^- - I|^2 (1 - H(\phi)) \, dx \, dy \]

\[ + \beta \int_{\Omega} \nabla u^+ \cdot \nabla H(\phi) \, dx \, dy + \beta \int_{\Omega} \nabla u^- \cdot \nabla (1 - H(\phi)) \, dx \, dy + \mu \int_{\Omega} \delta(\phi) \, \nabla \phi \, dx \, dy \]

Keep the contour fixed and minimize the energy: Euler-Lagrange equations

\[ u^+ - I = \beta \Delta u^+ \text{ on } \{\phi > 0\} \]  \[ \frac{\partial u^+}{\partial n} = 0 \text{ on } \{\phi = 0\} \]

\[ u^- - I = \beta \Delta u^- \text{ on } \{\phi < 0\} \]  \[ \frac{\partial u^-}{\partial n} = 0 \text{ on } \{\phi = 0\} \]

Keep \( u^+ \) and \( u^- \) fixed and minimize w.r.t. \( \phi \)

\[ \frac{\partial \phi}{\partial t} = \delta(\phi) \left[ \mu \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) - \alpha (u^+ - I)^2 + \alpha (u^- - I)^2 \right] \]

\[ - \beta |\nabla u^+|^2 + \beta |\nabla u^-|^2 \]

Difficultly: At each iteration, 2 pde on irregular domains need to be solved

need to extend \( u^+ \) and \( u^- \)
Li et al. propose a region-scalable fitting energy model:

\[
E(C, f_1(x), f_2(x)) = \sum_{i=1}^{2} \lambda_i \int \left[ \int_{\Omega_i} K_\sigma(x - y) |I(y) - f_i(x)|^2 \, dy \right] \, dx + \nu |C|
\]

The aim of the kernel function \( K_\sigma \) is to put heavier weights on points \( y \) which are close to the center point \( x \). For simplicity, a Gaussian kernel with a scale parameter \( \sigma > 0 \) was used:

\[
K_\sigma(u) = \frac{1}{2\pi\sigma^2} e^{-\frac{|u|^2}{2\sigma^2}}
\]

The level set formulation is:

\[
E_\varepsilon(\phi, f_1(x), f_2(x)) = \sum_{i=1}^{2} \lambda_i \int \left( \int K_\sigma(x - y) |I(y) - f_i(x)|^2 M^\varepsilon_1(\phi(y)) \, dy \right) \, dx + \nu \int |\nabla H_\varepsilon(\phi(x))| \, dx
\]

where \( M^\varepsilon_1(\phi) = H_\varepsilon(\phi) \) and \( M^\varepsilon_2(\phi) = 1 - H_\varepsilon(\phi) \).

A level set regularization term \( P(\phi) \) is used to preserve the regularity of the level set function \( \phi \):

\[
P(\phi) = \int \frac{1}{2} \left( |\nabla \phi(x)| - 1 \right)^2 \, dx
\]
There, the energy functional to minimize is:

\[ F(\phi, f_1, f_2) = E_\varepsilon(\phi, f_1, f_2) + \mu P(\phi) \]

Keep \( \phi \) fixed and minimize the energy:

\[ f_i(x) = \frac{K_\sigma(x)\ast [M_\varepsilon(\phi(x))I(x)]}{K_\sigma(x)\ast M_\varepsilon(\phi(x))}, \quad i = 1, 2 \]

Keep \( f_1 \) and \( f_2 \) fixed and minimize w.r.t. \( \phi \):

\[ \frac{\partial \phi}{\partial t} = -\delta_\varepsilon(\phi)(\lambda_1 e_1 - \lambda_2 e_2) + \nu \delta_\varepsilon(\phi) \text{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) + \mu \left(\nabla^2 \phi - \text{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)\right) \]

where \( \delta_\varepsilon \) is the derivative of \( H_\varepsilon \), and \( e_i \ (i = 1 \ or \ 2) \) is defined as:

\[ e_i(x) = \int K_\sigma(y - x)|I(x) - f_i(y)|^2 \, dy, \quad i = 1, 2 \]
Some Results for RSF Model

\( \mu = 0.001 \times 255^2, \tau = 0.1, \nu = 1, \sigma = 3.0, \lambda_1 = \lambda_2 = 1.0 \)
Some Results for RSF Model
Considering the gradient flow equation in the RSF model:

\[
\frac{\partial \phi}{\partial t} = -\delta_\varepsilon (\phi) (\lambda_1 e_1 - \lambda_2 e_2) + \nu \delta_\varepsilon (\phi) \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right) + \mu \left( \nabla^2 \phi - \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \right)
\]

Drop the last term and take \( \nu = 1 \):

\[
\frac{\partial \phi}{\partial t} = \delta_\varepsilon (\phi) \left( -\lambda_1 e_1 + \lambda_2 e_2 \right) + \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right)
\]

Following the idea from Chan et al., the stationary solution of the above equation coincides with the stationary solution of:

\[
\frac{\partial \phi}{\partial t} = \left( -\lambda_1 e_1 + \lambda_2 e_2 \right) + \text{div} \left( \frac{\nabla \phi}{|\nabla \phi|} \right)
\]

This simplified flow represents the gradient descent for minimization problem

\[
\min_{a_0 \leq \phi \leq b_0} E(\phi) = \min_{a_0 \leq \phi \leq b_0} |\nabla \phi|_1 + \langle \phi, r \rangle
\]

where the restriction \( a_0 \leq \phi \leq b_0 \) is to guarantee a unique global minimizer and \( r = \lambda_1 e_1 - \lambda_2 e_2 \)

Then the segmented region can be found for some \( \alpha \in (a_0, b_0) \):

\[
\Omega_1 = \{ x : \phi(x) > \alpha \}
\]
The new proposed Region-Scalable Fitting Energy

Replace the standard TV norm $TV(\phi) = \int |\nabla \phi| = |\nabla \phi|_1$ with the weighted version:

$$TV_g (\phi) = \int g |\nabla \phi| = |\nabla \phi|_g$$

where

$$g(\xi) = \frac{1}{1 + \beta |\xi|^2}$$

is the non-negative edge detector function.

Then the minimization problem becomes:

$$\min_{\phi \in \Omega} E(\phi) = \min_{\nabla \phi \in \Omega} |\nabla \phi|_g + \langle \phi, r \rangle$$

To apply the Split Bregman approach, an auxiliary variable $\tilde{d} \leftarrow \nabla \phi$ is introduced. Apply Bregman iteration to strictly enforce the constraint $\tilde{d} = \nabla \phi$, the resulting sequence of optimization problems is:

$$\begin{align*}
\left( \phi^{k+1}, \tilde{d}^{k+1} \right) & = \arg \min_{\phi \in \Omega} |\tilde{d}|_g + \langle \phi, r \rangle + \frac{\lambda}{2} \| \tilde{d} - \nabla \phi - \tilde{b}^k \|^2 \\
\tilde{b}^{k+1} & = \tilde{b}^k + \nabla \phi^{k+1} - \tilde{d}^{k+1}
\end{align*}$$

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Apply Split Bregman Method for Minimization

For fixed $\vec{d}$, minimize w.r.t. $\phi$:

$$\Delta \phi = \frac{r}{\lambda} + \nabla \cdot (\vec{d} - \vec{b}), \quad a_0 < \phi < b_0$$

Using central discretization for Laplace operator and backward difference for divergence operator, the numerical scheme is:

$$\begin{cases}
\alpha_{i,j} = d_{i-1,j}^x - d_{i,j}^x + d_{i,j}^y - d_{i-1,j}^y - \left(b_{i-1,j}^x - b_{i,j}^x + b_{i,j}^y - b_{i-1,j}^y\right) \\
\beta_{i,j} = \frac{1}{4} \left(\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i,j+1} - \frac{r}{\lambda} + \alpha_{i,j}\right) \\
\phi_{i,j} = \max \left\{\min \left\{\beta_{i,j}, b_0\right\}, a_0\right\}
\end{cases}$$

For fixed $\phi$, minimize w.r.t. $\vec{d}$:

$$\vec{d}^{k+1} = \text{shrink}_g \left(\vec{b}^k + \nabla \phi^{k+1}, \frac{1}{\lambda}\right) = \text{shrink} \left(\vec{b}^k + \nabla \phi^{k+1}, \frac{g}{\lambda}\right)$$

where

$$\text{shrink} \left(x, r\right) = \frac{x}{|x|} \max \left(|x| - r, 0\right)$$
Experimental Results (1):
Segmentation of a synthetic image

- **Comparison** between the proposed method and split Bregman on PC model
- **Column 1:** the original image and the initial contour
- **Column 2:** the result of our proposed method
- **Column 3:** the result of the split Bregman on PC model
Experimental Results (2):
Boundary extraction for four challenging inhomogeneous images

- Top row: original images with initial contours
- Bottom row: segmentation results with final contours
Efficiency demonstrated by comparing the iteration number and computation time with the original RSF model

<table>
<thead>
<tr>
<th></th>
<th>Image 1</th>
<th>Image 2</th>
<th>Image 3</th>
<th>Image 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our model</td>
<td>32(0.33)</td>
<td>67(1.13)</td>
<td>26(0.49)</td>
<td>48(0.70)</td>
</tr>
<tr>
<td>RSF model</td>
<td>200(1.40)</td>
<td>150(1.74)</td>
<td>300(3.72)</td>
<td>300(3.01)</td>
</tr>
</tbody>
</table>

- From this table, it is clear that our method is more efficient than the RSF model because we apply the split Bregman approach to the optimization problem.
Experimental Results (3):
Segmentation of three synthetic flower images with different distribution of intensities

- Row 1: piecewise constant image
- Row 2: inhomogeneous clean image
- Row 3: inhomogeneous image with noise
Experimental Results (4):
Detect boundary for a color image of flower

- The curve evolution process from the initial contour to the final contour is shown above.
References I

1. Implicit Active Contour/Surfaces Driven by Local Binary Fitting Energy  
   by Chunming Li, Chiu-Yen Kao, and Zhaohua Ding (IEEE CVPR 2007)
2. A Variational Level Set Method for Segmentation of Medical Images with Intensity Inhomogeneity  
   by Chunming Li, Chiu-Yen Kao, John C. Gore, and Zhaohua Ding (IEEE TIP, 2008)
3. Brain MR Image Segmentation Using Local and Global Intensity Fitting Active Contours/Surfaces  
   by Li Wang, Chunming Li, Quansen Sun, Deshen Xia, and Chiu-Yen Kao (MICCAI 2008)
4. Active Contours Driven by Local and Global Intensity Fitting Energy with Application to Brain  
   MR Image Segmentation  by Li Wang, Chunming Li, Quansen Sun, Deshen Xia, and Chiu-Yen  
   Kao (JCMIG, 2009)
5. Split Bregman Method for Minimization of region-Scalable Fitting Energy for Image Segmentation  
   by Yunyun Yang, Chunming Li, Chiu-Yen Kao, Stanley Osher, Advances in Visual Computing,  
   volume 6454 of Lecture Notes in Computer Sciences, pages 117-128, 2010
The End

Thank you for your attention!!

Questions??