A new approximation for effective Hamiltonians for homogenization of a class of Hamilton-Jacobo equations

Songting Luo, Yifeng Yu, Hongkai Zhao

University of California at Irvine

February 16, 2011

Homogenization of Hamilton-Jacobi equations

- For $\epsilon > 0$

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$

Here $H \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and is periodic in the x variable, i.e, $H(p, x+z) = H(p,x)$ for any $z \in \mathbb{Z}^n$.

Homogenization of Hamilton-Jacobi equations

- For $\epsilon > 0$
$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$
Here $H \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and is periodic in the x variable, i.e, $H(p, x+z) = H(p,x)$ for any $z \in \mathbb{Z}^n$.
- Question: what is the behavior of u^ϵ as $\epsilon \to 0$?

Homogenization of Hamilton-Jacobi equations

- For $\epsilon > 0$

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$

  Here $H \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and is periodic in the x variable, i.e, $H(p, x+z) = H(p,x)$ for any $z \in \mathbb{Z}^n$.
- Question: what is the behavior of u^ϵ as $\epsilon \to 0$?
- It was proved by Lions-Papanicoula-Varadhan [LPV](unpublished paper, 1988) the u^ϵ locally uniformly converge to $\bar{u}$ which is the unique solution of the effective equation

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 \\ \bar{u}(x,0) = g(x). \end{cases}$$

  Here $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is the so called "effective Hamiltonian".

Homogenization of Hamilton-Jacobi equations

- For $\epsilon > 0$
$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$
Here $H \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and is periodic in the x variable, i.e, $H(p, x+z) = H(p,x)$ for any $z \in \mathbb{Z}^n$.
- Question: what is the behavior of u^ϵ as $\epsilon \to 0$?
- It was proved by Lions-Papanicoula-Varadhan [LPV](unpublished paper, 1988) the u^ϵ locally uniformly converge to $\bar{u}$ which is the unique solution of the effective equation
$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 \\ \bar{u}(x,0) = g(x). \end{cases}$$
Here $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is the so called "effective Hamiltonian".
- $\bar{H}(p)$ is a nonlinear average of $H(p,x)$.

Homogenization of Hamilton-Jacobi equations

- For $\epsilon > 0$

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$

  Here $H \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and is periodic in the x variable, i.e, $H(p, x+z) = H(p,x)$ for any $z \in \mathbb{Z}^n$.
- Question: what is the behavior of u^ϵ as $\epsilon \to 0$?
- It was proved by Lions-Papanicoula-Varadhan [LPV](unpublished paper, 1988) the u^ϵ locally uniformly converge to $\bar{u}$ which is the unique solution of the effective equation

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 \\ \bar{u}(x,0) = g(x). \end{cases}$$

  Here $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is the so called "effective Hamiltonian".
- $\bar{H}(p)$ is a nonlinear average of $H(p,x)$. Caution: $\bar{H}(p) \neq \int_{\mathbb{T}^n} H(p,x)\, dx$.

Cell problem

Cell problem

Give $p \in \mathbb{R}^n$, there exists a UNIQUE number $\bar{H}(p)$ such that the following cell problem

$$H(p + Dw, x) = \bar{H}(p)$$

admits periodic solutions.

Cell problem

Give $p \in \mathbb{R}^n$, there exists a UNIQUE number $\bar{H}(p)$ such that the following cell problem

$$H(p + Dw, x) = \bar{H}(p)$$

admits periodic solutions.

$u_\epsilon \to \bar{u}$ uniformly. Nevertheless, Du^ϵ does not converge to $D\bar{u}$. Dw represents the oscillation of gradients.

Cell problem

Give $p \in \mathbb{R}^n$, there exists a UNIQUE number $\bar{H}(p)$ such that the following cell problem

$$H(p + Dw, x) = \bar{H}(p)$$

admits periodic solutions.

$u_\epsilon \to \bar{u}$ uniformly. Nevertheless, Du^ϵ does not converge to $D\bar{u}$. Dw represents the oscillation of gradients.

More precisely, if $p = D\bar{u}(x_0)$, then for $x \sim x_0$, we have the following formal asymptotic expansion.

$$u^\epsilon(x) = \bar{u}(x) + \epsilon w(\frac{x}{\epsilon}) + O(\epsilon^2).$$

Hence formally

$$u^\epsilon = \bar{u} + O(\epsilon).$$

Numerical computation of $\bar{H}(p)$: Motivations I

Numerical computation of $\bar{H}(p)$: Motivations I

Use $\bar{u}$ to approximate u^ϵ without resolving small scale ϵ. See the survey paper by Engquist-Souganidis [ES] (*Acta Numer. 2008*) and Achdou-Camilli-Capuzzo Dolcetta [ACC] (*Mathematical models and methods in applied sciences, 2008*).

Numerical computation of $\bar{H}(p)$: Motivations I

Use $\bar{u}$ to approximate u^ϵ without resolving small scale ϵ. See the survey paper by Engquist-Souganidis [ES] (*Acta Numer. 2008*) and Achdou-Camilli-Capuzzo Dolcetta [ACC] (*Mathematical models and methods in applied sciences, 2008*).

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$

Numerical computation of $\bar{H}(p)$: Motivations I

Use $\bar{u}$ to approximate u^ϵ without resolving small scale ϵ. See the survey paper by Engquist-Souganidis [ES] (*Acta Numer. 2008*) and Achdou-Camilli-Capuzzo Dolcetta [ACC] (*Mathematical models and methods in applied sciences, 2008*).

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 \\ \bar{u}(x,0) = g(x). \end{cases}$$

Numerical computation of $\bar{H}(p)$: Motivations I

Use $\bar{u}$ to approximate u^ϵ without resolving small scale ϵ. See the survey paper by Engquist-Souganidis [ES] (*Acta Numer. 2008*) and Achdou-Camilli-Capuzzo Dolcetta [ACC] (*Mathematical models and methods in applied sciences, 2008*).

$$\begin{cases} u^\epsilon_t + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \\ u^\epsilon(x,0) = g(x). \end{cases}$$

$$\begin{cases} \bar{u}_t + \bar{H}(D\bar{u}) = 0 \\ \bar{u}(x,0) = g(x). \end{cases}$$

Use $\bar{u}$ to approximate u^ϵ.

Numerical computation of the effective Hamiltonian: Motivations II

Numerical computation of the effective Hamiltonian: Motivations II

- Weak KAM theorem (Introduced by Fathi). A central project is to understand dynamical information encoded in the effective Hamiltonian.

Numerical computation of the effective Hamiltonian: Motivations II

- Weak KAM theorem (Introduced by Fathi). A central project is to understand dynamical information encoded in the effective Hamiltonian. The most important Hamiltonian is mechanical Hamiltonian, i.e

$$H(p,x) = \sum_{1 \le i,j \le n} a_{i,j}(x) p_i p_j + V(x).$$

Numerical computation of the effective Hamiltonian: Motivations II

- Weak KAM theorem (Introduced by Fathi). A central project is to understand dynamical information encoded in the effective Hamiltonian. The most important Hamiltonian is mechanical Hamiltonian, i.e

$$H(p, x) = \sum_{1 \leq i,j \leq n} a_{i,j}(x) p_i p_j + V(x).$$

- Turbulent combustion theory. A fundamental problem in turbulent combustion theory is to understand the so called "turbulent flame speed" which can be models by effective Hamiltonian.

Numerical computation of the effective Hamiltonian: Motivations II

- Weak KAM theorem (Introduced by Fathi). A central project is to understand dynamical information encoded in the effective Hamiltonian. The most important Hamiltonian is mechanical Hamiltonian, i.e

$$H(p,x) = \sum_{1 \le i,j \le n} a_{i,j}(x) p_i p_j + V(x).$$

- Turbulent combustion theory. A fundamental problem in turbulent combustion theory is to understand the so called "turbulent flame speed" which can be models by effective Hamiltonian. G-equation: $H(p,x) = s_L|p| + b(x) \cdot p$. (N.Peters, *Turbulent combustion*, book, 2000).

Numerical computation of the effective Hamiltonian: Motivations II

- Weak KAM theorem (Introduced by Fathi). A central project is to understand dynamical information encoded in the effective Hamiltonian. The most important Hamiltonian is mechanical Hamiltonian, i.e

$$H(p, x) = \sum_{1 \le i,j \le n} a_{i,j}(x) p_i p_j + V(x).$$

- Turbulent combustion theory. A fundamental problem in turbulent combustion theory is to understand the so called "turbulent flame speed" which can be models by effective Hamiltonian.
  G-equation: $H(p, x) = s_L|p| + b(x) \cdot p$. (N.Peters, *Turbulent combustion*, book, 2000).
  Majada-Souganidis model: $H(p, x) = |p|^2 + b(x) \cdot p$.

The large time method: J. L. Qian [Q] (*UCLA CAM reports, 2003*)

The large time method: J. L. Qian [Q] (*UCLA CAM reports, 2003*)

Given $p \in \mathbb{R}^n$. Let u be the solution of

$$\begin{cases} u_t + H(Du, x) = 0 \\ u(x,0) = p \cdot x. \end{cases}$$

The large time method: J. L. Qian [Q] (*UCLA CAM reports, 2003*)

Given $p \in \mathbb{R}^n$. Let u be the solution of

$$\begin{cases} u_t + H(Du, x) = 0 \\ u(x, 0) = p \cdot x. \end{cases}$$

By comparison principle, it is easy to see that

$$-\bar{H}(p)t + P \cdot x + w(x) - C \le u(x, t) \le -\bar{H}(p)t + P \cdot x + w(x) + C.$$

Here w is the solution of the cell problem

$$H(p + Dw, x) = \bar{H}(p).$$

The large time method: J. L. Qian [Q] (*UCLA CAM reports, 2003*)

Given $p \in \mathbb{R}^n$. Let u be the solution of

$$\begin{cases} u_t + H(Du, x) = 0 \\ u(x,0) = p \cdot x. \end{cases}$$

By comparison principle, it is easy to see that

$$-\bar{H}(p)t + P \cdot x + w(x) - C \leq u(x,t) \leq -\bar{H}(p)t + P \cdot x + w(x) + C.$$

Here w is the solution of the cell problem

$$H(p + Dw, x) = \bar{H}(p).$$

The error estimate follows immediately

$$\bar{H}(p) = \frac{-u(x,T)}{T} + O(\frac{1}{T}).$$

Other known methods

- Inf-max formula based method: Gomes-Oberman [GO] (*SIAM J. Control Optim. (2004)*). If $H = H(p, x)$ is convex in the p variable, then

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} H(p + D\phi, x).$$

Other known methods

- Inf-max formula based method: Gomes-Oberman [GO] (*SIAM J. Control Optim. (2004)*). If $H = H(p, x)$ is convex in the p variable, then
$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} H(p + D\phi, x).$$
- See Camilli-Capuzzo Dolcetta-Gomes [CDG] (*Appl. Math. Optim. 57 (2008)*) for errors estimates of several methods.

A new approach by Oberman-Takei-Vladimirsky for metric Hamiltonians: I

Suppose that the Hamiltonian is convex, positive and Homogeneous of degree in p variable, e.g

$$H(p, x) = c(x)|p|$$

for some positive periodic function $c(x)$. Oberman-Takei-Vladimirsky [OTV] (*Multiscale Modeling and Simulation, 2009*) introduced a complete new scheme to compute the effective Hamiltonian.

A new approach by Oberman-Takei-Vladimirsky for metric Hamiltonians: I

Suppose that the Hamiltonian is convex, positive and Homogeneous of degree in p variable, e.g

$$H(p, x) = c(x)|p|$$

for some positive periodic function $c(x)$. Oberman-Takei-Vladimirsky [OTV] (*Multiscale Modeling and Simulation, 2009*) introduced a complete new scheme to compute the effective Hamiltonian.

(1) $\bar{H}(p)$ is also convex, positive and Homogeneous of degree 1. Any such function can be represented by

$$\bar{H}(p) = \sup_{\{|\alpha|=1\}} c_\alpha p \cdot \alpha.$$

A new approach by Oberman-Takei-Vladimirsky for metric Hamiltonians: I

Suppose that the Hamiltonian is convex, positive and Homogeneous of degree in p variable, e.g

$$H(p,x) = c(x)|p|$$

for some positive periodic function $c(x)$. Oberman-Takei-Vladimirsky [OTV] (*Multiscale Modeling and Simulation, 2009*) introduced a complete new scheme to compute the effective Hamiltonian.
(1) $\bar{H}(p)$ is also convex, positive and Homogeneous of degree 1. Any such function can be represented by

$$\bar{H}(p) = \sup_{\{|\alpha|=1\}} c_\alpha p \cdot \alpha.$$

(2) Let $\bar{u}$ be the solution of

$$\begin{cases} \bar{H}(D\bar{u}) = 1 \\ \bar{u}(0) = 0. \end{cases}$$

A new approach by Oberman-Takei-Vladimirsky for metric Hamiltonians: II

Hopf-Lax formula implies

$$c_\alpha = \frac{1}{\bar{u}(\alpha)}.$$

A new approach by Oberman-Takei-Vladimirsky for metric Hamiltonians: II

Hopf-Lax formula implies

$$c_\alpha = \frac{1}{\bar{u}(\alpha)}.$$

$$\begin{cases} H(Du^\epsilon, \frac{x}{\epsilon}) = 1 \\ u^\epsilon(0) = 0. \end{cases}$$

Formally, $u_\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2) = \bar{u} + O(\epsilon)$.

A new approach by Oberman-Takei-Vladimirsky for metric Hamiltonians: II

Hopf-Lax formula implies

$$c_\alpha = \frac{1}{\bar{u}(\alpha)}.$$

$$\begin{cases} H(Du^\epsilon, \frac{x}{\epsilon}) = 1 \\ u^\epsilon(0) = 0. \end{cases}$$

Formally, $u_\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2) = \bar{u} + O(\epsilon)$. If we replace $\bar{u}$ with u^ϵ, i.e,

$$c_\alpha^\epsilon = \frac{1}{u^\epsilon(\alpha)}.$$

Then formally, $c_\alpha^\epsilon = c_\alpha + O(\epsilon)$.

Extend to other important Hamiltonians

Extend to other important Hamiltonians

(1) Mechanical Hamiltonian from weak KAM theory

$$H(p,x) = \sum_{1 \le i,j \le n} a_{i,j}(x) p_i p_j + V(x)$$

Extend to other important Hamiltonians

(1) Mechanical Hamiltonian from weak KAM theory

$$H(p,x) = \sum_{1\le i,j\le n} a_{i,j}(x)p_i p_j + V(x)$$

(2) Convective Hamiltonian (The Majada-Souganidis model) from the turbulent combustion theory.

$$H(p,x) = |p|^2 + b(x)\cdot p.$$

Extend to other important Hamiltonians

(1) Mechanical Hamiltonian from weak KAM theory

$$H(p,x) = \sum_{1\leq i,j\leq n} a_{i,j}(x)p_i p_j + V(x)$$

(2) Convective Hamiltonian (The Majada-Souganidis model) from the turbulent combustion theory.

$$H(p,x) = |p|^2 + b(x)\cdot p.$$

- The associated effective Hamiltonians are very complicated, in particular, is NOT homogeneous of any degree. We can NOT expect an elegant formula between solutions of the effective equation and the effective Hamiltonian as in the metric case.

Extend to other important Hamiltonians

(1) Mechanical Hamiltonian from weak KAM theory

$$H(p,x) = \sum_{1 \le i,j \le n} a_{i,j}(x) p_i p_j + V(x)$$

(2) Convective Hamiltonian (The Majada-Souganidis model) from the turbulent combustion theory.

$$H(p,x) = |p|^2 + b(x) \cdot p.$$

- The associated effective Hamiltonians are very complicated, in particular, is NOT homogeneous of any degree. We can NOT expect an elegant formula between solutions of the effective equation and the effective Hamiltonian as in the metric case.
- So to extend the Oberman-Takei-Vladimirsky approach, the key issue is to find a stable way to recover the effective Hamiltonian from solution of the effective equation.

An observation by Barron-Jensen

Recall the definition of viscosity solutions. u is a viscosity solution of $H(Du, x) = 0$ if

An observation by Barron-Jensen

Recall the definition of viscosity solutions. u is a viscosity solution of $H(Du, x) = 0$ if

- the test function ϕ touches u at x_0 from above $\Rightarrow H(D\phi(x_0), x_0) \leq 0$;

An observation by Barron-Jensen

Recall the definition of viscosity solutions. u is a viscosity solution of $H(Du, x) = 0$ if

- the test function ϕ touches u at x_0 from above $\Rightarrow H(D\phi(x_0), x_0) \leq 0$;
- the test function ϕ touches u at x_0 from below $\Rightarrow H(D\phi(x_0), x_0) \geq 0$;

An observation by Barron-Jensen

Recall the definition of viscosity solutions. u is a viscosity solution of $H(Du, x) = 0$ if

- the test function ϕ touches u at x_0 from above $\Rightarrow H(D\phi(x_0), x_0) \leq 0$;
- the test function ϕ touches u at x_0 from below $\Rightarrow H(D\phi(x_0), x_0) \geq 0$;

Barron-Jensen [BJ] (*Comm. Partial Differential Equations, 1990*) proved that if $H = H(p, x)$ is convex at the p variable, then $\geq$ is actually an $=$.

Luo-Yu-Zhao's scheme [LYZ] (*Preprint, 2010*)

Suppose $\bar{u}$ is the solution of

$$\begin{cases} \bar{H}(D\bar{u}) = f(x) \\ \bar{u}(0) = 0. \end{cases}$$

Choose suitable f such that $f \geq f(0) = \min \bar{H}$ and $\bar{u} \geq 0$ with superlinear growth. Then for any $p \in \mathbb{R}^n$, if $\bar{u} - p \cdot x$ attains minimum at x_0, then according to Barron-Jensen's observation

$$\bar{H}(p) = f(x_0).$$

Luo-Yu-Zhao's scheme [LYZ] (*Preprint, 2010*)

Suppose $\bar{u}$ is the solution of

$$\begin{cases} \bar{H}(D\bar{u}) = f(x) \\ \bar{u}(0) = 0. \end{cases}$$

Choose suitable f such that $f \geq f(0) = \min \bar{H}$ and $\bar{u} \geq 0$ with superlinear growth. Then for any $p \in \mathbb{R}^n$, if $\bar{u} - p \cdot x$ attains minimum at x_0, then according to Barron-Jensen's observation

$$\bar{H}(p) = f(x_0).$$

Consider the approximation

$$\begin{cases} H(Du^\epsilon, \frac{x}{\epsilon}) = f(x) \\ u^\epsilon(0) = 0. \end{cases}$$

and look at points where $u^\epsilon - p \cdot x$ attains minimum and evaluate f.

Stability: A rigorous error estimate

Theorem: (Luo-Yu-Zhao)

Suppose that

$$u^{\epsilon}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (u^{\epsilon} - p \cdot x) + \delta.$$

Then

$$|\bar{H}(p) - f(x_{\epsilon})| \leq O(\sqrt{\epsilon + \delta}).$$

Stability: A rigorous error estimate

Theorem: (Luo-Yu-Zhao)

Suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon \leq \min_{x \in \mathbb{R}^n}(u^\epsilon - p \cdot x) + \delta.$$

Then

$$|\bar{H}(p) - f(x_\epsilon)| \leq O(\sqrt{\epsilon + \delta}).$$

Heuristic argument: What we really need is to locate the place where $\bar{u} - p \cdot x$ attains minimum.

Stability: A rigorous error estimate

Theorem: (Luo-Yu-Zhao)

Suppose that

$$u^{\epsilon}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (u^{\epsilon} - p \cdot x) + \delta.$$

Then

$$|\bar{H}(p) - f(x_{\epsilon})| \leq O(\sqrt{\epsilon + \delta}).$$

Heuristic argument: What we really need is to locate the place where $\bar{u} - p \cdot x$ attains minimum. Formally, $u^{\epsilon} = \bar{u} + O(\epsilon)$. Hence

$$\bar{u}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (\bar{u} - p \cdot x) + \delta + \epsilon.$$

Stability: A rigorous error estimate

Theorem: (Luo-Yu-Zhao)

Suppose that

$$u^\epsilon(x_\epsilon) - p\cdot x_\epsilon \le \min_{x\in\mathbb{R}^n}(u^\epsilon - p\cdot x) + \delta.$$

Then

$$|\bar{H}(p) - f(x_\epsilon)| \le O(\sqrt{\epsilon+\delta}).$$

Heuristic argument: What we really need is to locate the place where $\bar{u} - p\cdot x$ attains minimum. Formally, $u^\epsilon = \bar{u} + O(\epsilon)$. Hence

$$\bar{u}(x_\epsilon) - p\cdot x_\epsilon \le \min_{x\in\mathbb{R}^n}(\bar{u} - p\cdot x) + \delta + \epsilon.$$

Think of the function $h(x) = |x|^2$. $h(\bar{x}) \le \tau \Rightarrow |\bar{x}| \le \sqrt{\tau}$.

Stability: A rigorous error estimate

Theorem: (Luo-Yu-Zhao)

Suppose that

$$u^{\epsilon}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (u^{\epsilon} - p \cdot x) + \delta.$$

Then

$$|\bar{H}(p) - f(x_{\epsilon})| \leq O(\sqrt{\epsilon + \delta}).$$

Heuristic argument: What we really need is to locate the place where $\bar{u} - p \cdot x$ attains minimum. Formally, $u^{\epsilon} = \bar{u} + O(\epsilon)$. Hence

$$\bar{u}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (\bar{u} - p \cdot x) + \delta + \epsilon.$$

Think of the function $h(x) = |x|^2$. $h(\bar{x}) \leq \tau \Rightarrow |\bar{x}| \leq \sqrt{\tau}$. Hence

$$|x_{\epsilon} - x_0| \leq O(\sqrt{\delta + \epsilon}).$$

Stability: A rigorous error estimate

Theorem: (Luo-Yu-Zhao)

Suppose that

$$u^{\epsilon}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (u^{\epsilon} - p \cdot x) + \delta.$$

Then

$$|\bar{H}(p) - f(x_{\epsilon})| \leq O(\sqrt{\epsilon + \delta}).$$

Heuristic argument: What we really need is to locate the place where $\bar{u} - p \cdot x$ attains minimum. Formally, $u^{\epsilon} = \bar{u} + O(\epsilon)$. Hence

$$\bar{u}(x_{\epsilon}) - p \cdot x_{\epsilon} \leq \min_{x \in \mathbb{R}^n} (\bar{u} - p \cdot x) + \delta + \epsilon.$$

Think of the function $h(x) = |x|^2$. $h(\bar{x}) \leq \tau \Rightarrow |\bar{x}| \leq \sqrt{\tau}$. Hence

$$|x_{\epsilon} - x_0| \leq O(\sqrt{\delta + \epsilon}).$$

$$|f(x_{\epsilon}) - \bar{H}(p)| = |f(x_{\epsilon}) - f(x_0)| \leq O(\sqrt{\delta + \epsilon}).$$

Numerical error: O(ϵ)!

The numerical error is $O(\epsilon)$!!

Numerical error: O(ϵ)!

The numerical error is $O(\epsilon)$!!

Question: What is the MIRACLE behind it?

Rapid oscillation is the hidden friend!!

$u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2)$. Now suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x).$$

Rapid oscillation is the hidden friend!!

$u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2)$. Now suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x).$$

Then

$$D\bar{u}(x_\epsilon) + D_y v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) + O(\epsilon) = p$$

and

$$D^2\bar{u}(x_\epsilon) + O(1) + \frac{1}{\epsilon} D_y^2 v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) \geq 0.$$

Rapid oscillation is the hidden friend!!

$u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2)$. Now suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x).$$

Then

$$D\bar{u}(x_\epsilon) + D_y v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) + O(\epsilon) = p$$

and

$$D^2\bar{u}(x_\epsilon) + O(1) + \frac{1}{\epsilon} D_y^2 v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) \geq 0.$$

Assume $x_\epsilon \to x_0$ and $\frac{x_\epsilon}{\epsilon} \to y_0$ mod $(\mathbb{Z}^n)$. Then $D\bar{u}(x_0) = p$,

Rapid oscillation is the hidden friend!!

$u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2)$. Now suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x).$$

Then

$$D\bar{u}(x_\epsilon) + D_y v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) + O(\epsilon) = p$$

and

$$D^2\bar{u}(x_\epsilon) + O(1) + \frac{1}{\epsilon} D_y^2 v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) \geq 0.$$

Assume $x_\epsilon \to x_0$ and $\frac{x_\epsilon}{\epsilon} \to y_0 \mod (\mathbb{Z}^n)$. Then $D\bar{u}(x_0) = p$,

$$D_y v(x_0, y_0) = 0$$

Rapid oscillation is the hidden friend!!

$u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2)$. Now suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x).$$

Then

$$D\bar{u}(x_\epsilon) + D_y v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) + O(\epsilon) = p$$

and

$$D^2\bar{u}(x_\epsilon) + O(1) + \frac{1}{\epsilon} D_y^2 v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) \geq 0.$$

Assume $x_\epsilon \to x_0$ and $\frac{x_\epsilon}{\epsilon} \to y_0$ mod $(\mathbb{Z}^n)$. Then $D\bar{u}(x_0) = p$,

$$D_y v(x_0, y_0) = 0$$

and

$$D_y^2 v(x_0, y_0) \geq 0.$$

Rapid oscillation is the hidden friend!!

$u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon}) + O(\epsilon^2)$. Now suppose that

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n}(u^\epsilon - p \cdot x).$$

Then

$$D\bar{u}(x_\epsilon) + D_y v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) + O(\epsilon) = p$$

and

$$D^2\bar{u}(x_\epsilon) + O(1) + \frac{1}{\epsilon} D_y^2 v(x_\epsilon, \frac{x_\epsilon}{\epsilon}) \geq 0.$$

Assume $x_\epsilon \to x_0$ and $\frac{x_\epsilon}{\epsilon} \to y_0 \bmod (\mathbb{Z}^n)$. Then $D\bar{u}(x_0) = p$,

$$D_y v(x_0, y_0) = 0$$

and

$$D_y^2 v(x_0, y_0) \geq 0.$$

This implies that $v(x_0, y)$ attains minimum at $y = y_0$!!!!!

The optimal error estimate $O(\epsilon)$: A formal proof.

For fixed $p \in \mathbb{R}^n$, assume that (1)

$$\bar{u}(x_0) - p \cdot x_0 = \min_{\mathbb{R}^n}\{\bar{u}(x) - p \cdot x\}$$

and $\bar{u}$ is C^2 near x_0 and nondegenerate

$$D^2\bar{u}(x_0) > 0.$$

The optimal error estimate $O(\epsilon)$: A formal proof.

For fixed $p \in \mathbb{R}^n$, assume that (1)

$$\bar{u}(x_0) - p \cdot x_0 = \min_{\mathbb{R}^n} \{\bar{u}(x) - p \cdot x\}$$

and $\bar{u}$ is C^2 near x_0 and nondegenerate

$$D^2 \bar{u}(x_0) > 0.$$

(2) $u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon})$ where v is C^1 and periodic in the y variable.

The optimal error estimate $O(\epsilon)$: A formal proof.

For fixed $p \in \mathbb{R}^n$, assume that (1)

$$\bar{u}(x_0) - p \cdot x_0 = \min_{\mathbb{R}^n} \{\bar{u}(x) - p \cdot x\}$$

and $\bar{u}$ is C^2 near x_0 and nondegenerate

$$D^2 \bar{u}(x_0) > 0.$$

(2) $u^\epsilon = \bar{u} + \epsilon v(x, \frac{x}{\epsilon})$ where v is C^1 and periodic in the y variable. Then we can rigorously show that

$$|x^\epsilon - x_0| \leq O(\epsilon)$$

where

$$u^\epsilon(x_\epsilon) - p \cdot x_\epsilon = \min_{\mathbb{R}^n} \{u^\epsilon(x) - p \cdot x\}.$$

Comparison with the large T method: a difficult task

Compare the computational complexity to recover the whole function $\bar{H}$ with accuracy γ. Recall the large T-method

$$\begin{cases} u_t + H(Du, x) = 0 \\ u(x,0) = p \cdot x \end{cases}$$

Then

$$\bar{H}(p) = -\frac{u(x,T)}{T} + O(\frac{1}{T}).$$

If we assume that $\bar{H}(p) \in C^2(\mathbb{R}^n)$ and use the linear interpolation, the complexity is $\gamma^{-\frac{3n}{2}-2}$ (forward Euler first order Godunov).

Comparison with the large T method: a difficult task

Compare the computational complexity to recover the whole function $\bar{H}$ with accuracy γ. Recall the large T-method

$$\begin{cases} u_t + H(Du, x) = 0 \\ u(x, 0) = p \cdot x \end{cases}$$

Then

$$\bar{H}(p) = -\frac{u(x, T)}{T} + O(\frac{1}{T}).$$

If we assume that $\bar{H}(p) \in C^2(\mathbb{R}^n)$ and use the linear interpolation, the complexity is $\gamma^{-\frac{3n}{2}-2}$ (forward Euler first order Godunov). The Luo-Yu-Zhao scheme

$$\begin{cases} H(Du^\epsilon, \frac{x}{\epsilon}) = f(x) \\ u^\epsilon(0) = 0. \end{cases}$$

Find the minimum points of $u^\epsilon - p \cdot x$ and evaluate f at those points to approximate $\bar{H}(p)$. If we assume the $O(\epsilon)$ error, the complexity is γ^{-2n}.

Comparison with the large T method: a difficult task

Compare the computational complexity to recover the whole function $\bar{H}$ with accuracy γ. Recall the large T-method

$$\begin{cases} u_t + H(Du, x) = 0 \\ u(x,0) = p \cdot x \end{cases}$$

Then

$$\bar{H}(p) = -\frac{u(x,T)}{T} + O(\frac{1}{T}).$$

If we assume that $\bar{H}(p) \in C^2(\mathbb{R}^n)$ and use the linear interpolation, the complexity is $\gamma^{-\frac{3n}{2}-2}$ (forward Euler first order Godunov). The Luo-Yu-Zhao scheme

$$\begin{cases} H(Du^\epsilon, \frac{x}{\epsilon}) = f(x) \\ u^\epsilon(0) = 0. \end{cases}$$

Find the minimum points of $u^\epsilon - p \cdot x$ and evaluate f at those points to approximate $\bar{H}(p)$. If we assume the $O(\epsilon)$ error, the complexity is γ^{-2n}. Note that $\gamma^{-2n} \leq \gamma^{-\frac{3n}{2}-2}$ for $n \leq 4$ which might be practically enough.

Important examples which are NOT covered by our scheme

- (Noncoercive Hamiltonians).

$$G_t + s_L|DG| + b(x) \cdot DG = 0.$$

The Hamiltonian is $H(p, x) = |p| + b(x) \cdot p$. It is convex but not coercive. The corresponding effective Hamiltonian

$$\bar{H}(p) = s_L|p + DG| + b(x) \cdot (P + DG)$$

is a model of the turbulent flame speed.

Important examples which are NOT covered by our scheme

- (Noncoercive Hamiltonians).

$$G_t + s_L|DG| + b(x) \cdot DG = 0.$$

The Hamiltonian is $H(p, x) = |p| + b(x) \cdot p$. It is convex but not coercive. The corresponding effective Hamiltonian

$$\bar{H}(p) = s_L|p + DG| + b(x) \cdot (P + DG)$$

is a model of the turbulent flame speed. The existence of $\bar{H}$ has been established only very recently by Xin-Yu [XY] (*Comm. Math. Sciences, 2010*) and Cardaliaguet-Nolen-Souganidis (*Archive. Ration. Mech, 2010*) independently through completely different approaches.

Important examples which are NOT covered by our scheme

- (Noncoercive Hamiltonians).

$$G_t + s_L|DG| + b(x) \cdot DG = 0.$$

The Hamiltonian is $H(p, x) = |p| + b(x) \cdot p$. It is convex but not coercive. The corresponding effective Hamiltonian

$$\bar{H}(p) = s_L|p + DG| + b(x) \cdot (P + DG)$$

is a model of the turbulent flame speed. The existence of $\bar{H}$ has been established only very recently by Xin-Yu [XY] (*Comm. Math. Sciences, 2010*) and Cardaliaguet-Nolen-Souganidis (*Archive. Ration. Mech, 2010*) independently through completely different approaches.

- (Nonconvex Hamiltonian) The G-equation with the strain term.

$$G_t + s_L|DG| + b(x) \cdot DG + \frac{DG}{|DG|} \cdot Db \cdot DG = 0.$$

The Hamiltonian is $H(p, x) = s_L|p| + b(x) \cdot p + \frac{p}{|p|} \cdot Db \cdot p$. The existence of effective Hamiltonian remains an open problem.