Fast convergent finite difference solvers for the elliptic Monge-Ampère equation

Adam Oberman

Simon Fraser University

BIRS
February 17, 2011
Joint work

- [Froese, O.] 2010 convergent scheme in arbitrary dim., proof of convergence of Newton’s method
- [Froese, O.] 2010 more accurate hybrid scheme, Newton’s method solver.
- [Froese] Optimal Transportation solver
Monge-Ampère equation

\[ \det(D^2u(x)) = f(x), \quad \text{for } x \text{ in } \Omega. \]  \hspace{1cm} (MA)

\[ u \text{ is convex,} \]  \hspace{1cm} (C)

\[ u(x) = g(x), \quad \text{for } x \text{ on } \partial\Omega. \]  \hspace{1cm} (D)

\( \det(D^2u) \), is the determinant of the Hessian of the function \( u \).
\( \Omega \subset \mathbb{R}^d \) is a convex bounded subset with boundary \( \partial\Omega \),
Example

\[ u(x) = \exp \left( \frac{|x|^2}{2} \right), \quad f(x) = (1 + |x|^2) \exp(|x|^2). \]

Figure: The solution \( u(x) \). The image of mapping \( y = \nabla u(x) \).
Application: Optimal Transportation Problem

Map from one domain onto another, with given volume distortion.

$$\det(D^2 u(x)) = f(x)$$

$$\nabla u(x) : A \to B$$

Figure: The image of mapping $y = \nabla u(x)$ [Froese]
Generate mappings with controlled volume distortion.

$$\det(D^2 u(x)) = \begin{cases} 1, & \text{in most of } \Omega \\ \text{Large}, & \text{elsewhere} \end{cases}$$

**Figure:** The image of mapping $y = \nabla u(x)$

(Also bounds on volume distortion in a larger variational problem.)
Early work:

- Oliker [OP88], converges to the Aleksandrov solution in two dimensions. Very small problem size.
- Benamou and Brenier [BB00] fluid mechanical approach for the optimal transportation problem.

Recent work (representative):

- Publicized by Glowinski at ICIAM 07. Dean and Glowinski [DG08, DG06, Glo09].
- Feng and Neilan, [FN09a, FN09b] and Neilan, Brenner, et. al.
- Loeper [LR05], in the periodic case (see also Frisch [ZPF10])
- Haber and Haker for Benamou-Brenier method.
Comments on related work

None of the other schemes have convergence proofs. Indeed, they all break down on singular solutions.

- A number of recent papers use other numerical methods, e.g. FEM to solve the equation.
- Proof of consistency and stability for smooth solutions [Neilan Brenner], [Bohmer]. Even in the smooth case, this is not a convergence proof.
- No other results for weak solutions.
- We provide evidence that non-monotone methods break down near singular solutions
- Solvers slow down near non-smooth solutions
Summary of Results

- A finite difference solver for the Monge-Ampère equation, which converges to viscosity solution (even for singular solutions).
- Proof of convergence for a monotone scheme
- Fast solver using modified Newton’s method, $O(M^{1.3})$
- A more accurate discretization away from singularities

Summary: fast, accurate solver for fully nonlinear equation, effort comparable to solving a linear PDE several (ten) times.
Analysis and weak solutions

- Linearization
- Definition of weak solutions
- Regularity theory
- Convexity
Lemma

Let \( u \in C^2 \). The linearization of the Monge-Ampère operator is elliptic if \( D^2 u \) is positive definite or, equivalently, if \( u \) is (strictly) convex.

Linearization of the Monge-Ampère operator, when \( u \in C^2 \):

\[
\nabla_M \det(D^2 u)(v) = \text{trace} \left( (D^2 u)_{adj} D^2(v) \right).
\]

Example (two dimensions)

\[
\nabla_M \det(D^2 u) v = u_{xx} v_{yy} + u_{yy} v_{xx} - 2 u_{xy} v_{xy}
\]
Regularity

The Monge-Ampère equation

\[ \det(D^2 u(x)) = f(x), \quad \text{for } x \text{ in } \Omega. \] (MA)

\[ u(x) = g(x), \quad \text{for } x \text{ on } \partial \Omega. \] (D)

\[ u \text{ is convex}, \] (C)

has a unique \( C^{2,\alpha} \) solution, see [CNS84, Urb86, Caf90] [Gut01] under the following conditions.

\[
\begin{cases}
\text{The domain } \Omega \text{ is strictly convex with boundary } \partial \Omega \in C^{2,\alpha}. \\
\text{The boundary values } g \in C^{2,\alpha}(\partial \Omega). \\
\text{The function } f \in C^\alpha(\Omega) \text{ is strictly positive.}
\end{cases}
\]

- Regularity determines precisely when a monotone scheme is needed
- Other methods break down (100 \times slower) when \( \max f / \min f > 40 \)
- Our methods fast independent of \( f \).
Viscosity solutions

Definition

Let $u \in C(\Omega)$ be convex and $f \geq 0$ be continuous. The function $u$ is a viscosity subsolution (supersolution) of the Monge-Ampère equation in $\Omega$ if whenever convex $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $(u - \phi)(x) \leq (\geq)(u - \phi)(x_0)$ for all $x$ in a neighbourhood of $x_0$, then we must have

$$\det(D^2 \phi(x_0)) \geq (\leq)f(x_0).$$

The function $u$ is a viscosity solution if it is both a viscosity subsolution and supersolution.
A PDE for convexity

Convexity:

\[ \lambda_1(D^2 u) \geq 0, \]

where \( \lambda_1[D^2 u] \) is the smallest eigenvalue of the Hessian of \( u \).

The convexity constraint can be absorbed into the PDE operator

\[ \det^+(M) = \prod_{j=1}^{d} \lambda_j^+ \] (1)

where \( M \) is a symmetric matrix, with eigenvalues, \( \lambda_1 \leq \ldots, \leq \lambda_n \) and

\[ x^+ = \max(x, 0). \]
Summary:

- Standard finite difference scheme
- Wide stencil schemes (in general)
- Local variational characterization of the operator
- Convergence theorem
- Hybrid discretization: more accuracy in regular regions. (lose convergence proof)
Convergence

Theorem (Barles-Souganidis convergence)

The solutions of a consistent, monotone finite difference scheme converge uniformly to the unique viscosity solution of (MA).

Idea: $F^\epsilon \to F$ (consistency)
$F^\epsilon[u^\epsilon] = f$ (approximate solutions).
Want: $u^\epsilon \to u$ (convergence).
Require: stability in $L^\infty$ via the comparison principle.
Remark: Most numerical schemes give stability in a weaker norm, which does not allow to pass to limit in nonlinear PDE.
Remark: require wide stencils to obtain a monotone discretization.
Lemma (Variational characterization of the determinant)

Let $A$ be a $d \times d$ symmetric positive definite matrix with eigenvalues $\lambda_j$ and let $V$ be the set of all orthonormal bases of $\mathbb{R}^d$:

$$V = \{(\nu_1, \ldots, \nu_d) \mid \nu_j \in \mathbb{R}^d, \nu_i \perp \nu_j \text{ if } i \neq j, \|\nu_j\|_2 = 1\}.$$ 

Then the determinant of $A$ is equivalent to

$$\prod_{j=1}^{d} \lambda_j = \min_{(\nu_1, \ldots, \nu_d) \in V} \prod_{j=1}^{d} \nu_j^T A \nu_j.$$
The finite difference operator in grid direction $\nu$,

$$D_{\nu\nu} u_i = \frac{1}{|\nu| h^2} (u(x_i + \nu h) + u(x_i - \nu h) - 2u(x_i)).$$

Additional term in the consistency error coming from the angular resolution $d\theta$ of the stencil.

(a) In the interior.  
(b) Near the boundary.

Figure: Wide stencils on a two dimensional grid.
For a $C^2$ function $u$:

$$\det^+(D^2 \phi) = \min_{\{\nu_1 \ldots \nu_d\} \in \mathcal{V}} \prod_{j=1}^{d} \left( \frac{\partial^2 \phi}{\partial \nu_j^2} \right)^+.$$ 

On a finite difference grid, $\mathcal{G}$ grid directions,

$$MA^M[u] \equiv \min_{\{\nu_1 \ldots \nu_d\} \in \mathcal{G}} \prod_{j=1}^{d} (D\nu_j \nu_j u)^+$$
Overview of solution methods.
Explicit Solution Method

Simplest,

\[ u^{n+1} = u^n + dt(MA[u^n] - f). \]

Converges if the monotone discretization is used.

Does not converge if standard finite differences are used: no selection principle for convex solution

Slow due to CFL condition

\[ dt = O(h^2). \]

This was the approach used in [Obe08].
Use identity for the Laplacian in two dimensions,

$$|\Delta u| = \sqrt{(\Delta u)^2} = \sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}}.$$  \hfill (2)

So if $u$ solves the Monge-Ampère equation, then

$$|\Delta u| = \sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + 2f} = \sqrt{|D^2 u|^2 + 2f}$$

Semi-implicit scheme

$$\Delta u^{n+1} = \sqrt{2f + |D^2 u^n|^2}$$  \hfill (3)
Challenging in singular case - like N.M for \((x^+)^2\) near 0.
To solve the discretized equation

\[ MA^H [u] = f \]

The corrector \(v^n\) solves the linear system

\[ \left( \nabla_u MA^H [u^n] \right) v^n = MA^H [u^n] - f. \]

**Theorem**

*Convergence of Newton’s method in continuous case under regularity assumptions (extension of [LR05]) and in the discrete case for the monotone scheme.*
Visualization of Computational results

- example where standard scheme fails
- visualization of sample solutions with different regularity.
Solution is surface of ball, with vertical tangent at one point of domain.

Example (unbounded gradient near the boundary point \((1, 1)\))

\[
\begin{align*}
  u(x) &= -\sqrt{2 - |x|^2}, \\
  f(x) &= 2 \left(2 - |x|^2\right)^{-2}.
\end{align*}
\] (4)
Failure of Newton’s method for natural finite differences

Solution in $[0, 1]^2$

$$u(x) = -\sqrt{2 - |x|^2}, \quad f(x) = 2\left(2 - |x|^2\right)^{-2}$$

Figure: The solution oscillates and becomes non-convex.
Mildly singular solution

Example ($C^1$)

$$u(x) = \frac{1}{2} \left( (|x - x_0| - 0.2)^+ \right)^2, \quad f(x) = \left( 1 - \frac{0.2}{|x - x_0|} \right)^+. \quad (5)$$
Most singular solution

Example (cone, non-differentiable)

\[ u(x) = \sqrt{|x - x_0|}, \quad f = \mu = \pi \delta_{x_0} \quad (6) \]

Approximate measure \( \mu \) by its average over ball of radius \( h/2 \),

\[ f^h = \begin{cases} 
4/h^2 & \text{for } |x - x_0| \leq h/2, \\
0 & \text{otherwise.}
\end{cases} \]
Quantitative Computational Results

Summary:

- tables of solution times: Newton method is fast. Other methods: speed may depend on regularity of solution
- tables of accuracy: Hybrid scheme is most accurate. On nonsmooth solutions, monotone scheme is more accurate than standard scheme, despite lower formal accuracy.
Compare: Gauss-Seidel, Semi-Implicit (Poisson), Newton.

<table>
<thead>
<tr>
<th>Method</th>
<th>$C^2,\alpha$</th>
<th>$C^1,\alpha$ (5) and (4)</th>
<th>$C^0,1$ (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss-Seidel</td>
<td>Moderate</td>
<td>Moderate</td>
<td>Moderate</td>
</tr>
<tr>
<td></td>
<td>($\sim O(M^{1.8})$)</td>
<td>($\sim O(M^{1.9})$)</td>
<td>($\sim O(M^2)$)</td>
</tr>
<tr>
<td>Poisson</td>
<td>Fast</td>
<td>Fast–Slow</td>
<td>Slow</td>
</tr>
<tr>
<td></td>
<td>($\sim O(M^{1.4})$)</td>
<td>($\sim O(M^{1.4})$–blow-up)</td>
<td>($\sim O(M^2)$–blow-up)</td>
</tr>
<tr>
<td>Newton</td>
<td>Fast</td>
<td>Fast</td>
<td>Fast</td>
</tr>
<tr>
<td></td>
<td>($\sim O(M^{1.3})$)</td>
<td>($\sim O(M^{1.3})$)</td>
<td>($\sim O(M^{1.3})$)</td>
</tr>
</tbody>
</table>

**Table:** The Newton solver is fastest in terms of absolute and order of magnitude solution time in each case.
## Computation time

<table>
<thead>
<tr>
<th>N</th>
<th>Newton</th>
<th>Its.</th>
<th>Newton (sec)</th>
<th>Poisson (sec)</th>
<th>Gauss-Seidel (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C^2$ Example</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>3</td>
<td>0.2</td>
<td>0.7</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>127</td>
<td>5</td>
<td>2.9</td>
<td>9.6</td>
<td>236.7</td>
<td></td>
</tr>
<tr>
<td>361</td>
<td>6</td>
<td>131.4</td>
<td>162.6</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C^1$ Example</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>4</td>
<td>0.4</td>
<td>1.1</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>127</td>
<td>11</td>
<td>5.7</td>
<td>256.8</td>
<td>145.5</td>
<td></td>
</tr>
<tr>
<td>361</td>
<td>20</td>
<td>200.0</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C^{0,1}$ (Lipschitz) Example</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>9</td>
<td>0.5</td>
<td>5.3</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>127</td>
<td>32</td>
<td>14.1</td>
<td>1758.2</td>
<td>373.9</td>
<td></td>
</tr>
<tr>
<td>361</td>
<td>29</td>
<td>280.2</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
### Accuracy: Max Error

#### $C^2$ Example

<table>
<thead>
<tr>
<th>N</th>
<th>Standard</th>
<th>Monotone</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>$7.14 \times 10^{-5}$</td>
<td>$89.09 \times 10^{-5}$</td>
<td>$24.45 \times 10^{-5}$</td>
</tr>
<tr>
<td>361</td>
<td>$0.05 \times 10^{-5}$</td>
<td>$44.00 \times 10^{-5}$</td>
<td>$0.46 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

#### $C^1$ Example

<table>
<thead>
<tr>
<th>N</th>
<th>Standard</th>
<th>Monotone</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>$2.6 \times 10^{-4}$</td>
<td>$17.5 \times 10^{-4}$</td>
<td>$12.2 \times 10^{-4}$</td>
</tr>
<tr>
<td>361</td>
<td>—</td>
<td>$7.0 \times 10^{-4}$</td>
<td>$0.7 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

#### Example with blow-up

<table>
<thead>
<tr>
<th>N</th>
<th>Standard</th>
<th>Monotone</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>$17.15 \times 10^{-3}$</td>
<td>$1.74 \times 10^{-3}$</td>
<td>$1.74 \times 10^{-3}$</td>
</tr>
<tr>
<td>361</td>
<td>$5.41 \times 10^{-3}$</td>
<td>$0.33 \times 10^{-3}$</td>
<td>$0.04 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

#### $C^{0,1}$ (Lipschitz) Example

<table>
<thead>
<tr>
<th>N</th>
<th>Standard</th>
<th>Monotone</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>$10 \times 10^{-3}$</td>
<td>$3 \times 10^{-3}$</td>
<td>$3 \times 10^{-3}$</td>
</tr>
<tr>
<td>361</td>
<td>—</td>
<td>$4 \times 10^{-3}$</td>
<td>$4 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
## Three dimensional Results

<table>
<thead>
<tr>
<th>N</th>
<th>Max Error</th>
<th>Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.0151</td>
<td>2</td>
<td>0.04</td>
</tr>
<tr>
<td>31</td>
<td>0.0111</td>
<td>5</td>
<td>86.63</td>
</tr>
</tbody>
</table>

### $C^2$ Example

<table>
<thead>
<tr>
<th>N</th>
<th>Max Error</th>
<th>Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.0034</td>
<td>1</td>
<td>0.02</td>
</tr>
<tr>
<td>31</td>
<td>0.0005</td>
<td>1</td>
<td>17.12</td>
</tr>
</tbody>
</table>

### $C^1$ Example

<table>
<thead>
<tr>
<th>N</th>
<th>Max Error</th>
<th>Iterations</th>
<th>CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>9.6 × 10^{-3}</td>
<td>1</td>
<td>0.03</td>
</tr>
<tr>
<td>31</td>
<td>2.9 × 10^{-3}</td>
<td>8</td>
<td>138.74</td>
</tr>
</tbody>
</table>

**Table:** Maximum error and computation time for the hybrid Newton's method on three representative examples.
Conclusions

Numerical methods for Monge-Ampère

- Even under conditions where solution is regular a naive scheme will not work, unless the convexity condition is enforced locally.
- For singular solutions, the equation becomes degenerate, and iterative solvers can break down.
- Using a monotone scheme resolves these problems.
- For increased accuracy, can use a hybrid scheme in regular regions of the solution.
- Monotonicity discretizations also prevent singularities in the gradient map, which is useful for applications.
End
Jean-David Benamou and Yann Brenier.

Luis A. Caffarelli.

L. Caffarelli, L. Nirenberg, and J. Spruck.

E. J. Dean and R. Glowinski.

Edward J. Dean and Roland Glowinski.

Xiaobing Feng and Michael Neilan.

Xiaobing Feng and Michael Neilan.

Roland Glowinski.
Numerical methods for fully nonlinear elliptic equations.

Cristian E. Gutiérrez.
*The Monge-Ampère equation.* 

Grégoire Loeper and Francesca Rapetti.
Numerical solution of the Monge-Ampère equation by a Newton’s algorithm. 

Adam M. Oberman.
Wide stencil finite difference schemes for the elliptic Monge-Ampère equation and functions of the eigenvalues of the Hessian. 

V. I. Oliker and L. D. Prussner.
On the numerical solution of the equation \((\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2) - (\partial^2 z/\partial x\partial y)^2 = f\) and its discretizations, I. 

John I. E. Urbas.
The generalized Dirichlet problem for equations of Monge-Ampère type. 

V. Zheligovsky, O. Podvigina, and U. Frisch.
The monge-ampère equation: Various forms and numerical solution. 