Log mixed Hodge theory ;
feeling relation with mirror symmetry

(joint work of Kato, Nakayama, and Usui)

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Fundamental diagram

$$
\begin{tikzcd}
 & & D_{SL(2),\mathrm{val}} \arrow[r] \arrow[d] & D_{BS,\mathrm{val}} \arrow[d] \\
D_{\Sigma,\mathrm{val}} \arrow[d] & D^{\sharp}_{\Sigma,\mathrm{val}} \arrow[l] \arrow[r] \arrow[d] & D_{SL(2)} & D_{BS} \\
D_{\Sigma} & D^{\sharp}_{\Sigma} \arrow[l] & &
\end{tikzcd}
$$

D : classifying space of MHS of type $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$

Period map

fine moduli of log MHS of type $\Phi = (\Lambda, \Sigma, \Gamma)$.

Theorem Given Φ with Γ neat.

Then, functor LMH_Φ is represented by $\Gamma \backslash D_\Sigma$ in $\mathcal{B}(\log)$,

i.e., $LMH_\Phi(\cdot) \xrightarrow{\sim} Map(\cdot, \Gamma \backslash D_\Sigma)$.

Category $\mathcal{B}(\log)$

Z: fs log anal. sp., $S \subset Z$.

$U \subset S$ open in <u>strong topology</u> of S in Z

$\Leftrightarrow$ $\forall f : {}^{\forall}A \to Z$ mor. of anal. sp.s s.t. $f(A) \subset S$, $f^{-1}(U)$ open in A.

$\mathcal{B}(\log)$: cat. of loc. ringed sp. S /$\mathbb{C}$ with fs log structure s.t. locally

$S \simeq$ strong subsp. of fs log anal. sp.

fs log anal. sp. is <u>log smooth</u>

$\Leftrightarrow$ locally, isom. to open set of toric variety.

smooth object of $\mathcal{B}(\log)$:

<u>log mfd</u> $\doteqdot$ log smooth fs log anal. sp. with slits, strong topology.

<u>Example</u>. $S = ((\mathbb{C} \times \mathbb{C}) \smallsetminus (\{0\} \times \mathbb{C})) \cup \{(0,0)\} \subset Z = \mathbb{C}^2$

Ringed space $(X^{\log}, \mathcal{O}_X^{\log})$

$X \in \mathcal{B}(\log)$. log str. $\alpha : M_X \to \mathcal{O}_X$ s.t. $\alpha^{-1}\mathcal{O}_X^\times \xrightarrow{\sim} \mathcal{O}_X^\times$, fs. (finitely gen. integral saturated)

$X^{\log} := \{(x, h) \mid x \in X,\ h : M_{X,x}^{gp} \to \mathbb{S}^1 \text{ s.t. } h(u) = \frac{u}{|u|}\ (u \in \mathcal{O}_{X,x}^\times)\}$.

weakest top. s.t. (1), (2) are continuous

(1) $\tau : X^{\log} \to X, \quad (x, h) \mapsto x$.

(2) $\forall U \underset{\text{open}}{\subset} X, \ \forall f \in \Gamma(U, M_X^{gp}), \ \tau^{-1}(U) \to \mathbb{S}^1, \ (x, h) \mapsto h(f)$.

Then, τ is proper, surjective.

$\tau^{-1}(x) = (\mathbb{S}^1)^r, \quad r := \operatorname{rank}(M_X^{gp}/\mathcal{O}_X^\times)_x$ (varies).

Define <u>sheaf of logarithms</u> $\mathcal{L} = \mathcal{L}_X$ of M_X^{gp} on X^{log} by

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tau^{-1}(M_X^{gp}) \arrow[d] & \ni (f \text{ at } (x,h)) \arrow[d, mapsto] \\ \mathrm{Cont}(\ ,i\mathbb{R}) \arrow[r, "\exp"] & \mathrm{Cont}(\ ,\mathbb{S}^1) & \ni h(f). \end{tikzcd}$$

$$\begin{tikzcd} f \arrow[d, mapsto] & \in \tau^{-1}(\mathcal{O}_X) \arrow[r, "\exp"] \arrow[d] & \tau^{-1}(\mathcal{O}_X^\times) \subset \tau^{-1}(M_X^{gp}) \\ \frac{1}{2}(f-\bar{f}) & \in \mathrm{Cont}(\ ,i\mathbb{R}) & \end{tikzcd}$$

induces $\iota : \tau^{-1}(\mathcal{O}_X) \to \mathcal{L}$.

$$\mathcal{O}_X^{log} := \frac{\tau^{-1}(\mathcal{O}_X) \otimes \mathrm{Sym}_{\mathbb{Z}}(\mathcal{L})}{(f \otimes 1 - 1 \otimes \iota(f) \mid f \in \tau^{-1}(\mathcal{O}_X))}.$$

Then, $\tau : (X^{log}, \mathcal{O}_X^{log}) \to (X, \mathcal{O}_X)$ mor. of ringed spaces $/\mathbb{C}$.

LMH (log mixed Hodge)

$S \in \mathcal{B}(\log)$

<u>pre-LMH</u> on S is $(\underbrace{H_{\mathbb{Z}}, W}_{\text{on } S^{\log}}, \underbrace{H_{\mathcal{O}}}_{\text{on } S})$ s.t.

isom. $\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (log RH corresp.),

$$F^p = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^p H_{\mathcal{O}},$$

$$\tau_* F^p = F^p H_{\mathcal{O}}.$$

pre-LMH on S is <u>LMH on S</u> if its pullback to each $s \in S$ is LMH on s in the following sense.

s : fs log point.

pre-LMH $(H_{\mathbb{Z}}, W, H_{\mathcal{O}})$ on s is LMH on s if

(1) Local monodromy of $(H_{\mathbb{R}}, W)$ is admissible. (next sheet)

(2) $\nabla F^{p} \subset \omega_s^{1,\log} \otimes F^{p-1}$, $\nabla := d \otimes 1_{H_{\mathbb{Z}}} : \mathcal{O}_s^{\log} \otimes H_{\mathbb{Z}} \to \omega_s^{1,\log} \otimes H_{\mathbb{Z}}$.

(3) $t \in s^{\log}$, $a : \mathcal{O}_{s,t}^{\log} \to \mathbb{C}$ specialization ($\mathbb{C}$-alg. hom),

$F(a) := \mathbb{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ on $H_{\mathbb{C},t}$.

$(H_{\mathbb{Z},t}, M(\tau, W), F(a))$ is MHS for $^{\forall}\tau \prec C(s) := \mathrm{Hom}(M_s/\mathcal{O}_s^{\times}, \mathbb{R}_{\geq 0}^{\mathrm{add}})$, $^{\forall}$sufficiently twisted a ⌋

Fix $(q_j)_{1 \leq j \leq n} \subset M_s \setminus \mathcal{O}_s^{\times}$ s.t. q_j mod $\mathcal{O}_s^{\times}$ generate $M_s/\mathcal{O}_s^{\times}$.

a is suff. twisted $\Leftrightarrow$ $\exp(a(\log(q_j)))$ suff. near 0.

nilpotent cone σ is admissible

$\Leftrightarrow$ $\forall \tau \prec \sigma$, $\exists M(\tau, W)$ satisfying (1) - (3):

(1) $\tau \prec \sigma$, $N \in \sigma \Rightarrow NM(\tau, W)_w \subset M(\tau, W)_w \ (\forall w)$.

(2) $\tau \prec \sigma$, $N \in \tau \Rightarrow NM(\tau, W)_w \subset M(\tau, W)_{w-2} \ (\forall w)$.

(3) $\tau \prec \sigma$, $N \in \sigma$, $\tau' \prec \sigma$: smallest containg τ and N

$\Rightarrow M(\tau', W) = M(N, M(\tau, W))$.

Examples

1) H' : MHS

$$\mathrm{Ext}^1(\mathbb{Z}, H') := \left(\begin{array}{l}\text{isom. classes of MHS } H \text{ s.t. } 0 \to H' \to H \to \mathbb{Z} \to 0 \\ \text{is exact in the category of MHS}\end{array}\right)$$

i.e., exactness of lattices, wt filts, Hodge filts

Case wt ≤ -1,

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathbb{Z}, H') & \overset{\text{bijection}}{\simeq} & H'_{\mathbb{Z}} \backslash H'_{\mathbb{C}} / F^0 H'_{\mathbb{C}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ H & \longmapsto & e_F - e_{\mathbb{Z}} \end{array}$$

e.g. $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}(1)) \simeq \mathbb{Z}(1) \backslash \mathbb{Z}(1)_{\mathbb{C}} / F^0 \mathbb{Z}(1)_{\mathbb{C}} = \mathbb{Z}(1) \backslash \mathbb{C} \underset{\sim}{\xrightarrow{\exp}} \mathbb{C}^\times$

e.g. X : smooth proj. var. $/\mathbb{C}$, $\quad H' := H^{2r-1}(X)(r)$.

$\mathrm{Ext}^1(\mathbb{Z}, H') = H^{2r-1}(X, \mathbb{Z}) \backslash H^{2r-1}(X, \mathbb{C}) / F^r H^{2r-1}(X, \mathbb{C})$.

r-th intermediate Jacobian of X.

2) $S \in \mathcal{B}(\log)$, H' : LMH on S'.

$\mathcal{E}xt^1(\mathbb{Z}, H')$: sheafification of functor (already sheaf if wt ≤ -1)

$$\mathcal{B}(\log)_{/S} \ni S' \mapsto \mathrm{Ext}^1_{\mathrm{LMH}/S'}(\mathbb{Z}, H'|_{S'})$$

$$:= \left(\begin{array}{l} \text{isom. classes of LMH } H \text{ on } S' \text{ s.t.} \\ 0 \to H' \to H \to \mathbb{Z} \to 0 \text{ is } \underline{\text{exact in the}} \\ \underline{\text{category of LMH}} \end{array} \right)$$

i.e., exactness of lattices, wt filts, Hodge filts.

This sheaf is not necessarily represented.

Using fans or weak fans, we construct object of $\mathcal{B}(\log)/S$ (models) which represent subsheaves of $\mathcal{E}xt^1(\mathbb{Z}, H')$.

Case wt ≤ -1, in particular for intermediate Jacobians, we contructed Néron models and connected Néron models.

3) Case $S = \operatorname{Spec} \mathbb{C}$ with trivial log str.

$\mathcal{E}xt^1(\mathbb{Z}, \mathbb{Z}(1)) \simeq \mathbb{G}_{m,\log}$ as sheaves on $\mathcal{B}(\log)$, where

$$\mathbb{G}_{m,\log}(S') := \Gamma(S', M^{gp}_{S'}).$$

This sheaf is not representable.

Subsheaf $\mathbb{G}_m$ of $\mathbb{G}_{m,\log}$, $\mathbb{G}_m(S') := \Gamma(S', \mathcal{O}^\times_{S'})$, is represented by Néron model = conn. Néron model = $\mathbb{C}^\times$.

Subsheaf $S' \mapsto \Gamma(S', M_{S'} \cup M^{-1}_{S'})$ of $\mathbb{G}_{m,\log}$ is represented by $\mathbb{P}^1(\mathbb{C})$.

Toric variety

σ : a rational nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$) (P.14)

$\Gamma < G_{\mathbb{Z}}$, $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$

$P(\sigma) := \mathrm{Hom}(\Gamma(\sigma), \mathbb{N})$ dual monoid

$\mathrm{toric}_\sigma := \mathrm{Hom}(P(\sigma), \mathbb{C}^{\mathrm{mult}}) \supset \mathrm{torus}_\sigma := \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbb{C}^\times)$

By $0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\mathbf{e}} \mathbb{C}^\times \to 1$,

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbb{Z}) & \to & \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbb{C}) & \xrightarrow{\mathbf{e}} & \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbb{C}^\times) & \to 1 \\ & \| & & \cong & & \| & \\ & \Gamma(\sigma)^{\mathrm{gp}} & & \sigma_{\mathbb{C}} & & \mathrm{torus}_\sigma & \\ & \text{fundamental group} & & \text{universal cover} & & & \end{array}$$

$$\mathbf{e}(z \otimes \log \gamma) := e^{2\pi i z} \otimes \gamma \quad (z \in \mathbb{C},\ \gamma \in \Gamma(\sigma)^{\mathrm{gp}})$$

For $\rho \prec \sigma$, $P(\rho) \twoheadleftarrow P(\sigma)$, $0_\rho \in \mathrm{toric}_\rho \hookrightarrow \mathrm{toric}_\sigma$ is the point

$$P(\rho) \ni f \mapsto \begin{cases} 1 & (f = 1) \\ 0 & (f \neq 1) \end{cases}$$

$:= \mathrm{toric}_\sigma = \{ \mathbf{e}(z)\cdot 0_\rho \mid \rho \prec \sigma,\ z \in \sigma_{\mathbb{C}}/(\rho_{\mathbb{C}} + \log \Gamma(\sigma)^{gp}) \}$

y $\mathbb{C} = \mathbb{R}_{\geq 0} \cdot \mathbb{S}^1$, (polar coordinate)

$^{\log} := \mathrm{Hom}(P(\sigma), \mathbb{R}_{\geq 0}^{\mathrm{mult}}) \times \mathrm{Hom}(P(\sigma), \mathbb{S}^1)$

$= \{(\mathbf{e}(iy)\cdot 0_\rho, \mathbf{e}(x)) \mid \rho \prec \sigma,\ x \in \sigma_{\mathbb{R}}/\log \Gamma(\sigma)^{gp},\ y \in \sigma_{\mathbb{R}}/\rho_{\mathbb{R}}\}$

$\downarrow \tau$

$= \mathrm{Hom}(P(\sigma), \mathbb{C}^{\mathrm{mult}})$

$\tau(\mathbf{e}(ib)\cdot 0_\rho, \mathbf{e}(a)) = \mathbf{e}(a+ib)\cdot 0_\rho.$

Since $0 \to \rho_{\mathbb{R}}/\Gamma(\rho)^{gp} \to \sigma_{\mathbb{R}}/\log\Gamma(\sigma)^{gp} \to \sigma_{\mathbb{R}}/(\rho_{\mathbb{R}} + \log\Gamma(\sigma)^{gp}) \to 0$,

$\tau^{-1}(\mathbf{e}(a+ib)\cdot 0_\rho) = \{ (\mathbf{e}(ib)\cdot 0_\rho, \mathbf{e}(a+x)) \mid x \in \rho_{\mathbb{R}}/\log\Gamma(\rho)^{gp} \}$

$\simeq (\mathbb{S}^1)^r, \quad r = r(\rho) := \mathrm{rank}\,\rho$ varies !

$\ldots_{\mathrm{val}} := \{ (A, V, z) \mid A \subset \sigma_{\mathbb{R}}$: $\mathbb{Q}$-linear subspace, $(\sigma \cap A)_{\mathbb{R}} = A_{\mathbb{R}}$.

$V \subset \underbrace{A^*}_{\text{cone}}$ s.t. $V \cap (-V) = \{0\}$, $V \cap (-V) = A^*$.

$\mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q})$ $\quad (\sigma \cap A)^{\vee} \subset V$ in A^*.

$f \quad z \in \sigma_{\mathbb{C}} / (A_{\mathbb{C}} + \log \Gamma(\sigma)^{gp}) \}$

$\ni f(A, V, z) = \mathbf{e}(z) \cdot 0_{\rho}$, $\rho \prec \sigma$: the smallest, having the above property

projective limit of log modifications

<u>Example</u> Case $\sigma = \mathbb{N}^2$.

$|S| := \{ \mathbf{e}(iy) \cdot 0_{\rho} \mid \rho \prec \sigma,\ y \in \sigma_{\mathbb{R}} / \rho_{\mathbb{R}} \} \supset |\mathrm{torus}|_{\sigma} := \{ \mathbf{e}(iy) \cdot 0_{\{0\}} \mid y \in \sigma_{\mathbb{R}} \}$

$|S|_{\mathrm{val}} :=$ (closure of $f^{-1}(|\mathrm{torus}|_{\sigma})$ in S_{val})

Weak fans

D : classifying space of MHS with pol. gr quotients for fixed data
$\Lambda := (H_0, W, (\langle\ ,\ \rangle_k)_k, (h^{p,q})_{p,q})$

$\check{D}$: "compact dual" of D

$G_A = \mathrm{Aut}(H_{0,A}, W, (\langle\ ,\ \rangle_k)_k)$, $\mathfrak{g}_A := \mathrm{Lie}\, G_A \quad (A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C})$

Nilpotent cone is sharp cone in $\mathfrak{g}_{\mathbb{R}}$, finitely generated, nilpotent, commutative.

nilpotent cone σ is admissible

$\Leftrightarrow$ $\forall N \in \sigma$, $\exists M(N, W)$ (depends only on the face of σ spanned by N)

σ : admissibe nilpotent cone.

$Z \subset \check{D}$ is a <u>σ-nilpotent orbit</u> if, for $F \in Z$,

(1) $Z = \exp(\sigma_{\mathbb{C}}) F$ (2) $NF^p \subset F^{p-1}$ $(\forall N \in \sigma, \forall p)$

(3) $N_1, \dots, N_n$: set of generators of σ.

$$\exp\left(\sum_j i y_j N_j\right) F \in D \quad \text{for} \quad y_j \gg 0 \quad (1 \le j \le n)$$

A <u>weak fan</u> Σ in $\mathfrak{g}_{\mathbb{Q}}$ is a set of rational nilp. cones in $\mathfrak{g}_{\mathbb{R}}$ s.t.

(1) $\forall \sigma \in \Sigma$ admissible.

(2) $\sigma \in \Sigma$, $\tau \prec \sigma \Rightarrow \tau \in \Sigma$.

(3) $\sigma, \sigma' \in \Sigma$, $\exists$ common interior point of σ and σ'.
$\exists F \in \check{D}$ s.t. (σ, F) and (σ', F) generate nilp. orbits
$\Rightarrow \sigma = \sigma'$.

In the definition of weak fan, (3) $\Leftrightarrow$ (3)$'$ $\Leftrightarrow$ (3)$''$:

(3)$'$ $\sigma, \sigma' \in \Sigma$, $\exists F \in \check{D}$ s.t. (σ, F), (σ', F), $(\sigma \cap \sigma', F)$ generate nilp. orbits $\Rightarrow$ $\sigma \cap \sigma' \prec \sigma$.

(3)$''$ τ : nilpotent cone, $F \in \check{D}$, (τ, F) generates nilp. orbit,
$A := \{\sigma \in \Sigma \mid \sigma \supset \tau, (\sigma, F) \text{ generates nilp. orbit}\} \neq \phi$
$\Rightarrow \exists \rho \in A$: smallest, and ρ is a face of any element of A.

Log mixed Hodge structure (universal one ass. to σ)

$= \mathrm{toric}_\sigma$

$_\sigma = (H_{\sigma,\mathbb{Z}}, W, (\langle\ ,\ \rangle_w)_w)$: canonical locally constant sheaf on $S^{\log}$

given by the represent. of $\pi_1(S^{\log}) = \Gamma(\sigma)^{gp} \subset G_{\mathbb{Z}}$ on $(H_0, W, (\langle\ ,\ \rangle_w)_w)$.

Claim $\exists 1\ \nu : \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\sigma,\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_0$ s.t.

(stalk of $H_{\sigma,\mathbb{Z}}$ at $1 = 0_{\{0\}} \in \mathrm{torus}_\sigma \subset S) = H_0$.

$(\gamma_j)_{1 \le j \le n}$: any basis of $\Gamma(\sigma)^{gp} = \pi_1(S^{\log}) = \pi_1((\mathbf{S}^1)^n) \simeq \mathbb{Z}^n$.

$(q_j)_{1 \le j \le n}$: the dual basis of $P(\sigma)^{gp} = \mathrm{Hom}(\Gamma(\sigma)^{gp}, \mathbb{Z}) \simeq \mathbb{Z}^n$.

$N_j := \log(\gamma_j) \in \mathrm{End}(H_{0,\mathbb{Q}})$

Define $\nu = \exp\left(-\sum_j \frac{1}{2\pi i} \log(q_j)_{1,0} \otimes N_j\right)$, where

$\log(q_j)_{1,0}$: the branch of the germ of $\log(q_j)$ at $1 = 0_{\{0\}} \in \mathrm{torus}_\sigma$

with value 0. $\dashv$

The universal pre-LMH H on $\check{E}_\sigma := \text{toric}_\sigma \times \check{D}$ is given by
$\mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathbb{Z}} H_\sigma = \mathcal{O}^{\log}_{\check{E}_\sigma} \otimes_{\mathcal{O}_{\check{E}_\sigma}} H_{\mathcal{O}}$, where $H_{\mathcal{O}} = \mathcal{O}_{\check{E}_\sigma} \otimes H_0$ is the locally free $\mathcal{O}_{\check{E}_\sigma}$-module come from that on $\check{D}$ endowed with the universal Hodge filtration F.

Define $E_\sigma := \{ x \in \check{E}_\sigma \mid H(x) \text{ is a LMH} \}$, where means for $t \in \tau^{-1}(x)$, $\mu_t : H_{\sigma,\mathbb{Z},t} \xrightarrow{\sim} H_0$, the following (1), (2) hold:

(1) Local monodromy cone α of H_σ at x is admissible.

(2) $\alpha \subset {}^{\exists}\rho \prec \sigma$ s.t. $(\rho, \mu_t(\mathbb{C} \otimes_{\mathcal{O}^{\log}_{x,t}} F_t))$ generates a nilpotent orbit.

Note that <u>slits</u> appear in E_σ because of the cond. (2).
(transversality

Moduli of LMH of type Φ

$= (\Lambda, \Sigma, \Gamma)$, Σ : weak fan in $\mathfrak{g}_{\mathbb{Q}}$, $\Gamma < G_{\mathbb{Z}}$,

trongly compatible, i.e., (1) $\sigma \in \Sigma$, $\gamma \in \Gamma \Rightarrow \mathrm{Ad}(\gamma)\sigma \in \Sigma$.

) $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$. Then σ is generated over $\mathbb{R}_{\geq 0}$ by $\log \Gamma(\sigma)$.

$D_\Sigma := \{(\sigma, Z) \mid$ nilp. orbit, $\sigma \in \Sigma$, $Z \subset \check{D}\}$.

$\in \Sigma$, $\mathrm{toric}_\sigma := \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^\vee])_{an}$

$$
\begin{array}{l}
E_\sigma \overset{\text{gen. nilp. orbit}}{\subset} \check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D} \\
\quad \downarrow \leftarrow \sigma_{\mathbb{C}}\text{-torsor (hardest)} \\
\Gamma(\sigma)^{gp} \backslash D_\sigma \\
\quad \downarrow \leftarrow \text{loc. isom.} \\
\Gamma \backslash D_\Sigma \quad \in \mathcal{B}(\log)
\end{array}
$$

The action of $h \in \sigma_{\mathbb{C}}$ on $(\mathbf{e}(a) \cdot 0_\rho, F) \in E_\sigma$ is $(\mathbf{e}(h+a) \cdot 0_\rho, \exp(-h)F)$.

The projection is $(\mathbf{e}(a) \cdot 0_\rho, F) \mapsto (\rho, \exp(\rho_{\mathbb{C}} + a)F)$.

$S \in \mathcal{B}(\log)$.

<u>LMH with pol. g.q. of type Φ on S</u> is LMH with pol. g.q.

$H = (H_{\mathbb{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ endowed with Γ-level structure

$\mu \in H^0(S^{\log}, \Gamma \backslash \mathrm{Isom}((H_{\mathbb{Z}}, W, (\langle\ ,\ \rangle_w)_w), (H_0, W, (\langle\ ,\ \rangle_w)_w)))$ s.t.

$\forall s \in S$, $t \in \tau^{-1}(s) = s^{\log}$, $\mu_t : H_{\mathbb{Z},t} \xrightarrow{\sim} H_0$.

$\alpha :=$ (local monodromy cone of $H_{\mathbb{Z}}$ at s)

$:= \mathrm{Im}\,(\mathrm{Hom}((M_S/\mathcal{O}_S^\times)_s, \mathbb{N}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbb{Z},t}))$.

$\mu_t \alpha \mu_t^{-1} \subset {}^{\exists}\sigma \in \Sigma$ s.t.

$(\sigma, \mu_t(\mathbb{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t))$ generates nilp. orbit.

(Then, the smallest such σ exists.)

<u>Theorem</u> (i) $\Gamma \backslash D_\Sigma \in \mathcal{B}(\log)$, Hausdorff.

If Γ is neat, $\Gamma \backslash D_\Sigma$ is log manifold.

(ii) On $\mathcal{B}(\log)$, $\Gamma \backslash D_\Sigma$ rep. functor LMH_Φ of LMH of type Φ.

<u>Period map</u> $S \in \mathcal{B}(\log)$.

$$LMH_\Phi(S) \xrightarrow{\sim} \mathrm{Map}(S, \Gamma \backslash D_\Sigma)$$

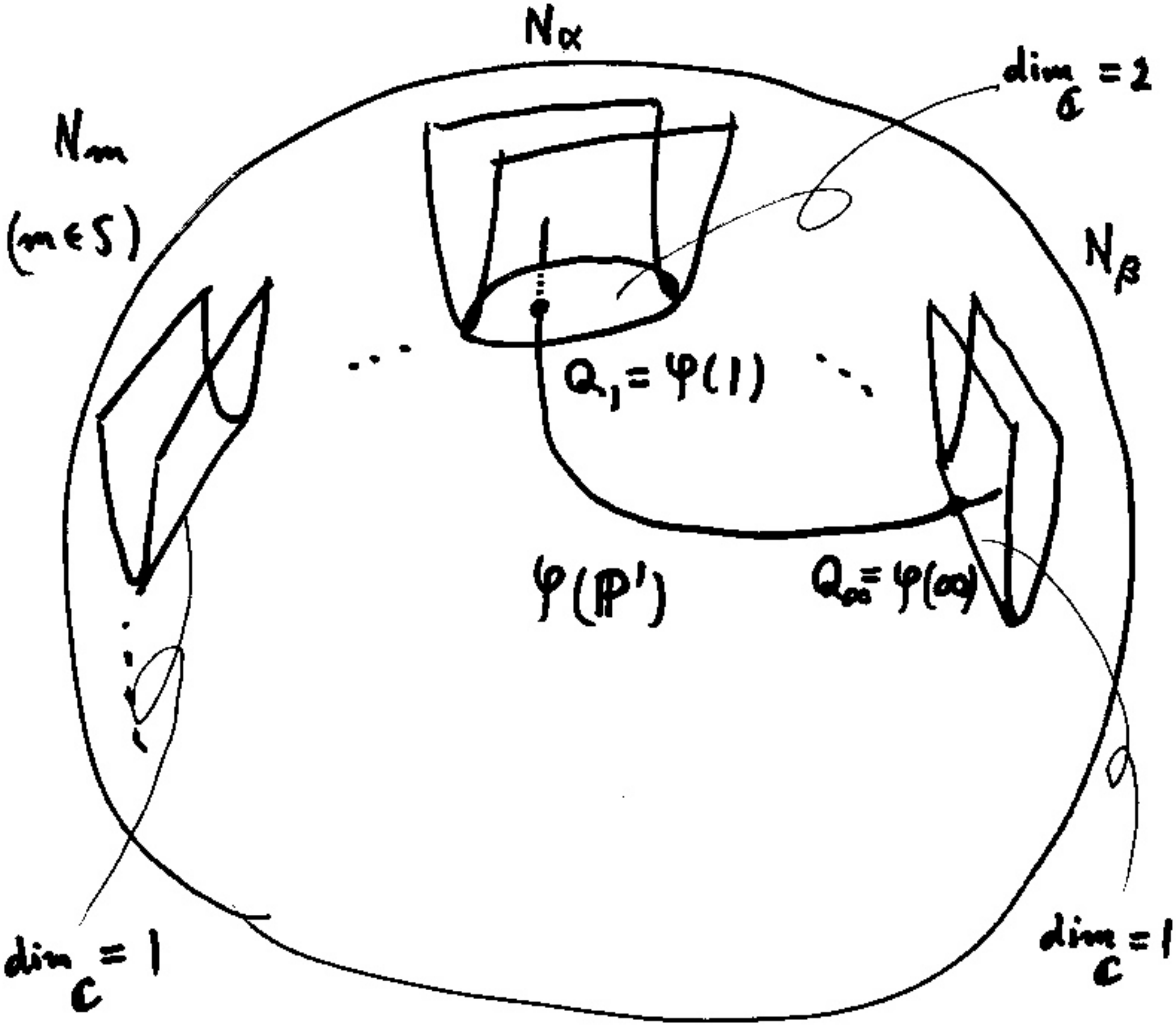
$$H \mapsto (S \ni s \mapsto (\sigma, \exp(\sigma_{\mathbb{C}})\mu_t(\mathbb{C} \otimes_{\mathcal{O}^{\log}_{S,t}} F_t)) \bmod \Gamma).$$

<u>Rem.</u> a nilp. orbit [CKS86] (pure), [KNU, II] (mixed)

$\leftrightarrow$ an IMHM (inf. mixed Hodge module) [Kashiwara 86]

$\leftrightarrow$ an LMH with pol. g. g. on an fs log point.

Example Quintic-mirror



$$\Gamma \backslash D_\Sigma, \quad \dim_{\mathbb{C}} = 4$$

$h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$, $h^{p,q} = 0$ otherwise.

Types of monodromy logarithms

0 :

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ (3,0) & (2,1) & (1,2) & (0,3) \end{array} \qquad \begin{array}{c} e_j \\ \downarrow \\ 0 \end{array} \quad (1 \leq j \leq 4)$$

N_α :

$$\begin{array}{c} \bullet \\ (3,0) \end{array} \qquad \begin{array}{c} \bullet \\ (2,2) \\ \downarrow \\ \bullet \\ (1,1) \end{array} \qquad \begin{array}{c} \bullet \\ (0,3) \end{array} \qquad \begin{array}{c} e_3 \\ \downarrow \\ e_1 \\ \downarrow \\ 0 \end{array} \qquad \begin{array}{c} \underbrace{e_2 \, , \, e_4} \\ \downarrow \\ 0 \end{array}$$

N_β :

$$\begin{array}{c} \bullet \\ (3,3) \\ \downarrow \\ \bullet \\ (2,2) \\ \downarrow \\ \bullet \\ (1,1) \\ \downarrow \\ \bullet \\ (0,0) \end{array} \qquad \begin{array}{c} e_4 \\ \downarrow \\ e_3 \\ \downarrow \\ -e_1 \\ \downarrow \\ e_2 \\ \downarrow \\ 0 \end{array}$$

N_m :

$$\begin{array}{c} \bullet \\ (3,1) \\ \downarrow \\ \bullet \\ (2,0) \end{array} \qquad \begin{array}{c} \bullet \\ (1,3) \\ \downarrow \\ \bullet \\ (0,2) \end{array} \qquad \begin{array}{c} e_1 \\ \downarrow \\ e_3 \end{array} \qquad \begin{array}{c} e_4 \\ \downarrow \\ -m e_2 \end{array}$$

$m \in S :=$ (square free positive integers).

Infinitesimal calculus

$S \in \mathcal{B}(\log)$ log manifold. $\quad \theta_S := \mathcal{H}om\,(\omega^1_S, \mathcal{O}_S)$.

<u>Theorem</u> Γ : neat.

$$\begin{array}{ccc}
\theta_{\Gamma\backslash D_\Sigma} & \xrightarrow{\sim} & \mathcal{E}nd\,(H_{\mathcal{O}}, W, (\langle\, ,\,\rangle_w)_w) / F^0 \mathcal{E}nd\,(\;\cdot\;) \\
\cup & & \cup \\
\theta^h_{\Gamma\backslash D_\Sigma} & \xrightarrow{\sim} & gr^{-1}_F\, \mathcal{E}nd\,(H_{\mathcal{O}}, W, (\langle\, ,\,\rangle_w)_w)
\end{array}$$

☺ $S := \Gamma\backslash D_\Sigma$, $\quad d \otimes 1_{H_{\mathbb{C}}} : \mathcal{O}^{\log}_S \otimes H_{\mathbb{C}} \to \omega^{1,\log}_S \otimes H_{\mathbb{C}}$.

Taking τ_*, $\quad \nabla : H_{\mathcal{O}} \to \omega^1_S \otimes H_{\mathcal{O}}$.

$$\theta_S \to \mathcal{E}nd\,(H_{\mathcal{O}}), \quad \delta \mapsto \nabla_\delta. \quad \lrcorner$$

Example. $X \xrightarrow{f} S := \Delta^{n+t}$ proj. mor. $A \subset X$: NCD s.t.

$$\begin{array}{ccl} X & \xrightarrow{f} & S := \Delta^{n+t} \\ \cup & & \cup \\ X^* & \to & S^* := (\Delta^*)^n \times \Delta^t \end{array}$$

$X^* \to S^*$ smooth, $A \cap X^*$: rel NC $/S^*$, $h : X^* \smallsetminus A \to S^*$.

Assume loc. monodromy of $H_{\mathbb{R}} := R^m h_*(\mathbb{R})$ along A is unipotent.

Then, VMHS with pol. g.g. on $H_{\mathbb{Z}} := R^m h_*(\mathbb{Z})$ on S^* canonically extends to LMH with pol. g.g. H on S of type Φ with M_S = log str. corresp. to divisor $S \smallsetminus S^*$, M_X : corresp. to A, Σ = face (loc. monodromy cone at $0 \in S$), $\Gamma = \pi_1(S^*) = \pi_1(S^{\log})$.

$\varphi = \varphi_H : S \to \Gamma \backslash D_\Sigma$

$$\begin{array}{ccc} \theta_S & \xrightarrow{d\varphi} & \varphi^* \theta^h_{\Gamma\backslash D_\Sigma} = \mathrm{gr}_F^{-1} \mathcal{E}nd\,(H_{\mathcal{O}}, W, (\langle\ ,\ \rangle_w)_w) \\ \text{K.S.} \downarrow & & \wr \\ R^1 f_* \theta_{X/S} & \xrightarrow{\text{coupling}} & \bigoplus_p \mathcal{H}om_{\mathcal{O}_S}\,(R^{m-p} f_* \omega^p_{X/S},\ R^{m-p+1} f_* \omega^{p-1}_{X/S}). \end{array}$$

Canonical splitting of MHS

(W, F) : MHS.

Then, uniquely $F = s(\exp(-\zeta)\exp(i\delta)gr^W(F))$, where

$s : gr^W \to H_{0,\mathbb{R}}$, $F' := gr^W(F)$.

$\delta, \zeta \in \mathfrak{L}(F') := \{\alpha \in \mathrm{End}_{\mathbb{R}}(gr^W) \mid \alpha H^{p,q}_{F'} \subset \bigoplus_{p'<p,\, q'<q} H^{p',q'}_{F'}\}$

$\zeta = \zeta(F', \delta)$: written as certain universal Lie polynomial of $\delta^{p,q}$ $(p, q \leq -1)$.

$spl_W : D \to spl(W)$ continuous.

Space $D_{SL(2)}$

Pure case, $SL(2)$-orbit in n variables is (ρ, φ),

$\rho : SL(2,\mathbb{C})^n \to G_{\mathbb{C}}$ def. $/\mathbb{R}$, $\varphi : \mathbb{P}^1(\mathbb{C})^n \to \check{D}$ equivariant s.t.

(1) $\varphi(\mathfrak{h}^n) \subset D$.

(2) $\rho_* F^p_z(\mathfrak{sl}(2,\mathbb{C})^{\oplus n}) \subset F^p_{\varphi(z)}(\mathfrak{g}_{\mathbb{C}})$ $\quad (\forall z \in \mathbb{P}^1(\mathbb{C})^n,\ \forall p)$.

Mixed case, nondeg. $SL(2)$-orbit of rk n is $((\rho_w, \varphi_w)_w, \mathbf{r})$

(ρ_w, φ_w) : $SL(2)$-orbit in n variables for gr^W_w, $\mathbf{r} \in D$ s.t.

(1) $\mathbf{r}(gr^W_w) = \varphi_w(i, \dots, i)$ $\quad (\forall w)$.

(2) $2 \leq \forall j \leq n$, $\exists w$ s.t. j-th comp. of ρ_w is nontrivial.

(3) $\mathbf{r} \in D_{\mathrm{spl}}$, $n \geq 1 \Rightarrow \exists w$ s.t. first comp. of ρ_w is nontrivial.

Given $((\rho_w, \varphi_w)_w, r)$. $\quad s_r := sp|_W(r)$.

Define $\tau : \mathbb{G}_{m,\mathbb{R}}^n \to \mathrm{Aut}_{\mathbb{R}}(H_{0,\mathbb{R}}, W)$ by

$$\tau(t_1, \dots, t_n) := s_r \circ \left(\bigoplus_w \left(\prod_{j=1}^n t_j\right)^w \rho_w(g_1, \dots, g_n) \text{ on } gr_w^W\right) \circ s_r^{-1}$$

with $g_j := \begin{pmatrix} (t_j \cdots t_n)^{-1} & 0 \\ 0 & (t_j \cdots t_n) \end{pmatrix}$.

$$W_w^{(j)} := \bigoplus_{k \le w} \{ v \in H_{0,\mathbb{R}} \mid \tau_j(a) = a^k v \ (\forall a \in \mathbb{R}^\times) \}.$$

τ_j ← j-th comp. of τ

$\{W^{(1)}, \dots, W^{(n)}\}$ ass. set of wt filt.s.

$p = ((\rho_w, \varphi_w)_w, r)$, $p' = ((\rho'_w, \varphi'_w)_w, r')$ are <u>equivalent</u>

$\Leftrightarrow \exists\, t \in \mathbb{R}_{>0}^n$ s.t.

$\rho'_w = \mathrm{Int}(gr_w^W(\tau(t))) \circ \rho_w$, $\varphi'_w = gr_w^W(\tau(t)) \circ \varphi_w$ $(\forall w)$, $r' = \tau(t) r$

τ, $\{W^{(j)}\}_j$ depend only on eq. class.

$D_{SL(2)} :=$ (set of eq. classes of nondeg. $SL(2)$-orbits of various rank s.t. ass $\{W^{(j)}\}_j$ is rational).

$D^{I}_{SL(2)}$, $D^{II}_{SL(2)}$: two topologies on $D_{SL(2)}$ (stronger, weaker)

(real anal. structure with boundaries)

Here we use this.

set Ψ of finite increasing filt.s on $H_{0,\mathbb{R}}$ is <u>admissible</u>

$\Leftrightarrow$ $\exists\, p \in D_{SL(2)}$ s.t. Ψ = ass. set of wt filts of p.

$D^{I}_{SL(2)}(\Psi) := \{ p \in D_{SL(2)} \mid$ ass set of wt filt.s of $p \subset \Psi \}$.

$\{D^{I}_{SL(2)}(\Psi)\}_{\Psi}$ is an open covering.

Associated SL(2)-orbit

Let $(N_1, \dots, N_n, F)$ generate a nilp. orbit.

nondeg. SL(2)-orbit $((\rho'_w, \varphi'_w)_w, r_1)$ is ass. as follows.

(ρ_w, φ_w): SL(2)-orbit of n var.s ass. to $(\mathrm{gr}^W_w(N_1), \dots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$,

$k := \min(\{j \mid 1 \le j \le n,\ N_j \ne 0\} \cup \{n+1\})$,

$J' := \{ j \mid 1 \le j \le n,$ j-th comp of ρ_w is nontriv for some $w\}$,

$$J := \begin{cases} J' \cup \{k\} & \text{if } k \ne n+1. \\ \phi & \text{if } k = n+1. \end{cases} \qquad J = \{a(1), \dots, a(r)\} \quad < \cdots <$$

$\rho'_w(g_{a(1)}, \dots, g_{a(r)}) := \rho_w(g_1, \dots, g_n)$. $\varphi'_w(z_{a(1)}, \dots, z_{a(r)}) := \varphi_w(z_1, \dots, z_n)$.

$s := \lim \mathrm{spl}_W(\sum_j i y_j N_j) F$ as $y_j / y_{j+1} \to \infty$, $y_{n+1} := 1$ (converges)

$\tau(t_1, \dots, t_n) := s \circ (\bigoplus_w (\prod_j t_j)^w \rho_w(g_1, \dots, g_n)$ on $\mathrm{gr}^W_w) \circ s^{-1}$

$r_1 := \lim \tau\left(\sqrt{\frac{y_2}{y_1}}, \dots, \sqrt{\frac{y_{n+1}}{y_n}}\right)^{-1} \exp(\sum_j i y_j N_j) F$ $\left(\begin{matrix}\text{torus-twist back} \\ \text{con}\end{matrix}\right)$.

Other method

Given $(N_1, \dots, N_n, F)$.

$W^{(0)} := W, \quad W^{(j)} := M(N_1 + \cdots + N_j, W) \quad (1 \leq j \leq n)$

$\mathbb{R}$-split MHS $(W^{(j)}, \hat{F}_{(j)})$ $(0 \leq j \leq n)$ defined as follows.

$(W^{(n)}, F)$ is MHS (by Deligne [Kash. 86]).

$(W^{(n)}, \hat{F}_{(n)})$: ass $\mathbb{R}$-split MHS.

$(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$: MHS

$(W^{(n-1)}, \hat{F}_{(n-1)})$: ass $\mathbb{R}$-split MHS.

. . . . . .

$s^{(j)} : gr^{W^{(j)}} \xrightarrow{\sim} H_{0,\mathbb{R}}$ given by $(W^{(j)}, \hat{F}_{(j)})$ $(0 \leq j \leq n)$.

$s^{(0)} = s$ before

$\tau_j : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{Aut}_{\mathbb{R}}(H_{0,\mathbb{R}}, W), \quad \tau_j(a)v := a^w v \quad (v \in s^{(j)}(gr_w^{W^{(j)}}),\ a \in \mathbb{R}^\times)$.

Valuative spaces

D_{val} = set of all (A, V, Z) s.t.

$A \subset \mathfrak{g}_{\mathbb{Q}}$: $\mathbb{Q}$-subspace

$V \cup (-V) = A^*$

$V \subset A^* := \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q})$ valuative sharp submonoid

$Z = \exp(A_{\mathbb{C}})F \subset \check{D}$ s.t. $\exists\, \tau \subset A_{\mathbb{R}}$: rat. cone,

$(\tau \cap A)^{\vee} \subset V$ in A^*, Z is a τ-nilp. orbit.

$D^{\#}_{\mathrm{val}}$ = set of all (A, V, Z) s.t.

"$Z = \exp(A_{\mathbb{C}})F$" above is replaced by "$Z = \exp(iA_{\mathbb{R}})F$".

$D^{\#}_{\mathrm{val}} \to D_{\mathrm{val}}$, $(A, V, Z) \mapsto (A, V, \exp(A_{\mathbb{C}})Z)$.

Given Σ : weak fan in $\mathfrak{g}_{\mathbb{Q}}$

$D_{\Sigma,\mathrm{val}} := \{(A,V,Z) \in D_{\mathrm{val}} \mid \exists \sigma \in \Sigma$ s.t. $(\sigma_{\cap} A)^{\vee} \subset V$ in A^*,
$\exp(\sigma_{\mathbb{C}})Z$ is σ-nilp. orbit $\}$.

$D^{\#}_{\Sigma,\mathrm{val}} := \{$ —— $D^{\#}_{\mathrm{val}} \mid$ ——, $\exp(i\sigma_{\mathbb{R}})Z$ —— $\}$.

$D^{\#}_{\Sigma,\mathrm{val}} \twoheadrightarrow D_{\Sigma,\mathrm{val}}$

<u>Lem</u>. $\forall (A,V,Z) \in D_{\Sigma,\mathrm{val}}$, $\exists \sigma \in \Sigma$: smallest sat. above cond.

$D_{\Sigma,\mathrm{val}} \to D_{\Sigma}$, $(A,V,Z) \mapsto (\sigma, \exp(\sigma_{\mathbb{C}})Z)$.

$D^{\#}_{\Sigma,\mathrm{val}} \to D^{\#}_{\Sigma}$, $($ —— $) \mapsto (-, \exp(i\sigma_{\mathbb{R}})\,-)$.

$$E_{\sigma,\mathrm{val}} := \mathrm{toric}_{\sigma,\mathrm{val}} \times_{\mathrm{toric}_\sigma} E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma,\mathrm{val}}.$$

$$(A, V, z, F) \mapsto (A, V, \exp(A_{\mathbf{C}}) \exp(z) F).$$

$$E^\#_{\sigma,\mathrm{val}} := |\mathrm{toric}|_{\sigma,\mathrm{val}} \times_{|\mathrm{toric}|_\sigma} E^\#_\sigma \to D^\#_{\sigma,\mathrm{val}}.$$

$$\begin{array}{ccc}
E_{\sigma,\mathrm{val}} & E^\#_{\sigma,\mathrm{val}} & (\sigma \in \Sigma) \\
\downarrow & \downarrow & \\
\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma,\mathrm{val}} & D^\#_{\sigma,\mathrm{val}} & \\
\downarrow & \downarrow & \\
\Gamma \backslash D_{\Sigma,\mathrm{val}} & D^\#_{\Sigma,\mathrm{val}} &
\end{array}$$

loc. ringed sp. / $\mathbf{C}$ topology   as before.

Map $\psi : D^{\#}_{val} \to D_{SL(2)}$

Theorem (i) For $p = (A, V, Z) \in D^{\#}_{val}$, $\exists (N_j)_{1 \le j \le n}$ "excellent basis" of $A_{\mathbb{R}}$
s.t. (1) $F \in Z \Rightarrow (N_1, \dots, N_n, F)$ gen. nilp. orbit
(2) $(N_j)_j : A^* = \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q}) \hookrightarrow \mathbb{R}^n$, $V =$ pullback of $\mathbb{R}^n_{\ge 0}$
w.r.t. lexco. order.
(ii) $\psi : D^{\#}_{val} \to D_{SL(2)}$, $p \mapsto$ (SL(2)-orbit ass. to $(N_1, \dots, N_n, F)$),
is well-defined.

Theorem Given weak fan Σ in $\mathfrak{g}_{\mathbb{Q}}$.
Then, $\psi : D^{\#}_{\Sigma, val} \to D^{I}_{SL(2)}$ is continuous

Comments

1. $\exists$ local fan.

   $\nexists$ fan for connected Néron model.

   $\exists$ weak fan for Néron model for $\forall$ gr W.

2. valuative formulation $S_{val} \to \Gamma \backslash D_{val,(\Gamma)}$,

   where $D_{val,(\Gamma)} \subset D_{val}$ is the union of $D_{\sigma,val}$ for $\sigma \in \mathcal{C}_\Gamma$

   with $\mathcal{C}_\Gamma :=$ set of all sharp rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ generated by the logarithms of a finite number of elems of a given $\Gamma < G_{\mathbb{Z}}$.

(reverse formulation to "cone conjecture")!