Hamilton-Jacobi equations with shocks arising from general Fokker-Planck equations: analysis and numerical approximation

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Workshop

Advancing numerical methods for viscosity solutions and applications

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Outline

- A class of Hamilton-Jacobi equations
  - Derived from Fokker-Planck equations
  - Develop propagating fronts with signal-dependent speed
- In particular, Hamilton-Jacobi equations arising from:
  - Relativistic Heat Fokker-Planck equation (Flux limited)
  - Relativistic Porous Media Fokker-Planck equation (Flux un-limited)
  - Relativistic Speed limited Fokker-Planck equation
- Numerical scheme and examples

Introduction

General **Fokker-Plank** equation

$$u_t = \mathrm{div}\Big(g(u, |\nabla u|)\nabla u\Big)$$

g(u,p) a non-negative scalar function of u and $|\nabla u|$.

- Models many physical phenomena related to **transport** processes
- If $g = g(u)$, represents a classical Fokker-Planck equation, (Transport in Statistical Mechanics)
- If $g = g(|\nabla u|)$, represents an anomalous diffusion equation, (Geometric Flows).
- If g is a positive constant $\nu > 0$ then represents the classical heat equation.
- If $g = g(u, |\nabla u|)$ is a non-negative bounded function then represents a **flux limited diffusion equation**.
- If $g = g(u, |\nabla u|)$ a non-negative function ...

Convective term of Fokker-Planck equations

Expanding the divergence of the Fokker-Planck equation we obtain

$$u_t = g_u(u, |\nabla u|)|\nabla u|^2 + g_u(u, |\nabla u|)\Delta u + \frac{g_p(u, |\nabla u|)}{|\nabla u|} L(\nabla u)$$

where

$$L(\nabla u) = \Big(\frac{\partial u}{\partial x}\Big)^2 + \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \Big(\frac{\partial u}{\partial y}\Big)^2 \frac{\partial^2 u}{\partial y^2}$$

We will study the convective term: **the Hamiltonian**

$$g_u(u, |\nabla u|)|\nabla u|^2$$

Convective term of Fokker-Planck equations

In particular we focus on

$$g(u, |\nabla u|) = f(u)\frac{r}{\sqrt{u^2 + r^2|\nabla u|^2}} \qquad\qquad r = \frac{\nu}{c}$$

The associated Fokker-Planck equations:

- $f(u) = cu \rightarrow$ Relativistic Heat Fokker-Planckequations (Flux limited)
- $f(u) = c\frac{u^2}{2} \rightarrow$ Relativistic porous media Fokker-Planck equation (Flux un-limited)

Relativistic Heat Equation (RHE)

With $f(u) = cu$, the Fokker-Planck equation becomes the

Relativistic Heat Equation

$$u_t = \nu \operatorname{div} \left(\frac{u \nabla u}{\sqrt{u^2 + (\frac{\nu}{c})^2 |\nabla u|^2}} \right)$$

Main contributions: P. Rosenau (1992), Y. Brenier (2003), F. Andreu, V. Caselles, JM Mazon et al (2005-2007), A. Marquina (2010)

■ **Motivation: Heat Equation**

Classical Heat Equation

The classical heat equation $u_t = \nu \Delta u$ can be written in divergence form as:

$$u_t + \mathrm{div}\,(u\vec{v}) \quad = \quad 0$$

where the **velocity field** $\vec{v}$ is defined as

$$\vec{v} = -\nu \frac{\nabla u}{u}, \qquad\qquad \nu > 0$$

proportional to ∇u (possibly unbounded)!

PROPERTIES:

- Velocity of heat transfer is <u>not limited</u>
- Diffusion propagates with **infinite** speed !!

$\rightarrow$ More realistic heat equation ??

Relativistic Heat Equation (RHE)

A way to control speed propagation:

- Limiting the flow velocity field by making it “relativistic" so the maximum velocity allowed is the speed of light $c > 0$

Rosenau (1992) proposes to weight the velocity field in the heat flux as

$$\nu \frac{\nabla u}{u} = \frac{-\vec{v}}{\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}}$$

$\big($ weightening with the dimensionless Lorentz factor $W = \frac{1}{\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}}\big)$

The equation becomes,

Relativistic Heat Equation (RHE)

$$u_t = \nu \, \mathrm{div} \left(\frac{u \nabla u}{\sqrt{u^2 + (\frac{\nu}{c})^2 |\nabla u|^2}} \right)$$

PROPERTIES:

- the solution is able to develop rarefaction waves, kinks and shocks
- shocks can not propagate faster than the speed of light c.
- Propagation occurs at **constant speed**

Relativistic Heat Equation (RHE)

The one-dimensional RHE can be expressed as

$$u_t = \nu\left(\frac{uu_x}{\sqrt{u^2 + r^2 u_x^2}}\right)_x, \qquad t > 0, u \geq 0$$

where $\nu > 0$ and $r = \frac{\nu}{c}$.

$$u_t = c\Big(\frac{ru_x}{\sqrt{u^2 + r^2 u_x^2}}\Big)^3 u_x + \nu\Big(\frac{u}{\sqrt{u^2 + r^2 u_x^2}}\Big)^3 u_{xx}$$

The **convective** part is a Hamilton-Jacobi term that depends on u and u_x

A non-conservative approach (1-D case)

We consider (what we call) the **Relativistic Hamilton-Jacobi heat** equation

$$u_t = c\left(\frac{ru_x}{\sqrt{u^2 + r^2u_x^2}}\right)^3 u_x$$

Around jump discontinuities or “large gradients" where $|u_x| >> u$, the ratio $\frac{u}{r|u_x|} << 1$ is small.

Defining $\text{sgn}(u_x) = \frac{u_x}{|u_x|}$ we re-write and obtain

$$u_t = c\frac{\text{sgn}(\text{u}_\text{x})}{\left(\sqrt{(\frac{u}{ru_x})^2 + 1}\right)^3} u_x$$

A non-conservative approach (1-D case)

Using the Taylor expansion of $(1+y)^{-\frac{3}{2}} = 1 - \frac{3}{2}y + \frac{15}{8}y^2 + O(y^3)$ (convergent for $|y| < 1$) for $y = \frac{u}{ru_x}$ we have

$$u_t \quad = \quad c\,\mathrm{sgn}(u_x)\Big(1 - \frac{3}{2}(\frac{u}{ru_x})^2 + \frac{15}{8}(\frac{u}{ru_x})^4 - \cdots\Big)u_x$$

Then, assuming $\frac{u}{r|u_x|} << 1$ the equation approaches to

$$u_t \quad \approx \quad c\,\mathrm{sgn}(u_x)u_x$$

Relativistic Hamilton-Jacobi heat equation:

- resembles **linear advection** equation around jump discontinuities
- propagates at **constant speed** $c > 0$ according to the direction prescribed by the sign of u_x.
- $\rightarrow$ is a convective term responsible of the development of waves, kinks and shocks in the solution where shocks will not propagate faster than the speed of light c.

Numerical method

$$u_t + H(u, u_x, u_y) = 0 \;\text{ where }\; H(u,p,q) := -G(u, \sqrt{p^2+q^2})(p^2+q^2)$$

Finite differences numerical scheme

$$u_{jk}^{n+1} = u_{jk}^n - \Delta t\ \tilde{h}\Big(u_{jk}^n, \frac{\Delta_-^x u_{jk}^n}{\Delta x}, \frac{\Delta_+^x u_{jk}^n}{\Delta x}, \frac{\Delta_-^y u_{jk}^n}{\Delta y}, \frac{\Delta_+^x u_{jk}^n}{\Delta y}\Big)$$

where $\tilde{h}$ is Lipschitz and consistent numerical Hamiltonian:

- **consistency:** $\tilde{h}(u,p,p,q,q) = H(u,p,q)$

$$\text{Notation}: \; \Delta_-^x u_{jk}^n = u_{jk}^n - u_{j-1,k}^n \qquad \Delta_+^x u_{jk}^n = u_{j+1k}^n - u_{j,k}^n$$

$$\Delta_-^y u_{jk}^n = u_{jk}^n - u_{j,k-1}^n \qquad \Delta_+^y u_{jk}^n = u_{j,k+1}^n - u_{j,k}^n$$

Numerical method

Local Lax-Friedrichs Hamiltonian

$$\tilde{h}^{LLF}(u,p^-,p^+,q^-,q^+) = H(u,\frac{p^- + p^+}{2},\frac{q^- + q^+}{2}) - \frac{\alpha_1}{2}(p^- - p^+) - \frac{\alpha_2}{2}(q^+ - q^-)$$

where

$$\alpha_1 = \max|H_p(u,p,q)|, \qquad \alpha_2 = \max|H_q(u,p,q)|$$

$$H_p = \frac{\partial H}{\partial p} \qquad H_q = \frac{\partial H}{\partial q}$$

The maxima are taken on the local intervals:

$$p \in I(p^-,p^+); q \in I(q^-,q^+), \;\; I(a,b) = [\min(a,b), \max(a,b)]$$

overall u in the domain.

Numerical Implementation

High order implementation of the numerical scheme

- In space: computing fifth order approximations of the arguments of the Hamiltonian
  - Weighted PowerENO5

- In time: explicit integration
  - Strong Stability Preserving Runge-Kutta method

Two square waves initial data

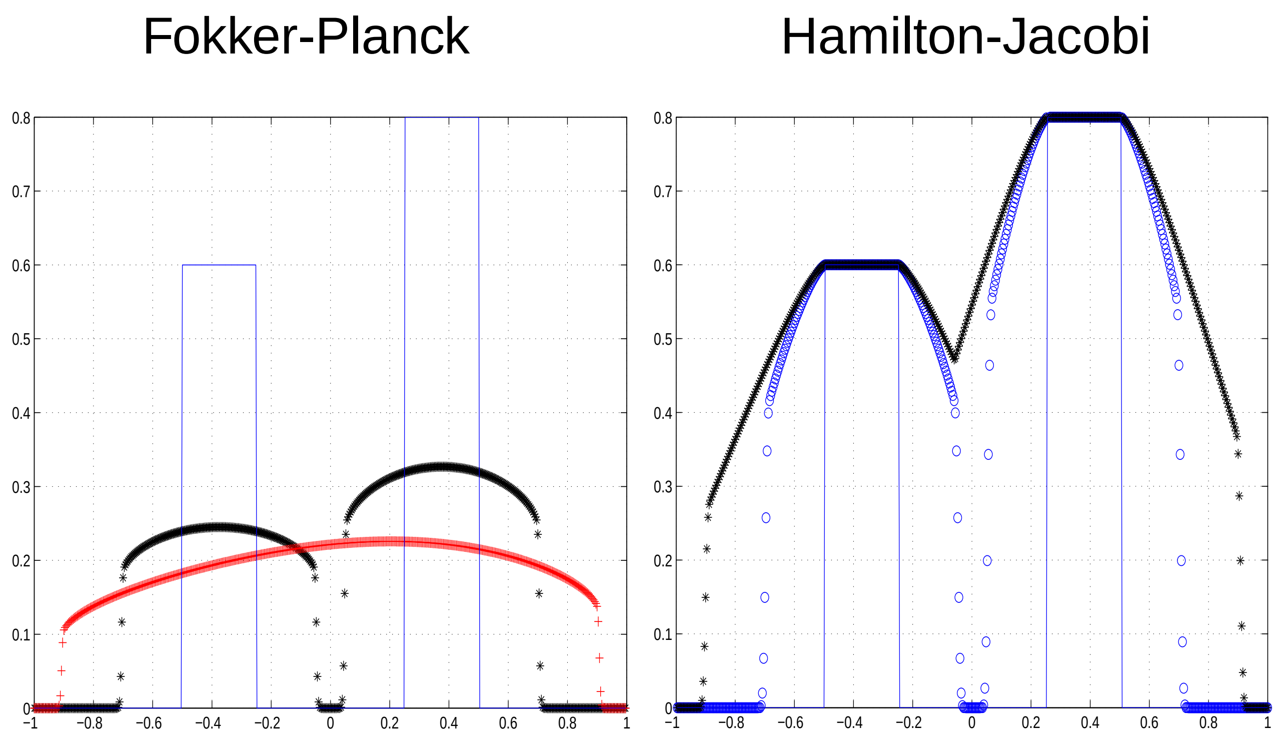
Two square waves initial data

$$u_0(x) = \begin{cases} 0.6 & -\frac{1}{2} \le x \le \frac{1}{4} \\ 0.8 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

- $x \in [-1, 1]$
- $t = 0.2$ $\qquad$ $t = 0.4$
- 500 points

Two square waves initial data

Relativistic Heat equation



Relativistic Porous-media equation

With $f(u) = c\frac{u^2}{2}$, the Fokker-Planck equation becomes the

Relativistic Porous-media equation

$$u_t = \operatorname{div}\left(\frac{\nu u^m \nabla u}{m\sqrt{u^2 + r^2|\nabla u|^2}}\right) \qquad \nu > 0,\ m > 1$$

We consider the case $m = 2$

then, the convective hyperbolic term, what we call
the **“relativistic porous-media Hamilton-Jacobi equation”**

$$u_t = cu\frac{r|\nabla u|^2}{\sqrt{u^2 + r^2|\nabla u|^2}}\left[2 - \frac{u^2}{u^2 + r^2|\nabla u|^2}\right]$$

Porous-media like Hamilton-Jacobi equation

In one-dimension can be expressed as

$$u_t = cu \frac{r|u_x|^2}{\sqrt{u^2 + r^2|u_x|^2}} \left[2 - \frac{u^2}{u^2 + r^2|u_x|^2} \right]$$

Double step initial data

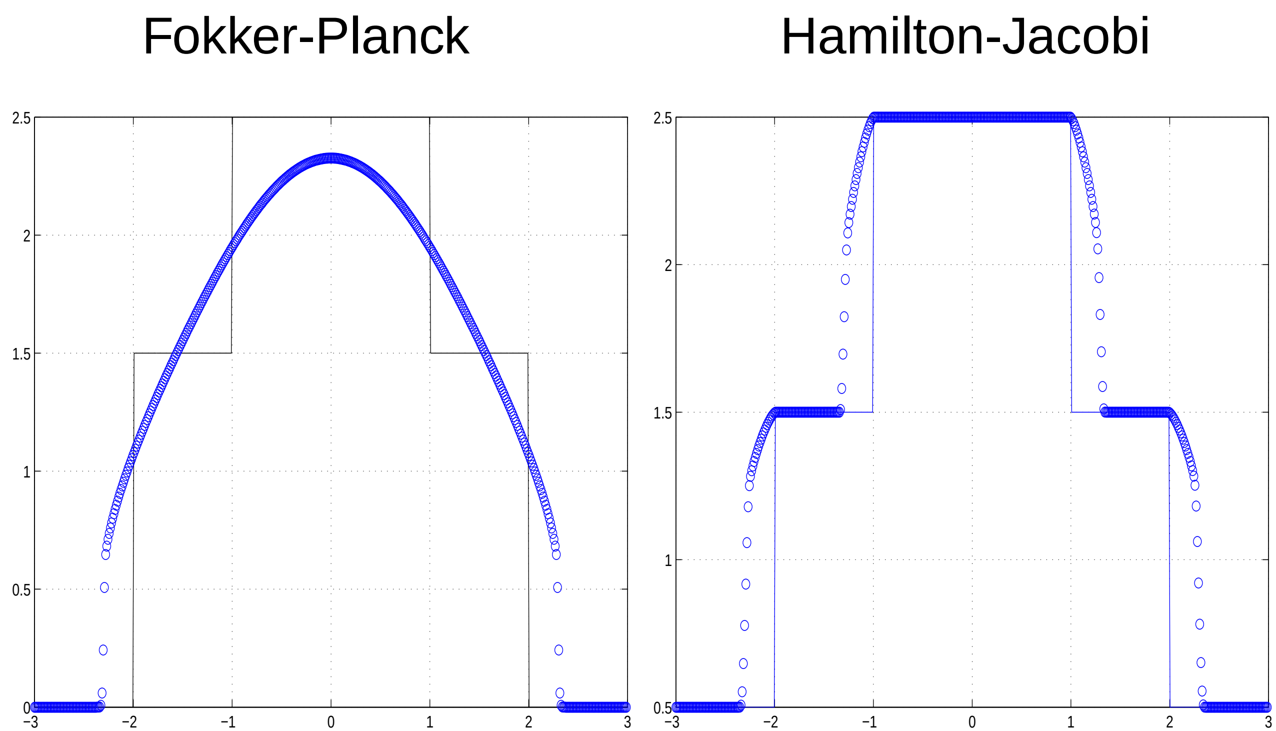
Double step initial data

$$u_0(x) = \begin{cases} 2.5 & |x| \leq 1 \\ 1.5 & -2 \leq x < -1 \\ 1.5 & 1 \leq x \leq 2 \\ 0.5 & \text{elsewhere} \end{cases}$$

- $x \in [-3, 3]$
- $t = 0.3$
- 500 points

Double step initial data

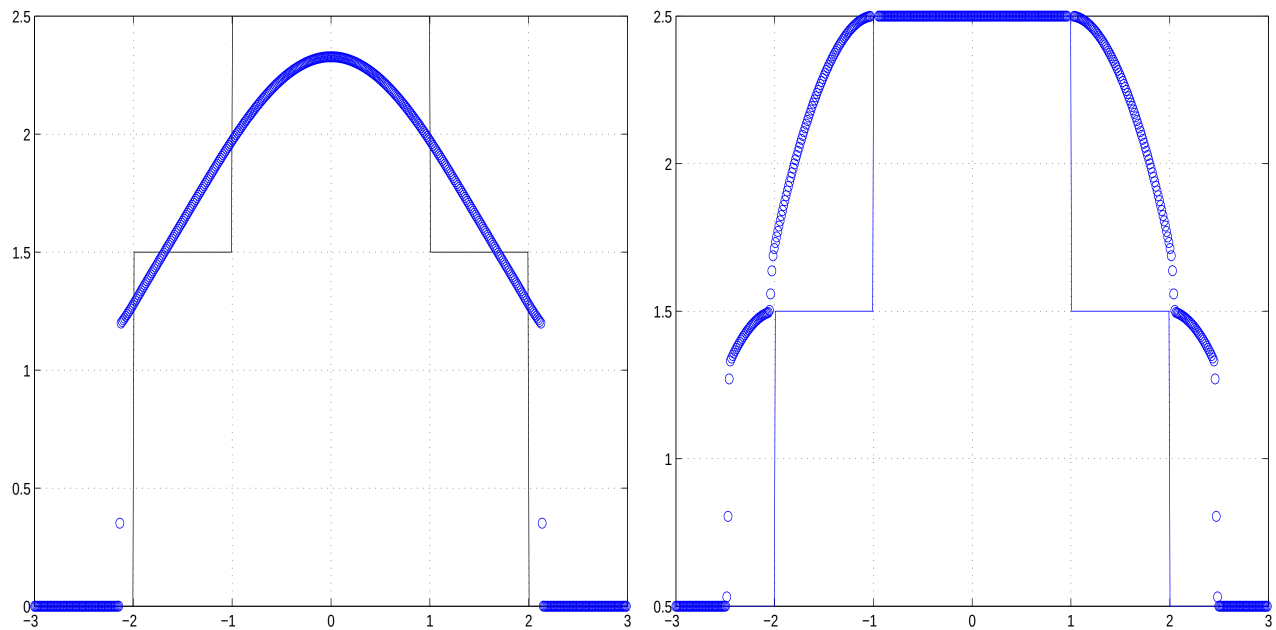
Relativistic Heat equation



Double step initial data

Relativistic Porous media equation

Fokker-Planck Hamilton-Jacobi



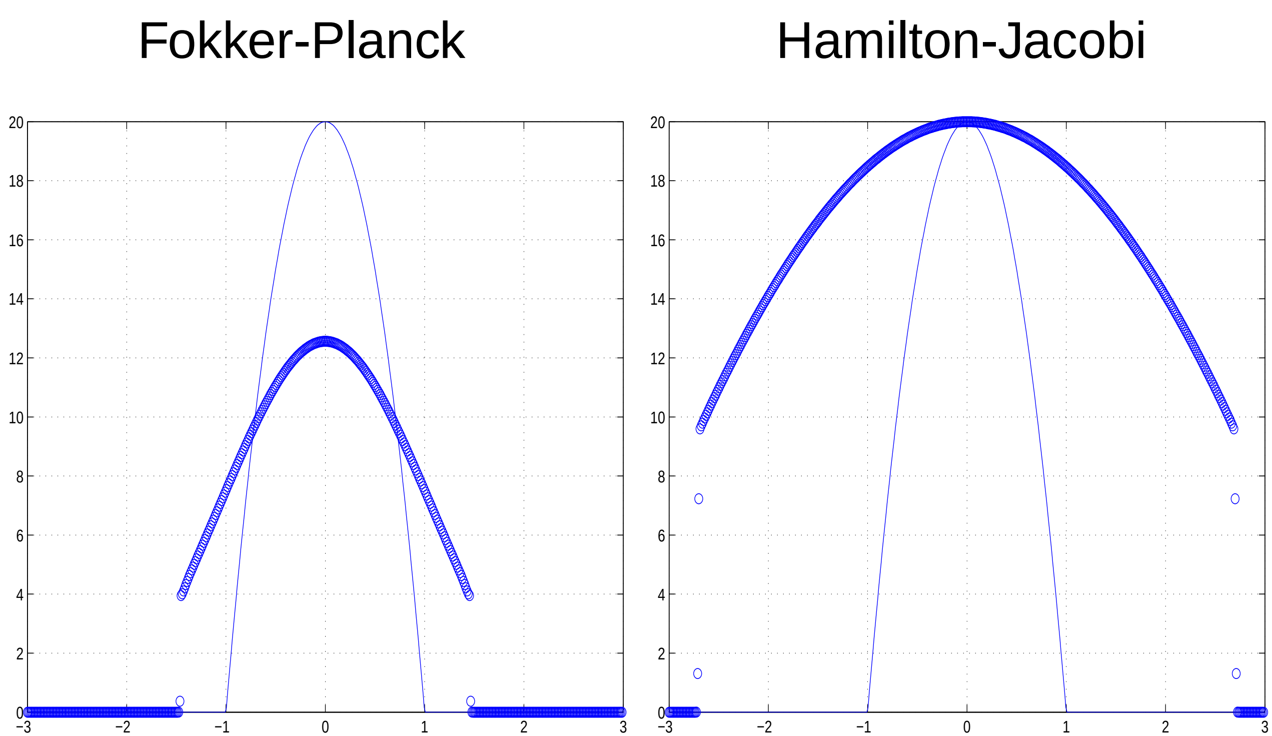
Continuous initial data

$$u_0(x) = \begin{cases} 0 & |x| \geq 1 \\ 20\max(1 - x^2, 0) & |x| < 1 \end{cases}$$

- $x \in [-3, 3]$
- $t = 0.3$
- 500 points

Continuous initial data

Relativistic Porous media equation



Speed limited Fokker-Planck equation

In one-dimension the Fokker-Planck is expressed as

$$u_t = \frac{r\, u_x^2}{\sqrt{u^2 + r^2 u_x^2}} \left(f'(u) - \frac{u f(u)}{u^2 + r^2 u_x^2} \right) + f(u) \frac{r\, u^2}{(u^2 + r^2 u_x^2)^{3/2}}\, u_{xx}$$

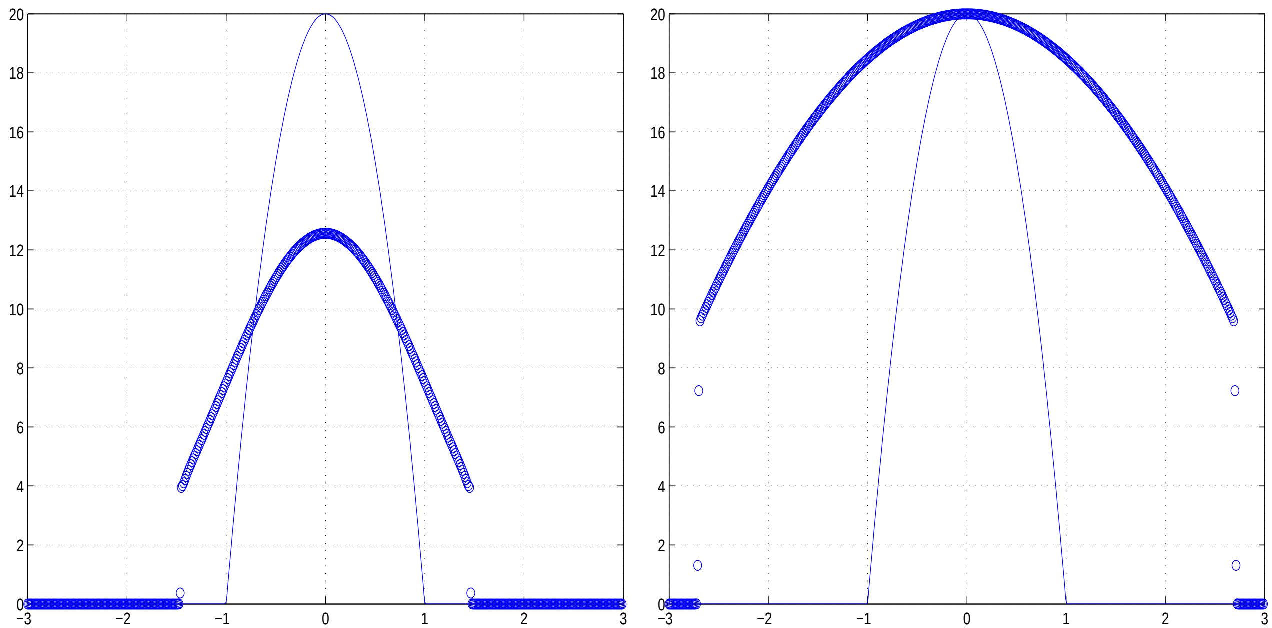
The 1D convective part written as a HJ equation is:

$$u_t \quad = \quad \frac{r\, u_x^2}{\sqrt{u^2 + r^2 u_x^2}} \left(f'(u) - \frac{u f(u)}{u^2 + r^2 u_x^2} \right)$$

Continuous initial data

Relativistic Porous media equation

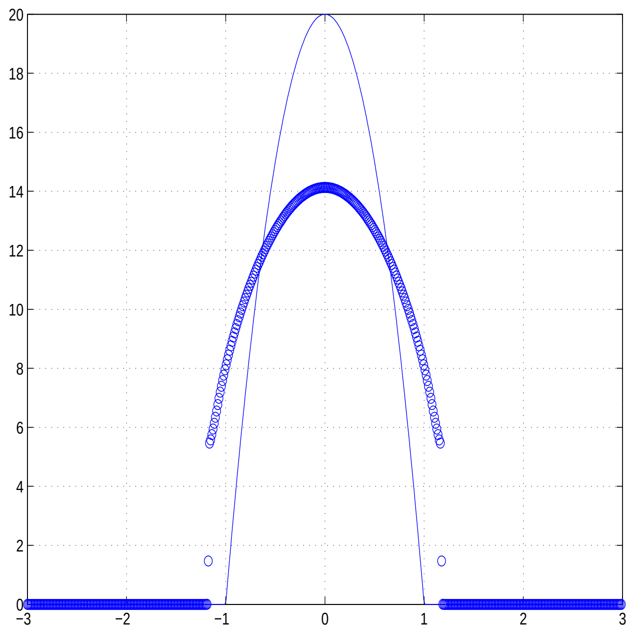
Fokker-Planck Hamilton-Jacobi



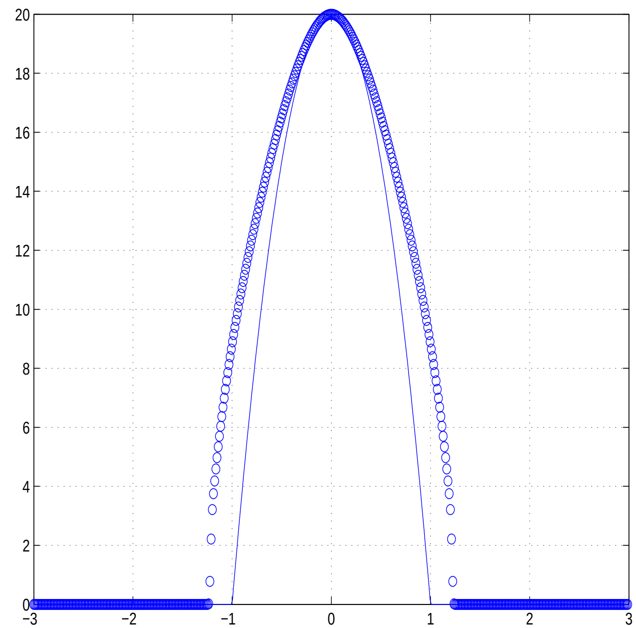
Continuous initial data

Relativistic Speed limited equation

Fokker-Planck



Hamilton-Jacobi



Thank you!