Some new cases of the Hodge Conjecture via graded matrix factorizations

Matthew Robert Ballard

Department of Mathematics
University of Wisconsin
ballard@math.wisc.edu

December 5, 2011

Attributions

Based on joint work with David Favero (Wien) and Ludmil Katzarkov (Miami and Wien), arXiv:1105.3177.

Statement of the main result

Let X be the complete intersection,

$$x^2u + y^2v + z^2w = xu^2 + yv^2 + zw^2 = 0,$$

in $\mathbb{P}^2_{\mathbb{C}}[x, y, z] \times \mathbb{P}^2_{\mathbb{C}}[u, v, w]$.

Theorem (B.-Favero-Katzarkov)

The Hodge Conjecture is true for $X^{\times n}, n \geq 0$: every rational (p, p)-cohomology class in $X^{\times n}$ is an algebraic class.

Relationship with the cubic fourfold

Hodge diamond of cubic fourfold and a $K3$ surface

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Relationship with the cubic fourfold

Hodge diamond of cubic fourfold and a $K3$ surface

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & {\color{red}1} & & 0 & & \\
 & 0 & & {\color{red}0} & & {\color{red}0} & & 0 & \\
0 & & {\color{red}1} & & {\color{red}20} & & {\color{red}1} & & 0 \\
 & 0 & & {\color{red}0} & & {\color{red}0} & & 0 & \\
 & & 0 & & {\color{red}1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Our example

X is a $K3$ surface and the cubic fourfold associated to it is

$$x^2u + y^2v + z^2w - xu^2 - yv^2 - zw^2 = 0$$

in $\mathbb{P}^5_{\mathbb{C}}[x, y, z, u, v, w]$. One can change variables to yield the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0$$

in $\mathbb{P}^5_{\mathbb{C}}$. This is the Fermat cubic fourfold. Let us denote it as Y.

Derived categories

Recall that, for a variety, Z, we have the category of bounded chain complexes of coherent sheaves on Z, $\mathrm{Chain}(\mathrm{coh}\, Z)$. A complex A, is **acyclic** if all its cohomology sheaves are zero. Let $\mathrm{Acyclic}(\mathrm{coh}\, Z)$ denote the full subcategory of acyclic complexes in $\mathrm{Chain}(\mathrm{coh}\, Z)$. For many natural reasons, we wish to create a new category by quotienting $\mathrm{Chain}(\mathrm{coh}\, Z)$ by $\mathrm{Acyclic}(\mathrm{coh}\, Z)$. The quotient is the derived category of coherent sheaves,

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z) = \mathrm{Chain}(\mathrm{coh}\, Z)/\,\mathrm{Acyclic}(\mathrm{coh}\, Z).$$

Kuznetsov's semi-orthogonal decomposition

Our observation involving the Hodge diamonds is the shadow of a theorem relating the derived categories of the K3 surface, X, and the cubic fourfold, Y.

Theorem (A. Kuznetsov)

There exists a semi-orthogonal decomposition,

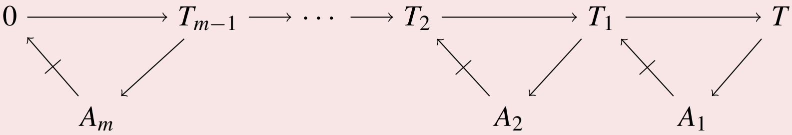
$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y) = \left\langle \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \right\rangle .$$

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Semi-orthogonal decompositions

Definition

A **semi-orthogonal decomposition** of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:



where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We shall denote a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$.

Orlov's semi-orthogonal decomposition

Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree, $d \leq n+1$, that defines a smooth hypersurface, X_f, in $\mathbb{P}^n_{\mathbb{C}}$.

Theorem (D. Orlov)

There is a semi-orthogonal decomposition

$$\mathrm{D^b}(\mathrm{coh}\, X_f) = \langle \mathrm{MF}(f, \mathbb{Z}), \mathcal{O}_{X_f}, \mathcal{O}_{X_f}(1), \dots, \mathcal{O}_{X_f}(n-d) \rangle.$$

In Orlov's theorem, $\mathrm{MF}(f, \mathbb{Z})$ is the category of graded matrix factorizations of f.

Matrix factorizations

Let A be a finitely-generated Abelian group and let R be an A-graded ring. Let $w \in R_d$ be a homogeneous element of degree, $d \in A$.

Definition

A **graded matrix factorization**, E, of the triple (R, w, A) is pair of A-graded R-module homomorphisms,

$$\phi_E : E_0 \to E_1 \quad , \quad \psi_E : E_1 \to E_0,$$

where E_0, E_1 are projective A-graded R-modules, the degree of ϕ is 0, the degree of ψ is d, and $\psi_E \circ \phi_E = w \,\mathrm{Id}_{E_0}, \phi_E \circ \psi_E = w \,\mathrm{Id}_{E_1}$.

Matrix factorizations

Given two graded matrix factorization, E and F, a **map**, $f : E \to F$, is a pair of A-graded R-module homomorphisms, $f_0 : E_0 \to F_0, f_1 : E_1 \to F_1$, of degree 0 so that the diagrams

$$\begin{CD} E_0 @>{\phi_E}>> E_1 \\ @V{f_0}VV @VV{f_1}V \\ F_0 @>{\phi_F}>> F_1 \end{CD} \qquad \begin{CD} E_1 @>{\psi_E}>> E_0 \\ @V{f_1}VV @VV{f_0}V \\ F_1 @>{\psi_F}>> F_0 \end{CD}$$

commute.

Matrix factorizations

A **homotopy**, h, between two maps, $f, g : E \to F$, is a pair of A-graded R-module homomorphisms, $h_0 : E_0 \to F_1, h_1 : E_1 \to F_0$ of degrees $-d$ and 0, respectively, satisfying

$$f_0 - g_0 = \psi_F \circ h_0 + h_1 \circ \phi_E \ , \ f_1 - g_1 = \phi_F \circ h_1 + h_0 \circ \psi_E.$$

Matrix factorizations

Definition

Given a triple (R, w, A), the **category of graded matrix factorizations** of w, $\mathrm{MF}(R, w, A)$, has as objects graded matrix factorizations and as morphisms homotopy classes of maps of graded matrix factorizations.

Our situation

In the case of the cubic fourfold, combining Kuznetsov's and Orlov's results shows that there is an equivalence

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \cong \mathrm{MF}(w, \mathbb{Z})$$

where X is our $K3$ surface in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ and

$$w = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3.$$

Matrix factorization descriptions of the self-products

Let $R_n = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]^{\otimes_{\mathbb{C}} n}$. Let A_n be the quotient of $\mathbb{Z}^n$ modulo the subgroup generated by $3e_i - 3e_j$, $i \neq j$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the i-th position.
Let $w^{\boxplus n} \in R_n$ be

$$\sum_{j=1}^{n} 1^{\otimes(j-1)} \otimes_{\mathbb{C}} w \otimes_{\mathbb{C}} 1^{\otimes(n-j)}.$$

Theorem (B.-Favero-Katzarkov)

There is an equivalence,

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X^{\times n}) \cong \mathrm{MF}(R_n, w^{\boxplus n}, A_n).$$

Main question

Can we formulate a version of the Hodge conjecture for graded matrix factorizations?

The Chern character

Let Z be a variety. The Chern character extends to an additive function,

$$\mathrm{ch} : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z) \to \mathrm{H}^*(Z, \mathbb{Q}).$$

The $\mathbb{Q}$-linear span of the image of ch is exactly the subspace of algebraic classes.

Hochschild homology

Let $\mathcal{O}_{\Delta Z}$ be the structure sheaf of the diagonal in $Z \times Z$.

Definition

The **Hochschild homology** of Z is the hypercohomology of

$$\mathcal{O}_{\Delta Z} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{Z \times Z}} \mathcal{O}_{\Delta Z}.$$

Set

$$\mathrm{HH}_i(Z) = \mathbb{H}^i \left(Z \times Z, \mathcal{O}_{\Delta Z} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{Z \times Z}} \mathcal{O}_{\Delta Z} \right).$$

Hochschild homology as Dolbeault cohomology

Proposition (R. Swan)

$$\mathrm{HH}_i(Z) \cong \bigoplus_{q-p=i} \mathrm{H}^{p,q}(Z).$$

Hodge package for nice triangulated categories

For "nice" triangulated categories, $\mathcal{T}$, such as all those in this talk, one can define Hochschild homology groups, $\mathrm{HH}_i(\mathcal{T})$ and a Chern character function,

$$\mathrm{ch} : \mathcal{T} \to \mathrm{HH}_0(\mathcal{T}),$$

which reduces to the previous definitions in the case that $\mathcal{T} = \mathrm{D^b}(\mathrm{coh}\, Z)$, under HKR isomorphism.

Hodge package for nice triangulated categories

For any "nice" exact functor, $F : \mathcal{T} \to \mathcal{S}$, there are functorial homomorphisms,

$$F_* : \mathrm{HH}_*(\mathcal{T}) \to \mathrm{HH}_*(\mathcal{S}),$$

such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F} & \mathcal{S} \\ \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} \\ \mathrm{HH}_0(\mathcal{T}) & \xrightarrow{F_0} & \mathrm{HH}_0(\mathcal{S}) \end{array}$$

commutes.

Hodge package for graded matrix factorizations

Special case: $w \in \mathbb{C}[y_0, \ldots, y_m] = R$ with an A-grading, m even, and w an isolated singularity. There is an action of $A^\vee$ on $\mathbb{A}^{m+1}$. We also assume that the identity is the only element of $A^\vee$ with fixed locus larger than the origin.

Proposition (T. Dyckerhoff, A. Căldărăru-J. Tu, A. Polishchuk-A. Vaintrob, B.-Favero-Katzarkov)

$$\mathrm{HH}_0(R, w, A) \cong (R/(\partial w))_{dm/2 - \sum_{i=0}^m \deg y_i} \oplus \bigoplus_{\substack{a \in A/(d) \\ a \neq 0}} \mathbb{C}.$$

Dyckerhoff, following A. Kapustin-Y. Li, also describes the Chern character map in the ungraded case. His description can be extended to the graded case.

Proving the main result

Theorem (B.-Favero-Katzarkov)

The Hodge Conjecture is true for $X^{\times n}, n \geq 0$: every rational (p,p)-cohomology class in $X^{\times n}$ is an algebraic class.

We prove the following statement: the image of the Chern character map,

$$\mathrm{ch} : \mathrm{MF}(R_n, w^{\boxplus n}, A_n) \to \mathrm{HH}_0(R_n, w^{\boxplus n}, A_n),$$

spans $\mathrm{HH}_0(R_n, w^{\boxplus n}, A_n)$.

Let us limit ourselves to the case $n = 2$ as the general case is similar but more complicated.

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Two missing cycles

We have

$$\mathrm{HH}_0(R_2, w^{\boxplus 2}, A_2) \cong$$
$$\left(\mathbb{C}[x_0, \ldots, x_5, y_0, \ldots, y_5]/(x_0^2, \ldots, x_5^2, y_0^2, \ldots, y_5^2)\right)_{12e_1 - 6e_2} \oplus \mathbb{C}^5.$$

The first component is spanned by the terms, $v \otimes_{\mathbb{C}} w$, with

$$v, w \in \mathbb{C}[x_0, \ldots, x_5]/(x_0^2, \ldots, x_5^2)$$

$\deg v, \deg w \in \{0, 3, 6\}$ and $\deg v + \deg w = 6$. To verify the Hodge Conjecture, we need to find matrix factorizations whose Chern characters are $1 \otimes_{\mathbb{C}} x_0 \cdots x_5$ and $x_0 \cdots x_5 \otimes_{\mathbb{C}} 1$.

Changing the grading

We have a homomorphism, $\mu : A_2 \to \mathbb{Z}$, which sends e_1, e_2 to 1. This induces a pair of adjoint functors:

$$\mathrm{Res} : \mathrm{MF}(R_2, w^{\boxplus 2}, A_2) \to \mathrm{MF}(R_2, w^{\boxplus 2}, \mathbb{Z})$$
$$\mathrm{Ind} : \mathrm{MF}(R_2, w^{\boxplus 2}, \mathbb{Z}) \to \mathrm{MF}(R_2, w^{\boxplus 2}, A_2).$$

One checks that
$(\mathrm{Ind} \circ \mathrm{Res})_0 : \mathrm{HH}_0(R_2, w^{\boxplus 2}, A_2) \to \mathrm{HH}_0(R_2, w^{\boxplus 2}, A_2)$ is multiplication by 3 on the component
$\left(\mathbb{C}[x_0, \ldots, x_5, y_0, \ldots, y_5]/(x_0^2, \ldots, x_5^2, y_0^2, \ldots, y_5^2)\right)_{12e_1 - 6e_2}$.

Changing the grading

Note that $w^{\boxplus 2}$ with $\mathbb{Z}$-grading defines the Fermat cubic 10-fold in $\mathbb{P}^{11}$. By Orlov's semi-orthogonal decomposition and a result of Ran, any element of

$$\mathrm{HH}_0(R_2, w^{\boxplus 2}, \mathbb{Z}) = \left(\mathbb{C}[x_0, \ldots, x_5, y_0, \ldots, y_5]/(x_0^2, \ldots, x_5^2, y_0^2, \ldots, y_5^2)\right)_6$$

lifts via ch to an object of $\mathrm{MF}(R_2, w^{\boxplus 2}, \mathbb{Z})$.
Thus, there exists a $\mathbb{Z}$-graded factorization, E, with
$\mathrm{ch}\, E = x_0 \cdots x_5 \otimes_{\mathbb{C}} 1$.

Changing the grading

By naturality of ch, the diagram

$$\begin{array}{ccccc}
\mathrm{MF}(R_2, w^{\boxplus 2}, A_2) & \xrightarrow{\mathrm{Res}} & \mathrm{MF}(R_2, w^{\boxplus 2}, \mathbb{Z}) & \xrightarrow{\mathrm{Ind}} & \mathrm{MF}(R_2, w^{\boxplus 2}, A_2) \\
\downarrow \mathrm{ch} & & \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\
\mathrm{HH}(R_2, w^{\boxplus 2}, A_2) & \xrightarrow{\mathrm{Res}_0} & \mathrm{HH}(R_2, w^{\boxplus 2}, \mathbb{Z}) & \xrightarrow{\mathrm{Ind}_0} & \mathrm{HH}(R_2, w^{\boxplus 2}, A_2)
\end{array}$$

commutes and we have

$$\mathrm{ch}(\mathrm{Ind}(E)) = 3x_0 \cdots x_5 \otimes_{\mathbb{C}} 1.$$

Current/future directions

1. These are arguments allow one to prove the following general statement: let A and B finitely-generated Abelian groups with A finite over B. The Hodge conjecture holds for $\mathrm{MF}(R, f, A)$ if and only if it holds $\mathrm{MF}(R, f, B)$.
2. Extend Orlov's semi-orthogonal decomposition. Such a statement is due to M. Herbst and J. Walcher for Calabi-Yau complete intersections in toric varieties. Extended to general complete intersections (B.-Favero-Katzarkov).
3. Define integral classes in $\mathrm{MF}(R, f, A)$. Return to Stokes' theorem for manifolds if LG model is over $\mathbb{C}$.
4. Realize Kuznetsov's semi-orthgonal decomposition as a statement about matrix factorizations and extend it.

Fin.