A DG solver for front propagation with obstacles

Olivier Bokanowski

Laboratory Jacques Louis Lions (Univ. Denis-Diderot, Paris)
Commands (INRIA Saclay) and Ensta Paris Tech

Joint work with:
Yingda CHENG (Austin, Univ of Texas, Dep. of Mathematics)
Chi-Wang SHU (Brown, division of Applied Mathematics)

1. HJ equation for Front Propagation with constraints

A front propagation model

- Consider an initial (closed) set $\Omega_0 \subset \mathbb{R}^n$, we want to compute the **reachable set**

$$\Omega_t := \{y_x^\alpha(t),\ \alpha \in L^\infty((0,t),\mathcal{A}),\ x \in \Omega_0\}.$$

where $y = y_x^\alpha(.)$ denotes the solution of the ODE:

$$\dot{y}(s) = f(y(s),\alpha(s)), \quad \text{a.e. } s \in (0,t)$$
$$y(0) = x$$

- **Front:** modelized by $\partial\Omega_t$
- **minimal time** problem: $\mathcal{T}(x) := \inf\{t \geq 0,\ \ x \in \Omega_t\}$
- **Target problem**
- **Capture basin set:** Replace $f(x,\alpha)$ by $Conv\{0, f(x,\alpha)\}$, ...

Level set approach

Let φ Lipschitz continuous, be such that

$$\Omega_0 = \{x,\ \varphi(x) \leq 0\}.$$

Let

$$u(t,x) := \inf\{\varphi(y_x^\alpha(-t)),\ \alpha \in \mathcal{U}\}$$

Proposition 1:

$$\Omega_t = \{x,\ u(t,x) \leq 0\}.$$

Proposition 2: We have a dynamic programming principle (DPP) and the following HJ equation [1]

$$\begin{cases} u_t + \max\limits_{a\in A}(f(x,a)\cdot\nabla u) = 0, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = \varphi(x), & x\in\mathbb{R}^n \end{cases}$$

[1] **Assumptions** (i) $\mathcal{A}$ compact, (ii) $f(x,A)$ convex, and (iii) $\exists L\ \forall x,y,a,\ |f(x,a)-f(y,a)| \leq L|x-y|$.

State constraints

- Let K be a nonempty closed set, we now want to compute

$$\Omega_t^K := \{y_x^\alpha(t),\ \alpha \in L^\infty(0,t),\ x \in \Omega_0,\ \Big(\mathbf{y}_\mathbf{x}^\alpha(\theta) \in \mathbf{K},\ \forall\theta \in [\mathbf{0},\mathbf{t}]\Big)\}.$$

- We still have $\Omega_t^K = \{x,\ u(t,x) \le 0\}$ where

$$u(t,x) := \begin{cases} \inf\Big\{\varphi(y_x^\alpha(-t)),\ \alpha \in L^\infty(0,t),\ \Big(\mathbf{y}_\mathbf{x}^\alpha(\theta) \in \mathbf{K},\ \forall\theta \in [\mathbf{0},\mathbf{t}]\Big)\Big\} \\ +\infty \quad \text{if there is no feasible trajectory} \end{cases}$$

- u **discontinuous**, no simple HJ equation for u ! [2]

[2] see however B.-Forcadel-Zidani, COCV 2010

Second way B.- Forcadel - Zidani SICON 2010

- Let g be Lipschitz constinuous and such that

$$\mathbf{g}(\mathbf{x}) \leq \mathbf{0} \Leftrightarrow \mathbf{x} \in \mathbf{K}.$$

Instead of u, we consider an "L^∞- penalized" problem

$$v(t,x) \quad := \quad \inf_{\alpha \in L^\infty(0,t)} \max\left(\varphi(y_x^\alpha(-t)),\ \max_{\theta\in[\mathbf{0},\mathbf{t}]} \mathbf{g}(\mathbf{y}_\mathbf{x}^\alpha(-\theta))\right)$$

- **Proposition 1.** $\{x,\ u(t,x) \leq 0\} = \{x, v(t,x) \leq 0\} = \Omega_t^K$.
- **Proposition 2.** v is the unique viscosity solution of:

$$\min\left(v_t + \max_{a\in A}(f(x,a)\cdot\nabla v),\ \mathbf{v} - \mathbf{g}(\mathbf{x})\right) = 0, \quad t > 0, x \in \mathbb{R}^n,$$
$$v(0,x) = \max(g(x), \varphi(x)), \quad x \in \mathbb{R}^n$$

- **Rem:** L^∞-cost was already considered by **Barron-Jensen.**

Application to minimal time

- **Minimal time with state constraints K:**

$$\mathcal{T}(x) := \inf\{t \geq 0, \quad x \in \Omega_t^K\}$$

(and $T(x) = +\infty$ if there is no feasible trajecories).

- **Proposition:**

$$\mathcal{T}(x) = \inf\{t \geq 0, v(t,x) \leq 0\}.$$

- **Application:** reconstruction of optimal trajectories

- without any controllability assumptions
- with/without obstacles

Very short & non exhaustive Literature

- **inward pointing condition:** Soner (86'), Cappuzzo-Dolcetta - Lions (90), Ishii-Koike (96), ...
- **outward pointing condition:** Frankowska-Plaskacz (00'), Frankowska-Vinter
- **No condition - Viability theory** (Aubin) : Cardaliaguet-Quincampoix-Saint-Pierre (97,00), Viability algorithm (Saint-Pierre, 94')
- **No condition - Penalization approach:** "Exact Penalization" Kurzhanski and Varayiya (2006).
- **Other works:** Kurzhanski-Mitchell-Varaiya (2006),
- **Two player games:** Bardi-Koike-Soravia

2 DG scheme
- linear + obstacle
- Non linear case

1. Variationnal formulation

Consider the $u_t + u_x = 0$ equation, with obstacle $g(x)$:

$$\min(u_t + u_x, u - g(x)) = 0 \tag{1}$$

This is equivalent to :

$$\Leftrightarrow \begin{cases} u_t + u_x \geq 0 \\ u - g(x) \geq 0 \\ (u_t + u_x).(u(t,x) - g(x)) = 0, \quad \text{a.e. } x \end{cases}$$

$$\Leftrightarrow \begin{cases} u_t + u_x \geq 0 \\ u - g(x) \geq 0 \\ (u_t + u_x,\ u(t,.) - g) = 0, \end{cases}$$

where $(.,.)$ denotes the scalar product on $L^2(0,1)$

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1. Variationnal formulation

(1') $\quad u \geq g$ and $u_t + u_x \geq 0,\ (u_t + u_x, u - g(x)) = 0.$

The variational formulation for (1') is : find $u(t,.) \geq g$ such that

$$\forall v \geq g, \quad (u_t + u_x, v - u(t,.)) \geq 0.$$

Proof: $\Rightarrow$: $v - u = (v - g) - (u - g)$ hence, $\forall v \geq g$,

$$(u_t + u_x, v - u) \;=\; (u_t + u_x, \underbrace{v - g}_{\geq 0}) + 0 \geq 0$$

$\Leftarrow$: $v = \varphi_n \geq g$, $\lim_{n\to\infty} \varphi_n(x_0) = +\infty \Rightarrow (u_x + u_x)(t, x_0) \geq 0.$
Taking $v = g$, we get $(u_t + u_x, g - u(t,.)) \geq 0$

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2. direct DG scheme (Cheng & Shu, JSC 2007)

- At first consider the case of

$$u_t + u_x = 0, \quad t > 0,\ x \in (0,1)$$

and with periodic boundary conditions.

- Given some mesh of $(0,1)$: $(x_{j-\frac{1}{2}})_j$, we introduce a space of discontinuous galering elements of degre k:

$$V_h = \left\{ v_h, \quad v_h \in P_k(I_j),\ \forall j \ \right\}, \qquad I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$$

where P_k are the polynomials of degree at most k.

- Notations: $(v_h)^{\pm}_{j-\frac{1}{2}} = v_h(x^{\pm}_{j-\frac{1}{2}})$, $[v_h]_{j-\frac{1}{2}} = v_h(x^{+}_{j-\frac{1}{2}}) - v_h(x^{-}_{j-\frac{1}{2}})$.

- Euler Forward DG formulation for $u_t + u_x = 0$:

Direct DG scheme, linear case

find u^{n+1} in V_h,

$$\int \frac{u^{n+1} - u^n}{\Delta t} v_h + \int u_x^n v_h + \sum_j a^+ [u^n]_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0, \quad \forall v_h \in V_h.$$

where a^+ is some constant such that $a^+ \geq 1$.

- Taking $a^+ = 1$, this is equivalent to the classical DG scheme.
- We may write formally the scheme as

$$\left(\frac{u^{n+1} - u^n}{\Delta t} + \mathcal{H}(u^n), v_h\right) = 0, \quad \forall v_h \in V_h,$$

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$$(\frac{u^{n+1} - u^n}{\Delta t} + \mathcal{H}(u^n), v_h) = 0, \quad \forall v_h \in V_h,$$

- Equivalent vector formulation:

$$u^n(x) = \sum_{\alpha=0,\ldots,k} U_\alpha^{n,i} \varphi_\alpha(x), \quad \text{and} \quad U^{n,i} = \begin{pmatrix} U_0^{n,i} \\ \vdots \\ U_k^{n,i} \end{pmatrix}$$

where $(\varphi_\alpha(x))_{\alpha=0,\ldots,k}$ is some basis of P_k.

- Then the scheme becomes :

$$M \frac{U^{n+1,i} - U^{n,i}}{\Delta t} + AU^{n,i} + BU^{n,i-1} = 0 \in \mathbb{R}^{k+1}$$

where M is the mass matrix: $M_{\alpha,\beta} = (\varphi_\alpha, \varphi_\beta)$.

- In the end, we get an explicit formula $U_\alpha^{n+1,i} = F(U^n)_{i,\alpha}$.

3. Direct DG scheme for the obstacle case (B. - Cheng - Shu, Preprint 2010)

- Since $v \geq g$ is a little bit strong for polynomials, we introduce

$$V_h^g := \{v \in V_h, \quad "v \geq g"\}$$

where

$$"v \geq g" \Leftrightarrow v(x_\alpha^i) \geq g(x_\alpha^i), \quad \forall i, \alpha$$

and where $(x_\alpha^i)_{\alpha=0,\dots,k}$ are the $k+1$ **gauss points** on cell I_i.

Direct DG scheme, obstacle case

find u^{n+1} in V_h, $"u^{n+1} \geq g"$,

$$\left(\frac{u^{n+1}-u^n}{\Delta t} + h(u^n), v_h - u^{n+1}\right) \geq 0, \quad \forall v_h \in V_h^{\mathbf{g}}$$

4. Simplification

- In matrix form, the problem becomes ($\forall i$):

$$\sum_\alpha (M\frac{U^{n+1,i}-U^{n,i}}{\Delta t} + AU^{n,i} + BU^{n,i-1})_\alpha \,(V_\alpha - U_\alpha^{n+1,i}) \geq 0,$$

$$\forall V_\alpha \geq g(x_\alpha^i),$$

- As in the continuous case, it is equivalent to ($\forall i$),

$$\min\left((M\frac{U^{n+1,i}-U^{n,i}}{\Delta t}+AU^{n,i}+BU^{n,i-1})_\alpha,\; U_\alpha^{n+1,i}-g(x_\alpha^i)\right) \geq 0, \quad \forall\alpha$$

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This is still a non-linear system to solve !

• Simple idea: consider the dual basis associated to the gaussian points: s.t. $\varphi_\alpha(x_\beta) = \delta_{\alpha\beta}$. Then

$$M = \mathrm{diag}(w_0, \ldots, w_k) \quad \text{with } w_\alpha > 0$$

• Now the system becomes ($\forall i$):

$$\min\left(w_\alpha \frac{U_\alpha^{n+1,i} - U_\alpha^{n,i}}{\Delta t} + (AU^{n,i} + BU^{n,i-1})_\alpha,\ U_\alpha^{n+1,i} - g(x_\alpha^i) \right) \geq 0.$$

... which can be solved explicitly :

• **Remark:** This is similar with a Finite Difference Euler Forward scheme for $\min(u_t + u_x, u - g(x)) = 0$!

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$$U_\alpha^{n+1,i} = \max\left(U_\alpha^{n,i} - \frac{\Delta t}{w_\alpha}(AU^{n,i} + BU^{n,i-1})_\alpha,\ g(x_\alpha^i) \right)$$

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$$U_\alpha^{n+1,i} = \max\left(\underbrace{U_\alpha^{n,i} - \frac{\Delta t}{w_\alpha}(AU^{n,i} + BU^{n,i-1})_\alpha}_{F(U^n)_\alpha^i},\ g(x_\alpha^i) \right)$$

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5. Non linear + obstacle : (Cheng-Shu JSC 07', B.-Cheng-Shu SJSC)

- For $u_t + H(x, u_x) = 0$, consider any DG scheme, for instance:
- $\forall v \in V_h$,

$$\int_{I_j} \left\{ (u_h)_t + H(x, (u_h)_x) \right\} v + H^{1,+}_{j-\frac{1}{2}} [\tilde{u}_h]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + H^{1,-}_{j+\frac{1}{2}} [\tilde{u}_h]_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} = 0$$

where

$$H^{1,+}_{j-\frac{1}{2}} := \max\left(0, \max_{x \in I_{j-\frac{1}{2}}} \frac{\partial H}{\partial u_x}(x_{j-\frac{1}{2}}, u_{hx}(x))\right)$$

$$H^{1,-}_{j+\frac{1}{2}} := \min\left(0, \min_{x \in I_{j+\frac{1}{2}}} \frac{\partial H}{\partial u_x}(x_{j+\frac{1}{2}}, u_{hx}(x))\right)$$

These terms are for **STABILITY**.

6. The scheme in 2d

- Consider Q_k elements generated by

$$x_1^p x_2^q, \quad 0 \le p, q \le k$$

- We take an explicit and stable $TVD - RK3$ scheme,

$$U^{n+1} = F(U^n).$$

- The full scheme reads

$$U_\alpha^{n+1,i} = \max\left(F(U^n)_\alpha^i,\ g(x_\alpha^i)\right)$$

where $x_\alpha^i = (x_{\alpha_1}^{i_1}, x_{\alpha_2}^{i_2})$ (using $1 - -d$ gauss points)

3. Numerical results
    - Without obstacles
    - With obstacles

A - Without obstacles

Good long time behavior

$$\begin{cases} \varphi_t + f(\mathbf{x}) \cdot \nabla\varphi = 0, & \mathbf{x} \in \Omega,\ t \in [0, T] \\ \varphi(0, \mathbf{x}) = \varphi^0(\mathbf{x}) \end{cases}$$

with $\Omega \subset \mathbb{R}^2$.

$\varphi^0 : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function such that

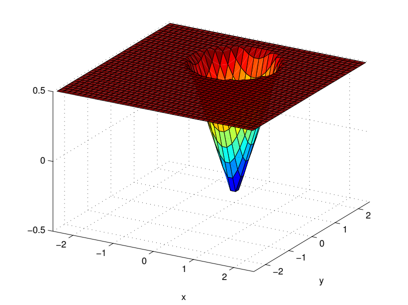
$$\Omega_0 \text{ (target)} \equiv \quad \{x,\ \varphi^0(x) \leq 0\}$$

1. Rotation of a circle

Dynamics: $f(x, y) := 2\pi(-y, x)$

Initial data:

$$\varphi^0(x, y) = \min(r_0, \|x - x_A\|_2 - r_0),\ r_0 = 0.5,\ A = (0, 1)$$



$\mathbf{P}^2$: Local error (region s.t. $|\varphi(t,.)| < 0.15$), Hausdorff distance

t = 1

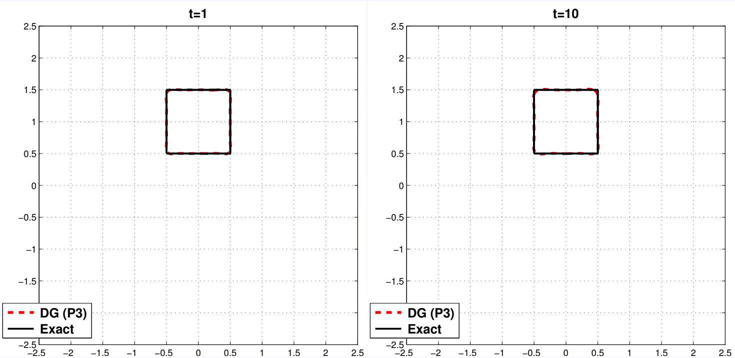
N_x	Δx	L^1-error	order	L^2-error	order	L^∞-error	order	$\mathbf{d}_H$	order
10	0.5	1.03e-2	-	1.34e-2	-	3.84e-2	-	3.29e-2	-
20	0.25	4.27e-3	**1.2**	5.36e-3	**1.3**	1.76e-2	**1.1**	9.86e-3	**1.7**
40	0.125	4.28e-4	**3.3**	5.66e-4	**3.2**	2.90e-3	**2.6**	1.64e-3	**2.5**
80	0.0675	4.76e-5	**3.1**	6.22e-5	**3.1**	2.55e-4	**3.5**	1.33e-4	**3.6**

t = 10

N_x	Δx	L^1-error	order	L^2-error	order	L^∞-error	order	$\mathbf{d}_H$	order
10	0.5	4.66e-2	-	5.62e-2	-	1.30e-1	-	1.17e-1	-
20	0.25	8.59e-3	**2.4**	1.01e-2	**2.4**	2.33e-2	**2.4**	1.19e-2	**3.3**
40	0.125	1.65e-3	**2.3**	1.99e-3	**2.3**	6.09e-3	**1.9**	3.33e-3	**1.8**
80	0.0675	2.31e-4	**2.8**	2.91e-4	**2.7**	7.89e-4	**2.9**	2.73e-4	**3.6**

Hausdorff distance: $d_H(A,B) := \max(\max_{a\in A} d(a,B), \max_{b\in B} d(b,A))$.

2. Rotation of a square



Rotation of a square. $t = 1$ (left), and $t = 10$ (right), with P^3 and $N_x = N_y = 40$ ($\equiv 125^2$ values)

We observe

- P3 is better to well catch the corners
- First order (but the solution is only Lipschitz continuous)
- Very good long time behavior

3. Deformation test

- We consider

$$f(t,x,y) := sign\,(T-t)\,\overbrace{\max(1-\|\mathbf{x}\|_2, 0)}^{a(\|\mathbf{x}\|)}\begin{pmatrix} -2\pi\, y \\ 2\pi\, x \end{pmatrix}$$

where $\|\mathbf{x}\|_2 := \sqrt{x^2+y^2}$ and

$$\varphi^0(x,y) = \min(\max(y,-1),1). \qquad (2)$$

The function φ^0 has a 0-level set which is the x axis:

$$\{\varphi^0 = 0\} \equiv \{y = 0\}$$

- Exact solution for $t \le T$:

$$u(t,\mathbf{x}) := u_0(R_{-2\pi t a(\mathbf{x})}\mathbf{x}) \quad \text{where} \quad R_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

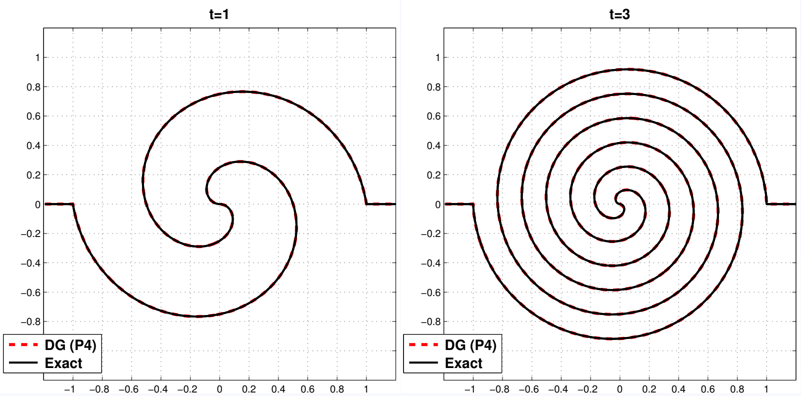


Figure: Plots at times $t = 1$, $t = 3$, with P^4 and 24×24 mesh cells.

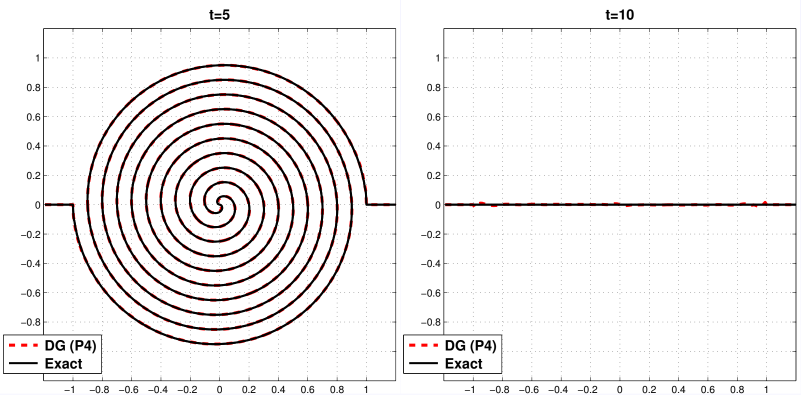


Figure: Plots at times $t = 5$ and $t = 10$ (return to initial data) - with P^4 and 24×24 mesh cells ($\simeq 100^2$ values)

Example coming from $\ddot{x} = \alpha$: $\mathbf{u_t - yu_x + |u_y| = 0}$

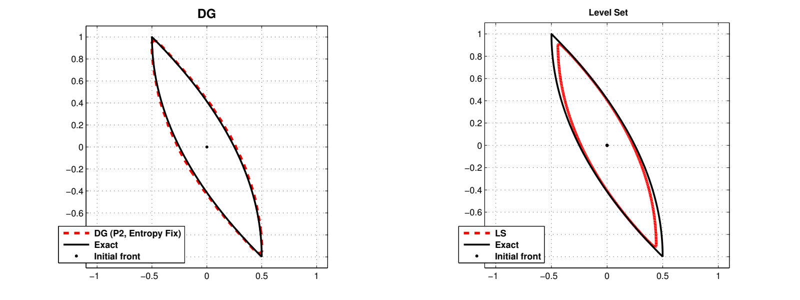


Figure: Comparison at time $t = 1.0$: DG scheme with 44^2 cells, P^2 (left) and traditional level set method using a second order Lax-Friedrich type scheme (right) with 401^2 mesh cells

B - With obstacles

Example 1 (1–d, linear + obstacle)

We first consider a one-dimensional test:

$$\min(u_t + u_x, \mathbf{u} - \mathbf{g}(\mathbf{x})) = 0, \quad t > 0, \; x \in [-1, 1], \tag{3}$$

$$u(0, x) = u_0(x), \quad x \in [-1, 1], \tag{4}$$

with periodic boundary conditions and $g(x) := \sin(\pi x)$, $u_0(x) := 0.5 + \sin(\pi x)$. In that case, for times $0 \leq t \leq 1$, the exact solution can be computed analytically.

The numerical solution agrees well with the exact solution everywhere.

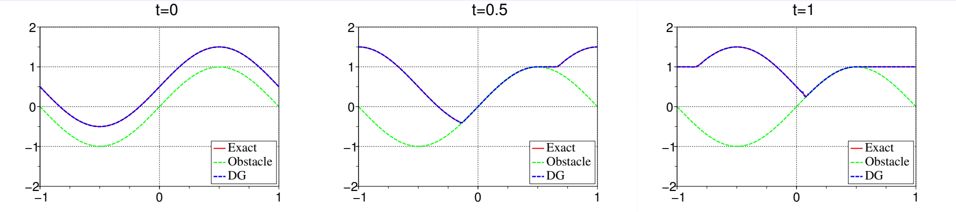


Figure: Example 1, times $t = 0$ (initial data), $t = 0.5$ and $t = 1$, using P^2 elements with $N_x = 20$ mesh cells (obstacle : green dotted line)

Table: Example 1. $t = 0.5$. P^2 elements (error at distance $d = 0.1$ away from singular points)

N_x	Δx	L^1-error	order	L^2-error	order	L^∞-error	order
40	5.00e-2	3.34e-05	2.41	1.01e-04	1.98	7.02e-04	2.20
80	2.50e-2	1.77e-06	4.24	3.64e-06	4.79	2.82e-05	4.64
160	1.25e-2	1.78e-07	3.31	2.91e-07	3.64	2.40e-06	3.55
320	6.25e-3	2.13e-08	3.06	3.43e-08	3.08	1.28e-07	4.23
640	3.13e-3	2.66e-09	3.00	4.28e-09	3.00	1.60e-08	3.00
1280	1.56e-3	3.32e-10	3.00	5.35e-10	3.00	2.00e-09	3.00

Example 2 (1–d, nonlinear + obstacle)

We consider a one-dimensional test with a nonlinear Hamiltonian:

$$\min(u_t + |u_x|, u - g(x)) = 0, \quad t > 0,\ x \in [-1, 1], \tag{5}$$

$$u(0, x) = u_0(x), \quad x \in \Omega, \tag{6}$$

with periodic boundary conditions and $g(x) := \sin(\pi x)$, $u_0(x) := 0.5 + \sin(\pi x)$. In this particular case, the exact solution is given by:

$$u(t, x) = \max(\bar{u}(t, x),\ g(x))$$

where $\bar{u}$ is the solution of the Eikonal equation $u_t + |u_x| = 0$ and can be computed analytically.

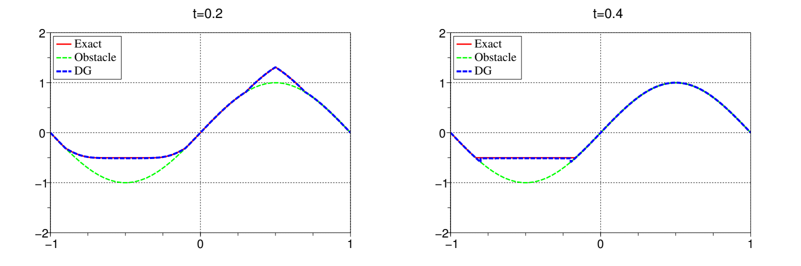


Figure: Example 2, numerical and exact solutions at times $t = 0.2$ and $t = 0.4$, $N_x = 20$, using P^2 (obstacle : green dotted line).

$\Rightarrow$ good agreement with the exact solution.

Example 3 (2–d, linear + obstacle, accuracy test)

The equation solved is

$$\min(u_t + \frac{1}{2}u_x + \frac{1}{2}u_y, u - g(x,y)) = 0, \quad t > 0,\ (x,y) \in \Omega, \tag{7}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in \Omega, \tag{8}$$

where $g(x,y) := \sin(\pi(x+y))$, $u_0(x,y) = 0.5 + g(x,y)$, and $\Omega = [-1,1]^2$ with periodic boundary conditions. The exact solution is known :

$$u(t,x,y) = u^{(1)}(t, x+y)$$

(where $u^{(1)}$ is the exact solution for 1–d Example 1).
The errors are computed away from the singular zone :

$$\{(x,y) \in \Omega,\ 1 \le i \le 3,\ d(x+y-s_i, 2\mathbb{Z}) \ge \delta)\} \quad (\delta = 0.1)$$

Table: Example 3. $t = 0.5$. Q^2 elements.

N_x	Δx	L^1-error	order	L^2-error	order	L^∞-error	order
10	2.00e-1	7.70e-03	-	1.03e-02	-	1.04e-01	-
20	1.00e-1	9.27e-04	3.05	1.28e-03	3.01	8.71e-03	3.58
40	5.00e-2	9.48e-05	3.29	1.67e-04	2.94	1.04e-03	3.06
80	2.50e-2	7.15e-06	3.73	1.11e-05	3.91	1.02e-04	3.34

$\Rightarrow$ We observe optimal convergence rate in this example.

Example 4 (2–d, linear + obstacle)

The initial data is $u_0(\mathbf{x}) := \|\mathbf{x} - (-0.5, 0)\|_2 - 0.3$.
The obstacle is coded by $g(\mathbf{x}) := 0.25 - \|\mathbf{x} - (0, 0.25)\|_2$.
The problem is

$$\min(u_t + u_x, u - g(x, y)) = 0, \quad t > 0, \ (x, y) \in \Omega, \tag{9}$$

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Omega, \tag{10}$$

on $\Omega := [-1, 1]^2$ with periodic boundary conditions.

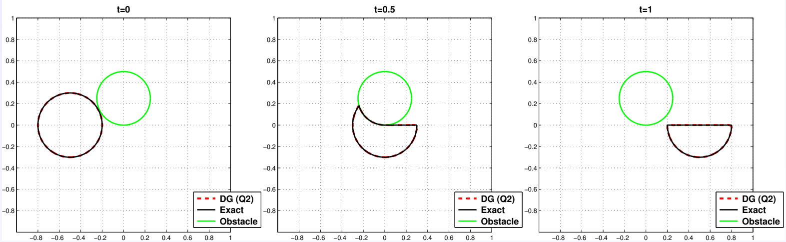


Figure: Example 4($N_x = N_y = 40$), times $t \in \{0,\ 0.5,\ 1\}$

Example 5 (2–d, linear + obstacle, variable coefficients)

We consider

$$f(x,y) := \begin{pmatrix} -2\pi\, y \\ 2\pi\, x \end{pmatrix} \max(1 - \|\mathbf{x}\|_2, 0)$$

where $\|\mathbf{x}\|_2 := \sqrt{x^2 + y^2}$ and with a Lipschitz continuous initial data u_0:

$$u_0(x,y) = \min(\max(y, -1), 1). \tag{11}$$

The function u_0 has a 0-level set which is the x axis: $\{\mathbf{x} = (x,y) \in \mathbb{R}^2 \mid y = 0\}$. When there is no obstacle function, the exact solution is known.

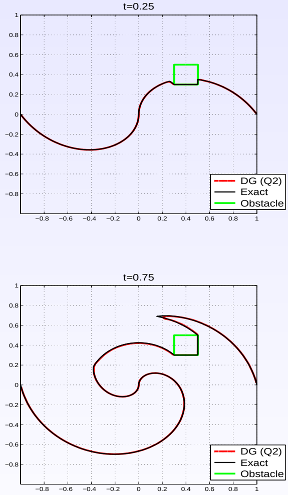


Figure: Q^2 and 40×40 mesh cells.

Example 6 (2–d, nonlinear)

The problem is

$$\min(u_t + \max\left(0, 2\pi(-y, x)\cdot\nabla u\right), u - g(x, y)) = 0, \quad (12)$$
$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Omega, \quad (13)$$

Domain $\Omega := [-2, 2]^2$,
Initial data : $u_0(x, y) := \|(x, y) - (1, 0)\|_2 - 0.5$,
Obstacle : $g(x, y) := 0.5 - \|(x, y) - (0,\ 0.5)\|_2$

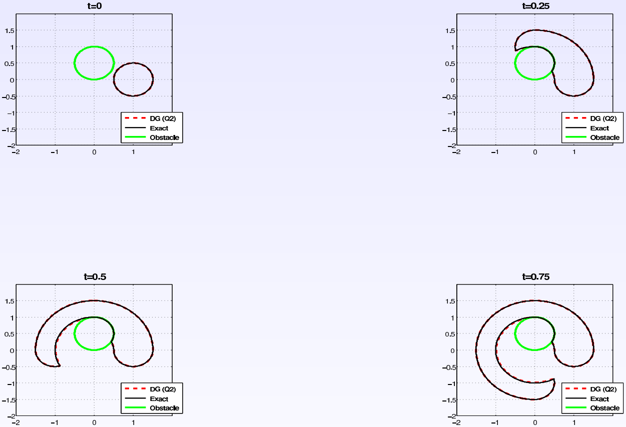


Figure: Example 6, $t \in \{0,\ 0.25,\ 0.5,\ 0.75\}$, Q^2, 80×80 cells.

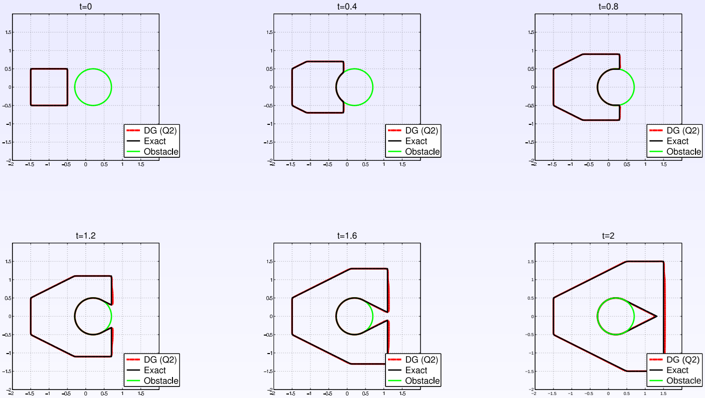
More complex example

We consider the problem

$$\min\left(u_t + \max\left(0, u_x + \frac{1}{2}|u_y|\right), u - g(x,y)\right) = 0, \quad t > 0, \tag{14}$$

$$u(0,x,y) = u_0(x,y), \quad x \in \Omega, \tag{15}$$

with $u_0(\mathbf{x}) := \|\mathbf{x} - (-1.0,\ 0)\|_\infty - 0.5$ and $g(\mathbf{x}) := \min\left(0.25, \|\mathbf{x} - (0.2,\ 0)\|_2 - 0.5\right)$, corresponding to a square initial data and a disk obstacle.
In this example the “entropy fix" is needed.


t=0
t=0.4
t=0.8
t=1.2
t=1.6
t=2
DG (Q2)
Exact
Obstacle

Example 8 - Narrow band algorithm

.

- define a “cutoff" value ($C := 2\Delta x$),
- The initial data u_0 is transformed into

$$\tilde{u}_0(x, y) := \min(C, \max(-C, u_0(x, y))).$$

- At each time step, (i) for each cell (centered at (x_i, y_j)) :

$$nlogo^0_{i,j} := \begin{cases} 1 & \text{if } |u^n(x_i, y_j)| \leq 0.99\ C, \\ 0 & \text{otherwise} \end{cases}$$

- (ii) for all index i, j, compute

$$nlogo_{i,j} := \max(nlogo^0_{i,j}, nlogo^0_{i,j\pm1}, nlogo^0_{i\pm1,j})$$

- (iii) Do the DG computations only on cells (i, j) such that $nlogo_{i,j} = 1$.

Narrow band example

- We consider

$$u_t + 2\pi(-y, x) \cdot \nabla u = 0, \quad t > 0,\ (x, y) \in \Omega, \tag{16a}$$

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Omega, \tag{16b}$$

and same initial data u_0 as for the rotation example.

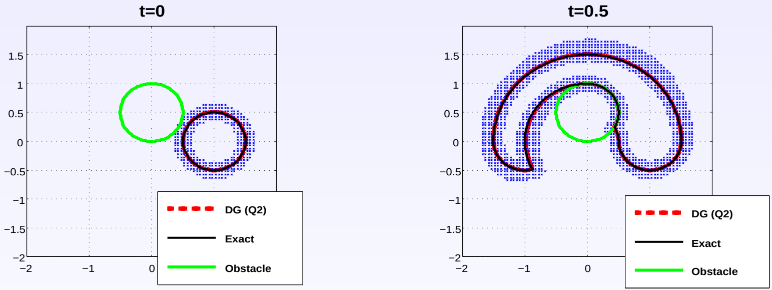

t=0
t=0.5
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1
0
DG (Q2)
Exact
Obstacle

Table: (Example 8) comparison of CPU times (in sec.) for full and narrow band approaches for (**??**), $t = 0.5$

N_x	full	"order"	narrow band	"order"	Gain (full / band)
20	8.1 s	-	6.9 s	-	1.17
40	45.2 s	5.58	17.1 s	2.47	2.64
80	347.4 s	7.68	83.4 s	4.87	4.16
160	2705.3 s	7.78	386.0 s	4.62	7.00

The “order" is computed as the ratio of CPU times $time(N_x)/time(N_x/2)$.

FUTUR WORK :

- improvement of the narrow band approach
- convergence proof (linear + obstacle case)
- applications to optimal control (higher dimensional problems)