Fast convergent finite difference solvers for the elliptic Monge-Ampère equation

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BIRS
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Joint work

- [O.] 2008. Convergent scheme in two dim. Explicit solver.
- [Froese, Benamou, O.] 2010. Standard finite difference schemes in two dimensions. Two solvers (explicit/semi-implicit), both enforcing convexity.
- [Froese, O.] 2010 convergent scheme in arbitrary dim., proof of convergence of Newton's method
- [Froese, O.] 2010 more accurate hybrid scheme, Newton's method solver.
- [Froese] Optimal Transportation solver

Monge-Ampère equation

$$\det(D^2 u(x)) = f(x), \quad \text{for } x \text{ in } \Omega. \tag{MA}$$

$$u \text{ is convex,} \tag{C}$$

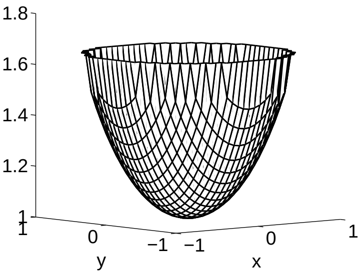
$$u(x) = g(x), \quad \text{for } x \text{ on } \partial\Omega. \tag{D}$$

$\det(D^2 u)$, is the determinant of the Hessian of the function u.
$\Omega \subset \mathbb{R}^d$ is a convex bounded subset with boundary $\partial\Omega$,

Visualization of solution and gradient map

Example

$$u(\mathbf{x}) = \exp\left(\frac{|\mathbf{x}|^2}{2}\right), \qquad f(\mathbf{x}) = (1 + |\mathbf{x}|^2)\exp(|\mathbf{x}|^2).$$



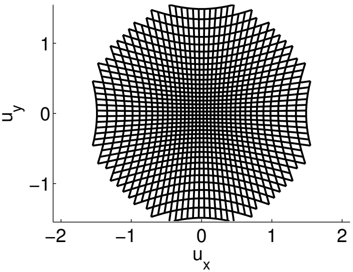


Figure: The solution $u(x)$. The image of mapping $y = \nabla u(x)$

Application: Optimal Transportation Problem

Map from one domain onto another, with given volume distortion.

$$\det(D^2 u(x)) = f(x)$$

$$\nabla u(x) : A \to B$$

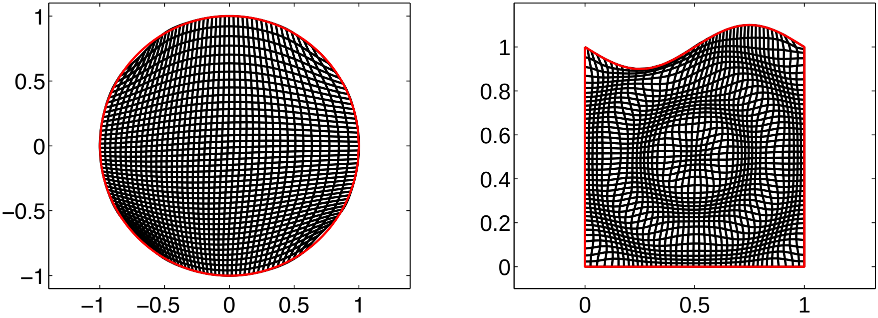


Figure: The image of mapping $y = \nabla u(x)$ [Froese]

Application: mappings with controlled volume distortion

Generate mappings with controlled volume distortion.

$$\det(D^2 u(x)) = \begin{cases} 1, & \text{in most of } \Omega \\ \text{Large}, & \text{elsewhere} \end{cases}$$

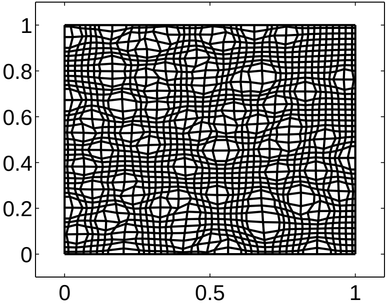


Figure: The image of mapping $y = \nabla u(x)$

(Also bounds on volume distortion in a larger variational problem.)

Related numerical works

Early work:

- Oliker [OP88], converges to the Aleksandrov solution in two dimensions. Very small problem size.
- Benamou and Brenier [BB00] fluid mechanical approach for the optimal transportation problem.

Recent work (representative):

- Publicized by Glowinski at ICIAM 07. Dean and Glowinski [DG08, DG06, Glo09].
- Feng and Neilan, [FN09a, FN09b] and Neilan, Brenner, et. al.
- Loeper [LR05], in the periodic case (see also Frisch [ZPF10])
- Haber and Haker for Benamou-Brenier method.

Comments on related work

None of the other schemes have convergence proofs. Indeed, they all break down on singular solutions.

- A number of recent papers use other numerical methods, e.g. FEM to solve the equation.
- Proof of consistency and stability for smooth solutions [Neilan Brenner], [Bohmer]. Even in the smooth case, this is not a convergence proof.
- No other results for weak solutions.
- We provide evidence that non-monotone methods break down near singular solutions
- Solvers slow down near non-smooth solutions

Summary of Results

- A finite difference solver for the Monge-Ampère equation, which converges to viscosity solution (even for singular solutions).
- Proof of convergence for a monotone scheme
- Fast solver using modified Newton's method, $\mathcal{O}(M^{1.3})$
- A more accurate discretization away from singularities

Summary: fast, accurate solver for fully nonlinear equation, effort comparable to solving a linear PDE several (ten) times.

Analysis and weak solutions

- Linearization
- Definition of weak solutions
- Regularity theory
- Convexity

Linearization and ellipticity

Lemma

Let $u \in C^2$. The linearization of the Monge-Ampère operator is elliptic if $D^2 u$ is positive definite or, equivalently, if u is (strictly) convex.

Linearization of the Monge-Ampère operator, when $u \in C^2$:

$$\nabla_M \det(D^2 u)(v) = \operatorname{trace}\left((D^2 u)_{adj} D^2(v)\right).$$

Example (two dimensions)

$$\nabla_M \det(D^2 u) v = u_{xx} v_{yy} + u_{yy} v_{xx} - 2 u_{xy} v_{xy}$$

Regularity

The Monge-Ampère equation

$$\det(D^2 u(x)) = f(x), \quad \text{for } x \text{ in } \Omega. \tag{MA}$$

$$u(x) = g(x), \quad \text{for } x \text{ on } \partial\Omega. \tag{D}$$

$$u \text{ is convex,} \tag{C}$$

has a unique $C^{2,\alpha}$ solution, see [CNS84, Urb86, Caf90] [Gut01] under the following conditions.

$$\begin{cases} \text{The domain } \Omega \text{ is strictly convex with boundary } \partial\Omega \in C^{2,\alpha}. \\ \text{The boundary values } g \in C^{2,\alpha}(\partial\Omega). \\ \text{The function } f \in C^{\alpha}(\Omega) \text{ is strictly positive.} \end{cases}$$

- *Regularity determines precisely when a monotone scheme is needed*
- *Other methods break down (100 $\times$ slower) when* $\max f / \min f > 40$
- *Our methods fast independent of f.*

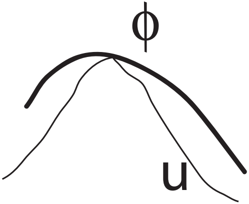
Viscosity solutions

Definition

Let $u \in C(\Omega)$ be convex and $f \geq 0$ be continuous. The function u is a *viscosity subsolution (supersolution)* of the Monge-Ampère equation in Ω if whenever convex $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $(u - \phi)(x) \leq (\geq)(u - \phi)(x_0)$ for all x in a neighbourhood of x_0, then we must have

$$\det(D^2\phi(x_0)) \geq (\leq) f(x_0).$$

The function u is a *viscosity solution* if it is both a viscosity subsolution and supersolution.



A PDE for convexity

Convexity:

$$\lambda_1(D^2u) \geq 0,$$

where $\lambda_1[D^2u]$ is the smallest eigenvalue of the Hessian of u.
The convexity constraint can be absorbed into the PDE operator

$$\det{}^+(M) = \prod_{j=1}^{d} \lambda_j^+ \tag{1}$$

where M is a symmetric matrix, with eigenvalues, $\lambda_1 \leq \ldots, \leq \lambda_n$ and

$$x^+ = \max(x, 0).$$

Finite Difference Discretization

Summary:

- Standard finite difference scheme
- Wide stencil schemes (in general)
- Local variational characterization of the operator
- Convergence theorem
- Hybrid discretization: more accuracy in regular regions. (lose convergence proof)

Convergence

Theorem (Barles-Souganidis convergence)

The solutions of a consistent, monotone finite difference scheme converge uniformly to the unique viscosity solution of (MA)*.*

Idea: $F^\epsilon \to F$ (consistency)
$F^\epsilon[u^e] = f$ (approximate solutions).
Want: $u^\epsilon \to u$ (convergence).
Require: stability in L^∞ via the comparison principle.
Remark: Most numerical schemes give stability in a weaker norm, which does not allow to pass to limit in nonlinear PDE.
Remark: require wide stencils to obtain a monotone discretization.

Variational characterization of the determinant

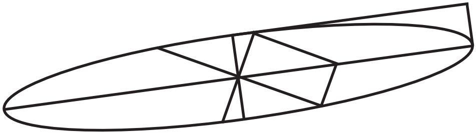
Lemma (Variational characterization of the determinant)

Let A be a $d \times d$ symmetric positive definite matrix with eigenvalues λ_j and let V be the set of all orthonormal bases of $\mathbb{R}^d$:

$$V = \{(\nu_1, \ldots, \nu_d) \mid \nu_j \in \mathbb{R}^d, \nu_i \perp \nu_j \text{ if } i \neq j, \|\nu_j\|_2 = 1\}.$$

Then the determinant of A is equivalent to

$$\prod_{j=1}^{d} \lambda_j = \min_{(\nu_1,\ldots,\nu_d)\in V} \prod_{j=1}^{d} \nu_j^T A \nu_j.$$

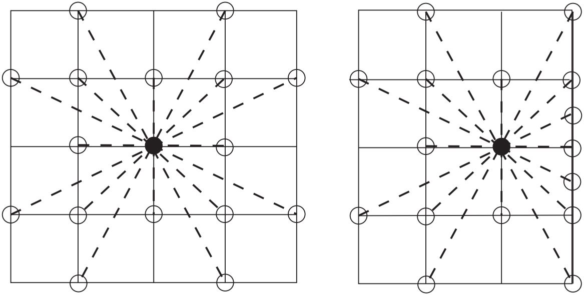


Wide stencils

The finite difference operator in grid direction ν,

$$\mathcal{D}_{\nu\nu} u_i = \frac{1}{|\nu|\, h^2} \left(u(x_i + \nu h) + u(x_i - \nu h) - 2u(x_i) \right).$$

Additional term in the consistency error coming from the angular resolution $d\theta$ of the stencil.



(a) In the interior. (b) Near the boundary.

Figure: Wide stencils on a two dimensional grid.

Discretization of convexified Monge-Ampère operator

For a C^2 function u:

$$\det{}^+(D^2\phi) = \min_{\{\nu_1 \ldots \nu_d\} \in V} \prod_{j=1}^{d} \left(\frac{\partial^2 \phi}{\partial \nu_j^2} \right)^+ .$$

On a finite difference grid, $\mathcal{G}$ grid directions,

$$MA^M[u] \equiv \min_{\{\nu_1 \ldots \nu_d\} \in \mathcal{G}} \prod_{j=1}^{d} \left(\mathcal{D}_{\nu_j \nu_j} u \right)^+ \qquad (MA)^M$$

Solution Methods

Overview of solution methods.

Explicit Solution Method

Simplest,

$$u^{n+1} = u^n + dt(MA[u^n] - f).$$

Converges if the monotone discretization is used.

Does not converge if standard finite differences are used: *no selection principle for convex solution*

Slow due to CFL condition

$$dt = \mathcal{O}(h^2).$$

This was the approach used in [Obe08].

Semi-Implicit solution method (2d)

Use identity for the Laplacian in two dimensions,

$$|\Delta u| = \sqrt{(\Delta u)^2} = \sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}}. \tag{2}$$

So if u solves the Monge-Ampère equation, then

$$|\Delta u| = \sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + 2f} = \sqrt{|D^2 u|^2 + 2f}$$

Semi-implicit scheme

$$\Delta u^{n+1} = \sqrt{2f + |D^2 u^n|^2} \tag{3}$$

Implementation of Newton's method

Challenging in singular case - like N.M for $(x^+)^2$ *near 0.*
To solve the discretized equation

$$MA^H[u] = f$$

The corrector v^n solves the linear system

$$\left(\nabla_u MA^H[u^n]\right) v^n = MA^H[u^n] - f.$$

Theorem

Convergence of Newton's method in continuous case under regularity assumptions (extension of [LR05]) and in the discrete case for the monotone scheme.

Visualization of Computational results

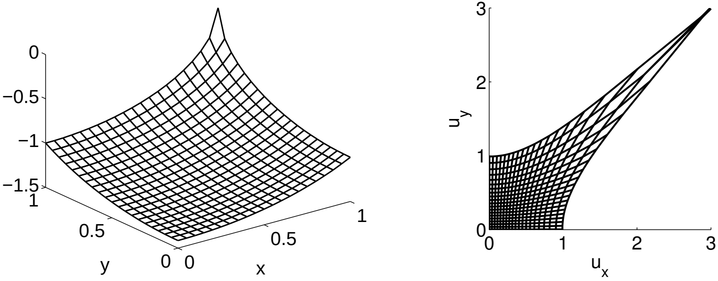
- example where standard scheme fails
- visualization of sample solutions with different regularity.

Singularity in gradient

Solution is surface of ball, with vertical tangent at one point of domain.

Example (unbounded gradient near the boundary point (1, 1))

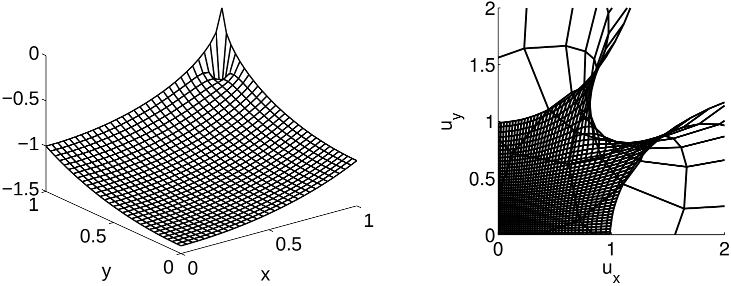
$$u(\mathbf{x}) = -\sqrt{2 - |\mathbf{x}|^2}, \qquad f(\mathbf{x}) = 2\left(2 - |\mathbf{x}|^2\right)^{-2}. \qquad (4)$$



Failure of Newton's method for natural finite differences

Solution in $[0,1]^2$

$$u(\mathbf{x}) = -\sqrt{2 - |\mathbf{x}|^2}, \qquad f(\mathbf{x}) = 2\left(2 - |\mathbf{x}|^2\right)^{-2}$$



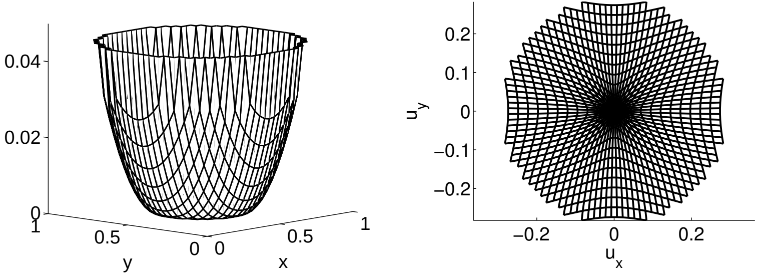
(a) Solution after two iterations (b) Gradient map after two iterations

Figure: The solution oscillates and becomes non-convex.

Mildly singular solution

Example (C^1)

$$u(\mathbf{x}) = \frac{1}{2}\left(\left(|\mathbf{x}-\mathbf{x}_0| - 0.2\right)^+\right)^2, \quad f(\mathbf{x}) = \left(1 - \frac{0.2}{|\mathbf{x}-\mathbf{x}_0|}\right)^+. \quad (5)$$



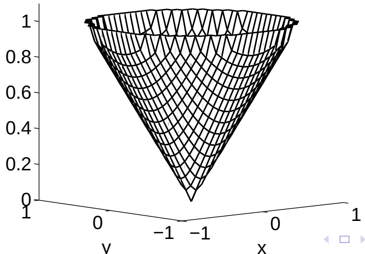
Most singular solution

Example (cone, non-differentiable)

$$u(\mathbf{x}) = \sqrt{|\mathbf{x} - \mathbf{x}_0|}, \qquad f = \mu = \pi\,\delta_{\mathbf{x}_0} \tag{6}$$

Approximate measure μ by its average over ball of radius $h/2$,

$$f^h = \begin{cases} 4/h^2 & \text{for } |\mathbf{x} - \mathbf{x}_0| \le h/2, \\ 0 & \text{otherwise.} \end{cases}$$



Quantitative Computational Results

Summary:

- tables of solution times: Newton method is fast. Other methods: speed may depend on regularity of solution
- tables of accuracy: Hybrid scheme is most accurate. On nonsmooth solutions, monotone scheme is more accurate that standard scheme, despite lower formal accuracy.

Order of magnitude computation time

Compare: Gauss-Seidel, Semi-Implicit (Poisson), Newton.

Method	Regularity of Solution $C^{2,\alpha}$	$C^{1,\alpha}$ (5) and (4)	$C^{0,1}$ (6)
Gauss-Seidel	Moderate ($\sim \mathcal{O}(M^{1.8})$)	Moderate ($\sim \mathcal{O}(M^{1.9})$)	Moderate ($\sim \mathcal{O}(M^{2})$)
Poisson	Fast ($\sim \mathcal{O}(M^{1.4})$	Fast–Slow ($\sim \mathcal{O}(M^{1.4})$–blow-up)	Slow ($\sim \mathcal{O}(M^{2})$–blow-up)
Newton	Fast ($\sim \mathcal{O}(M^{1.3})$)	Fast ($\sim \mathcal{O}(M^{1.3})$)	Fast ($\sim \mathcal{O}(M^{1.3})$)

Table: The Newton solver is fastest in terms of absolute and order of magnitude solution time in each case.

Computation time

C^2 Example

N	Newton Its.	Newton (sec)	Poisson (sec)	Gauss-Seidel (sec)
31	3	0.2	0.7	2.2
127	5	2.9	9.6	236.7
361	6	131.4	162.6	—

C^1 Example

N	Newton Its.	Newton (sec)	Poisson (sec)	Gauss-Seidel (sec)
31	4	0.4	1.1	0.8
127	11	5.7	256.8	145.5
361	20	200.0	—	—

$C^{0,1}$ (Lipschitz) Example

N	Newton Its.	Newton (sec)	Poisson (sec)	Gauss-Seidel (sec)
31	9	0.5	5.3	0.8
127	32	14.1	1758.2	373.9
361	29	280.2	—	—

Accuracy: Max Error

C^2 Example

N	Standard	Monotone	Hybrid
31	7.14×10^{-5}	89.09×10^{-5}	24.45×10^{-5}
361	0.05×10^{-5}	44.00×10^{-5}	0.46×10^{-5}

C^1 Example

N	Standard	Monotone	Hybrid
31	2.6×10^{-4}	17.5×10^{-4}	12.2×10^{-4}
361	—	7.0×10^{-4}	0.7×10^{-4}

Example with blow-up

N	Standard	Monotone	Hybrid
31	17.15×10^{-3}	1.74×10^{-3}	1.74×10^{-3}
361	5.41×10^{-3}	0.33×10^{-3}	0.04×10^{-3}

$C^{0,1}$ (Lipschitz) Example

N	Standard	Monotone	Hybrid
31	10×10^{-3}	3×10^{-3}	3×10^{-3}
361	—	4×10^{-3}	4×10^{-3}

Three dimensional Results

C^2 Example

N	Max Error	Iterations	CPU Time (s)
7	0.0151	2	0.04
31	0.0111	5	86.63

C^1 Example

N	Max Error	Iterations	CPU Time (s)
7	0.0034	1	0.02
31	0.0005	1	17.12

Example with Blow-up

N	Max Error	Iterations	CPU Time (s)
7	9.6×10^{-3}	1	0.03
31	2.9×10^{-3}	8	138.74

Table: Maximum error and computation time for the hybrid Newton's method on three representative examples.

Conclusions

Numerical methods for Monge-Ampère

- Even under conditions where solution is regular a naive scheme will not work, unless the convexity condition is enforced locally
- For singular solutions, the equation becomes degenerate, and iterative solvers can break down
- Using a monotone scheme resolves these problems.
- For increased accuracy, can use a hybrid scheme in regular regions of the solution.
- Monotonicity discretizations also prevent singularities in the gradient map, which is useful for applications.

End

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