Some Localised Pattern Problems

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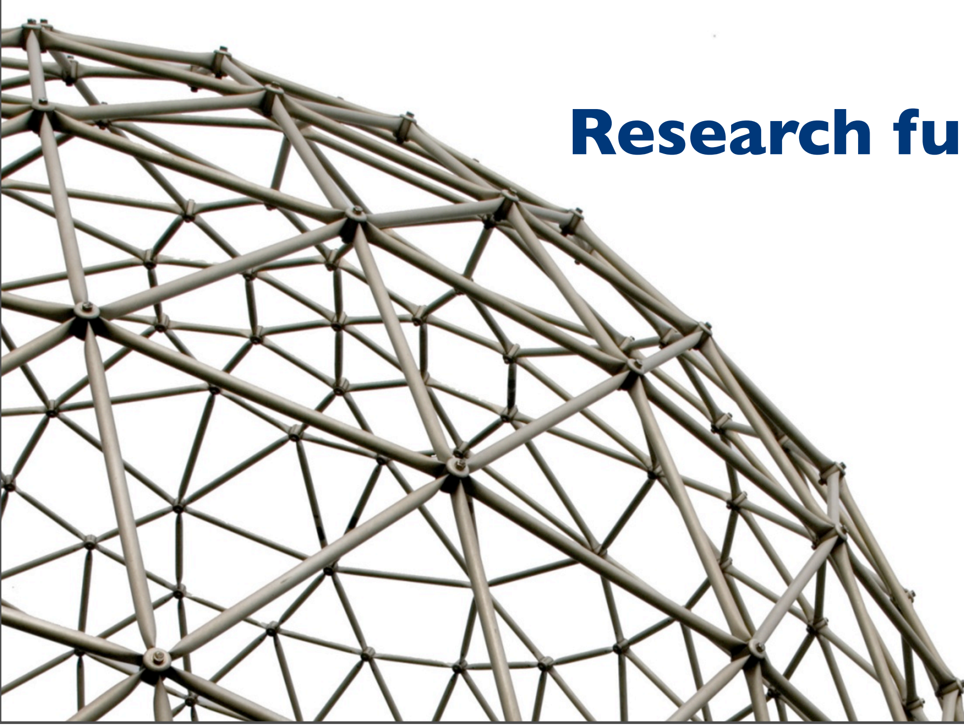

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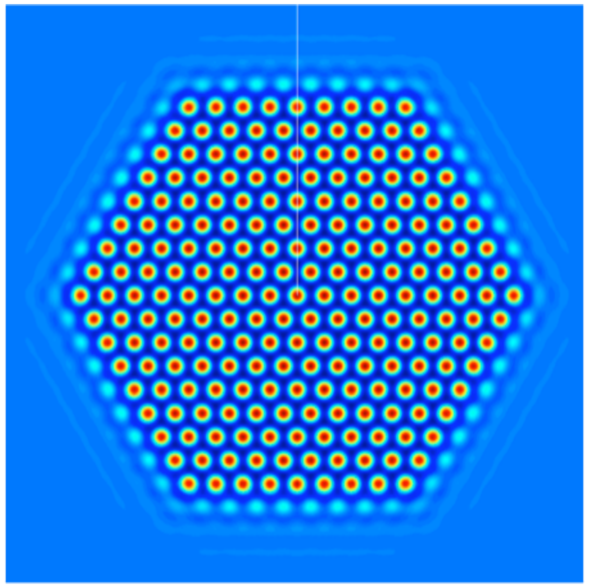
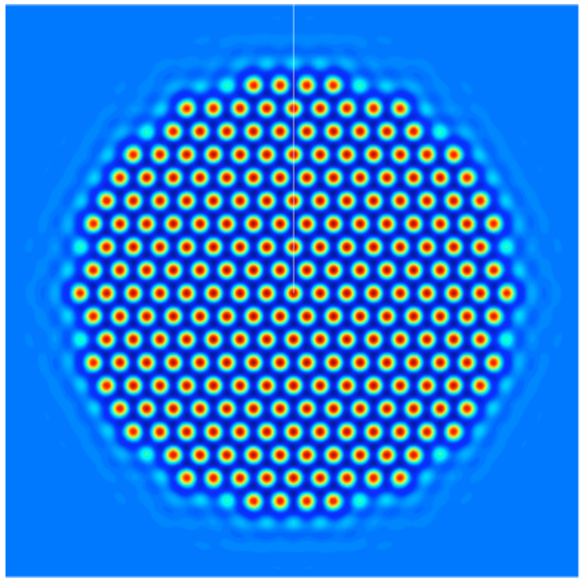
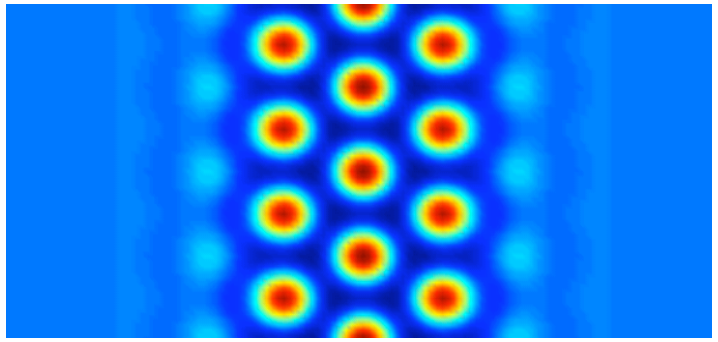
- **Wulff Structures and localised patches**
- **Snaking near the singular limit**
- **Ferrofluids**

Large scale patch shape

- **Hexagon patches related to Hexagon fronts**



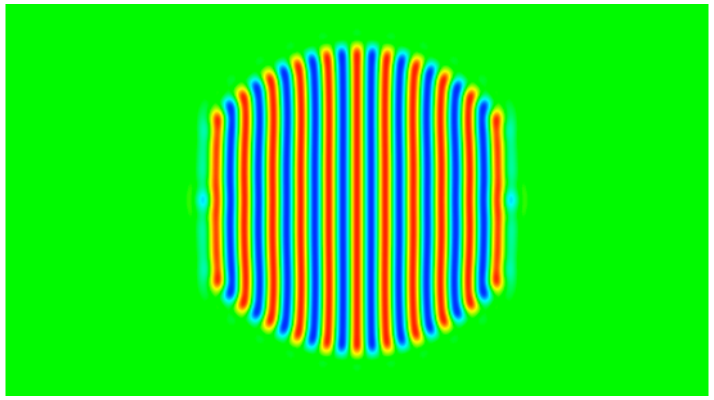




hexagon pulses

Localised hexagon patches

- **What about other types of patches?**



Localised roll patches

Localised square patches

Wulff Construction

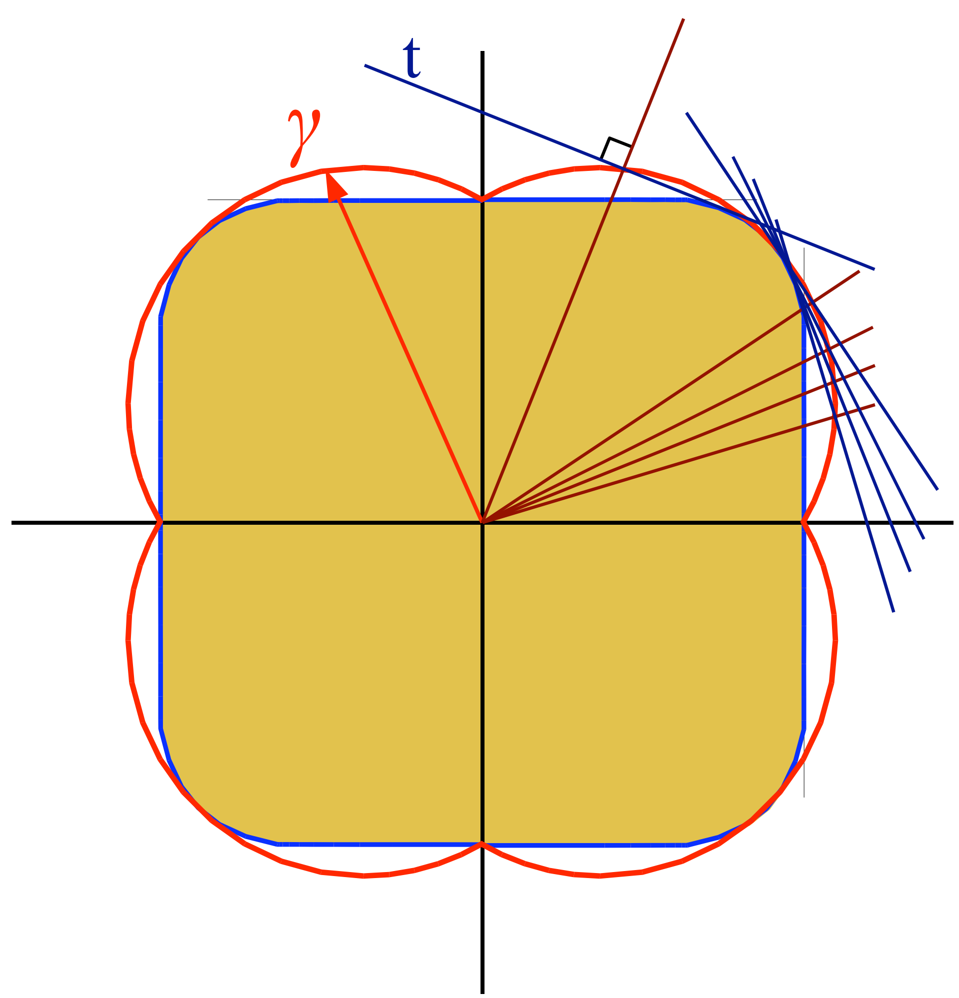
- **Yields preferred crystalline shape of fixed volume inside a separate phase**
- **Minimise:**

$$\min \int \gamma dA$$

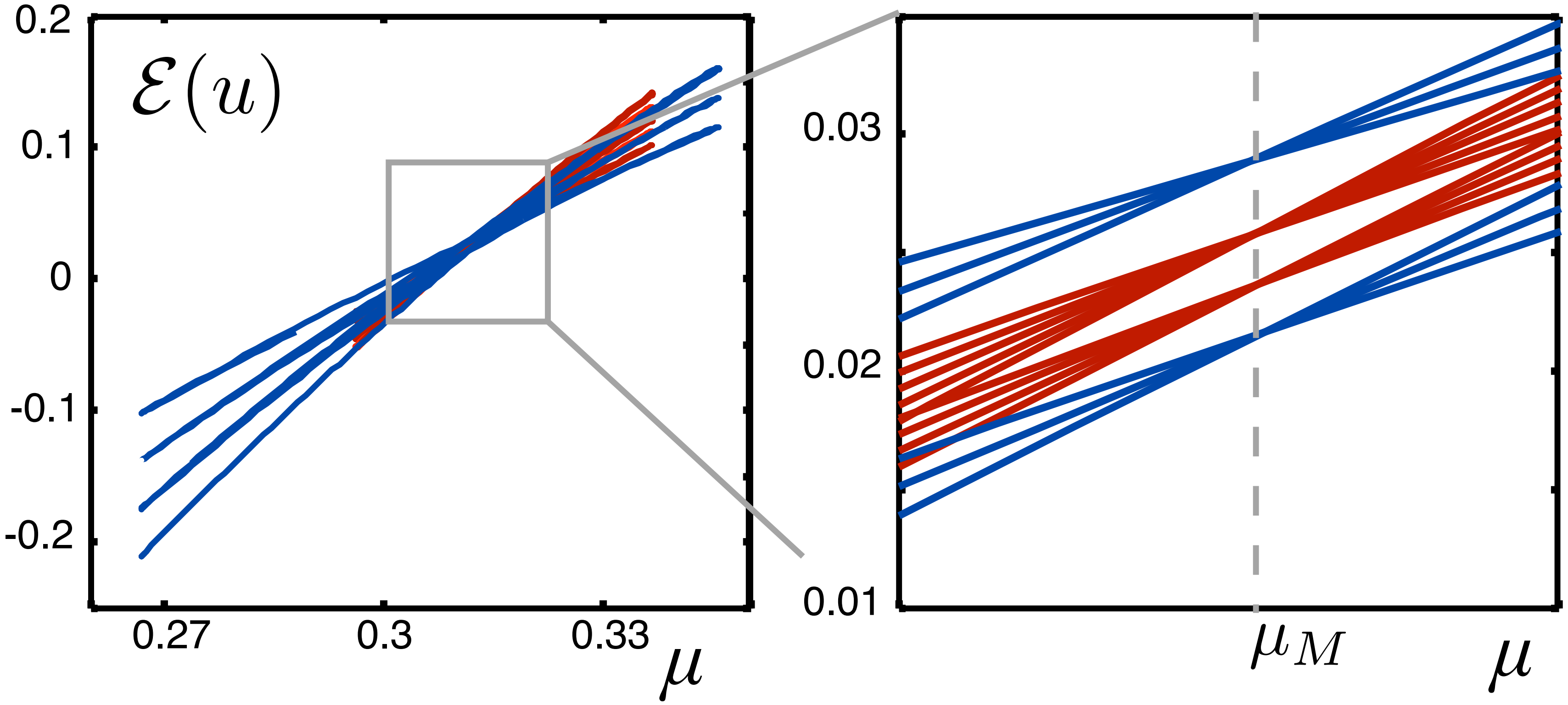
$\gamma(\hat{n})$ **Interfacial energy**

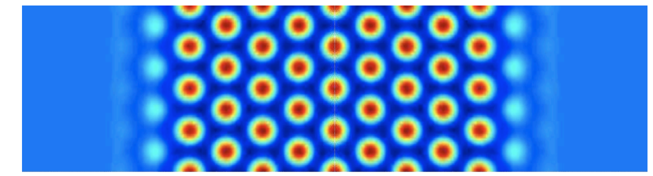
$\hat{n}$ **Orientation of interface**

- **Wulff Construction (Herring 1953)**

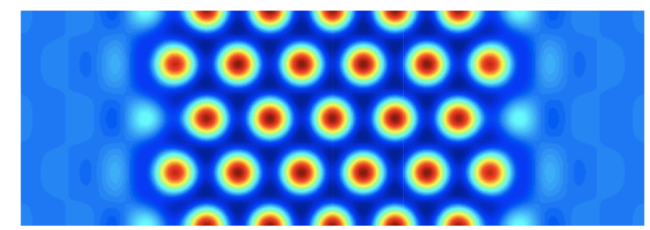


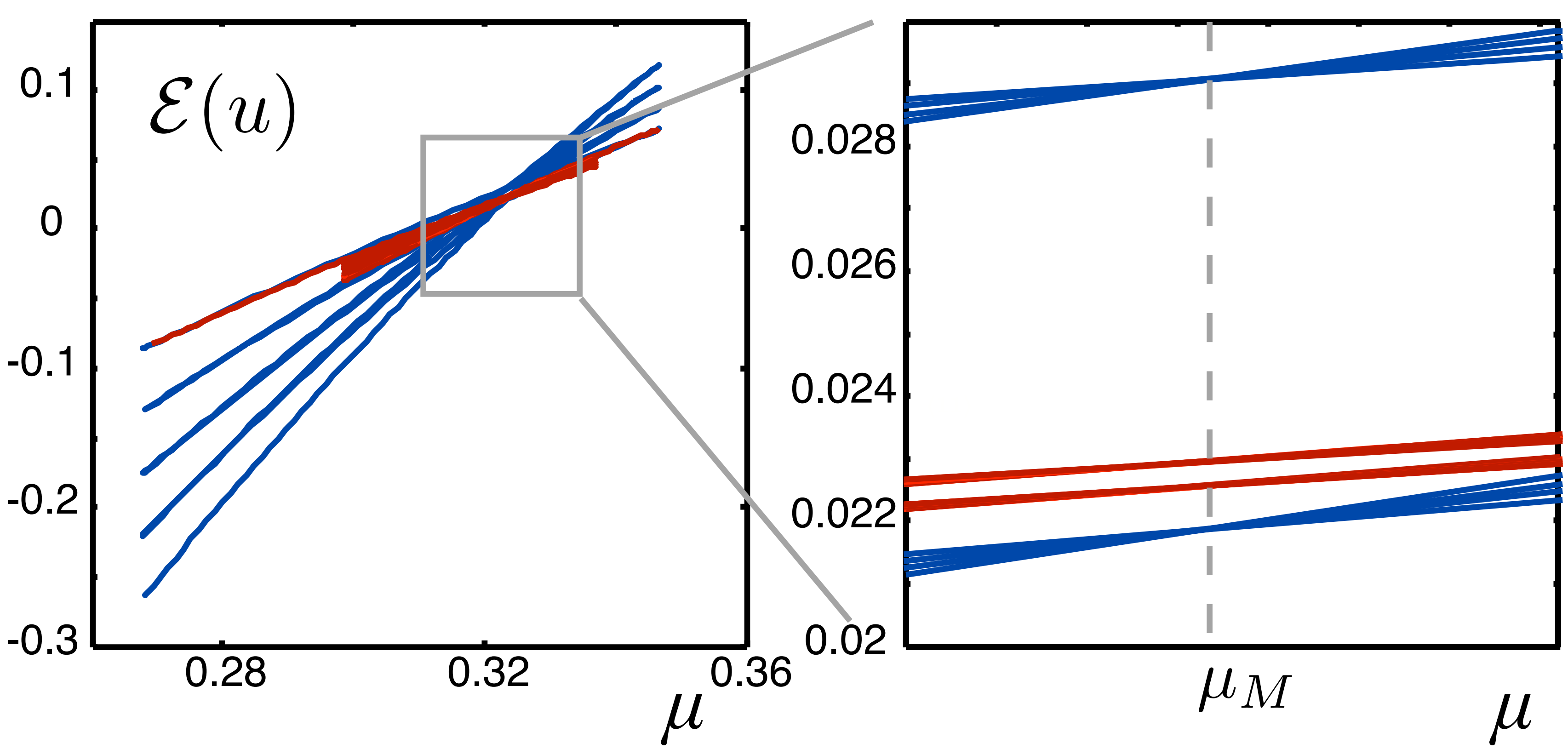
Energy of planar hexagons



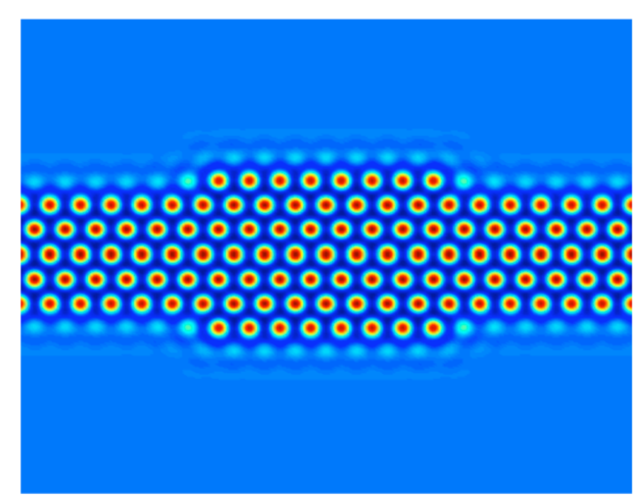


[10] hexagon front
[11] hexagon front



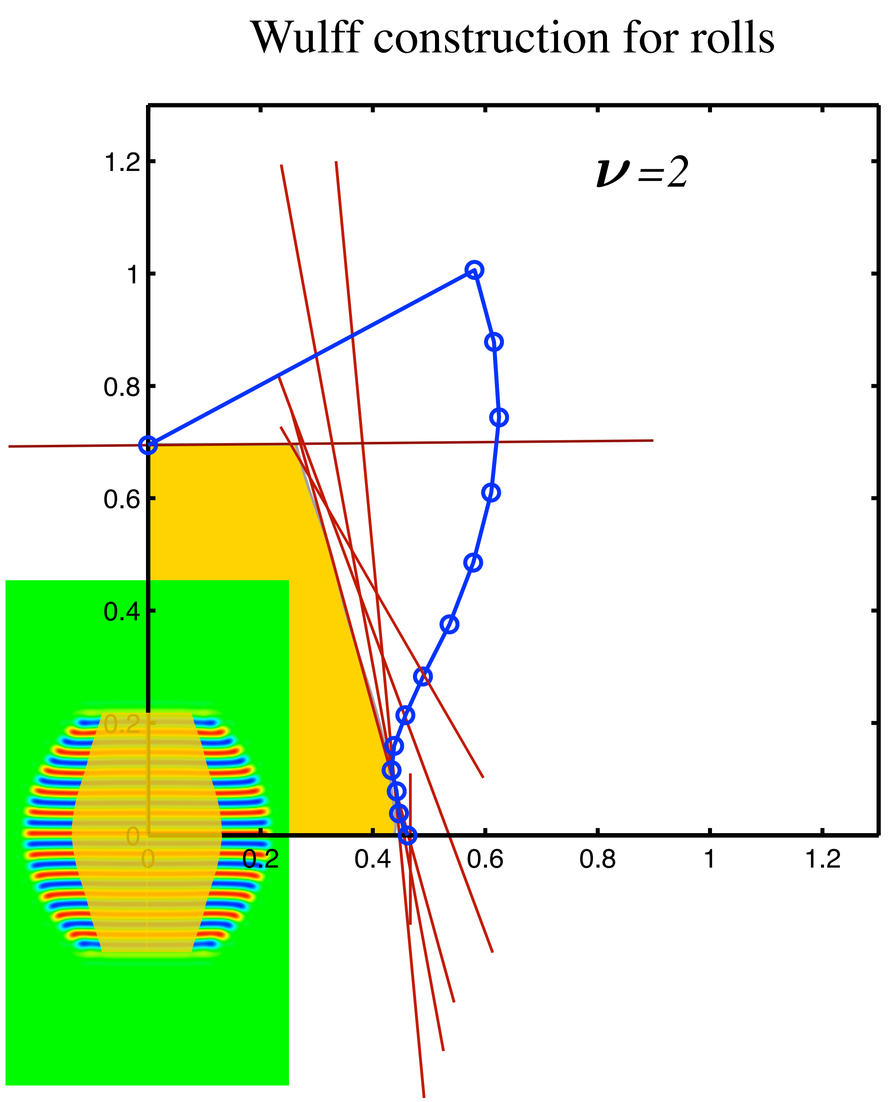


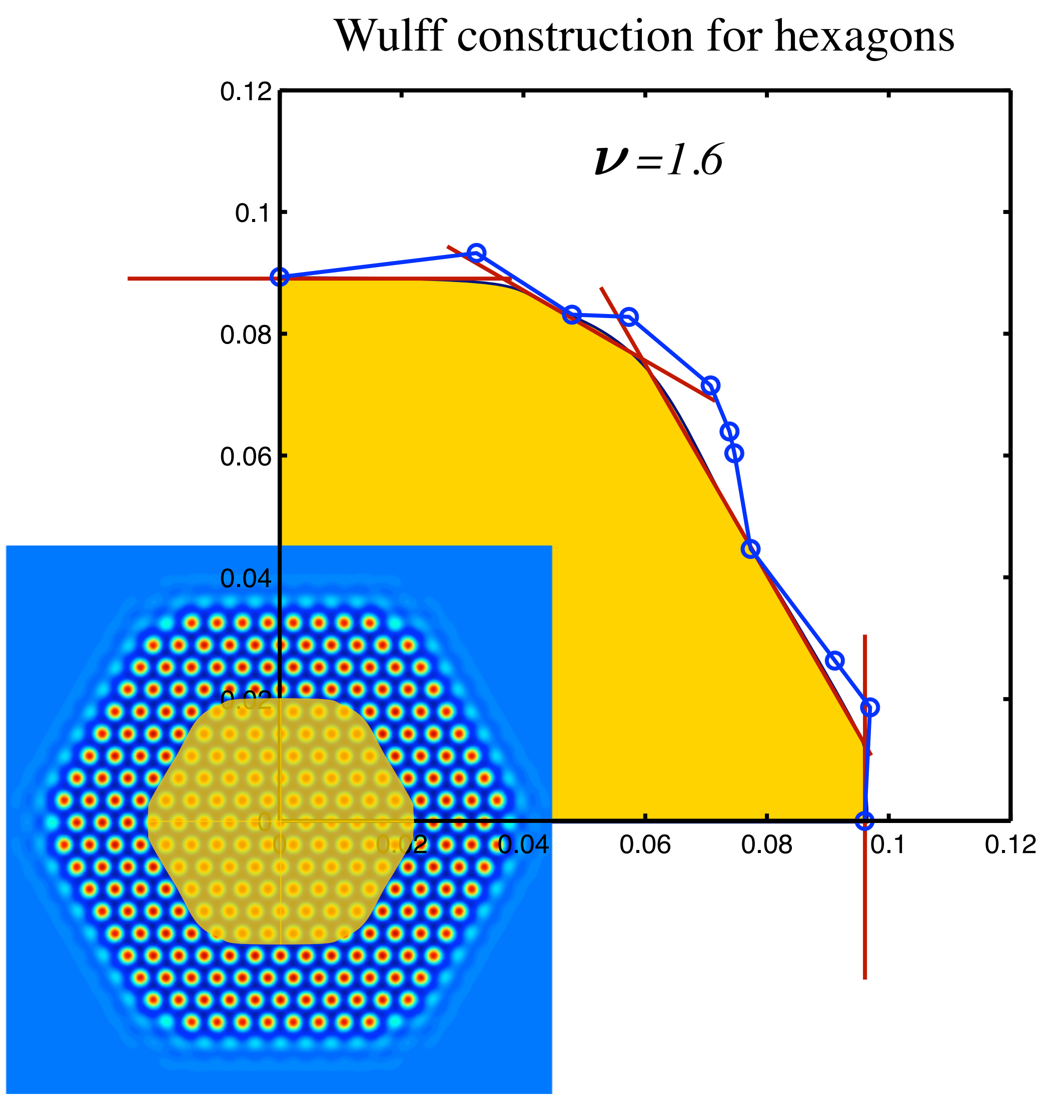
Hexagon front [10]
Almost [10] front



Wulff Constructions

- **Compute fronts in Skew coordinates**





- **Near co-dim 2 point, asymptotics -> hexagon Wulff construction circle**
- **What happens for other localised patches?**

Snaking near the singular limit

- **Crime hotspot model:**
  - **Short et al. (Math. Models Meth. Appl. Sci. 1249 (18) 2008)**
- **Non-dimensional Mean field equations with:**

$$A_t = \eta\nabla^2 A - A + A_0 + \rho A,$$
$$\rho_t = \nabla\cdot\left(\nabla\rho - \frac{2\rho}{A}\nabla A\right) - \rho A + \overline{A} - A_0$$

Spatially Homogenous equilibrium:

$$A = \overline{A}, \qquad \rho = 1 - \frac{A_0}{\overline{A}}$$

Stable: $A_0 > A_0^{\mathrm{crit}}$

Turing instability: $A_0^{\mathrm{crit}} = \frac{2}{3}\overline{A} - \frac{1}{3}\eta\overline{A}^2 - \frac{2}{3}\overline{A}\sqrt{\eta\overline{A}}$

1D Subcritical: $\sqrt{\eta\overline{A}} \lesssim 0.157$

- **Steady problem can be re-written as a 4th-order PDE:**

$$-\eta B_{xxxx} - 2B_{xx} + 2\frac{B_x^2}{B+\bar{A}} + 4\eta\frac{B_x B_{xxx}}{B+\bar{A}} - 6\eta\frac{B_x^2 B_{xx}}{(B+\bar{A})^2} - 6A_0\frac{B_x^2}{(B+\bar{A})^2} + 3\eta\frac{B_{xx}^2}{B+\bar{A}} + 3A_0\frac{B_{xx}}{B+\bar{A}} - \bar{A}B - B^2 + (B+\bar{A})\eta B_{xx} = 0$$

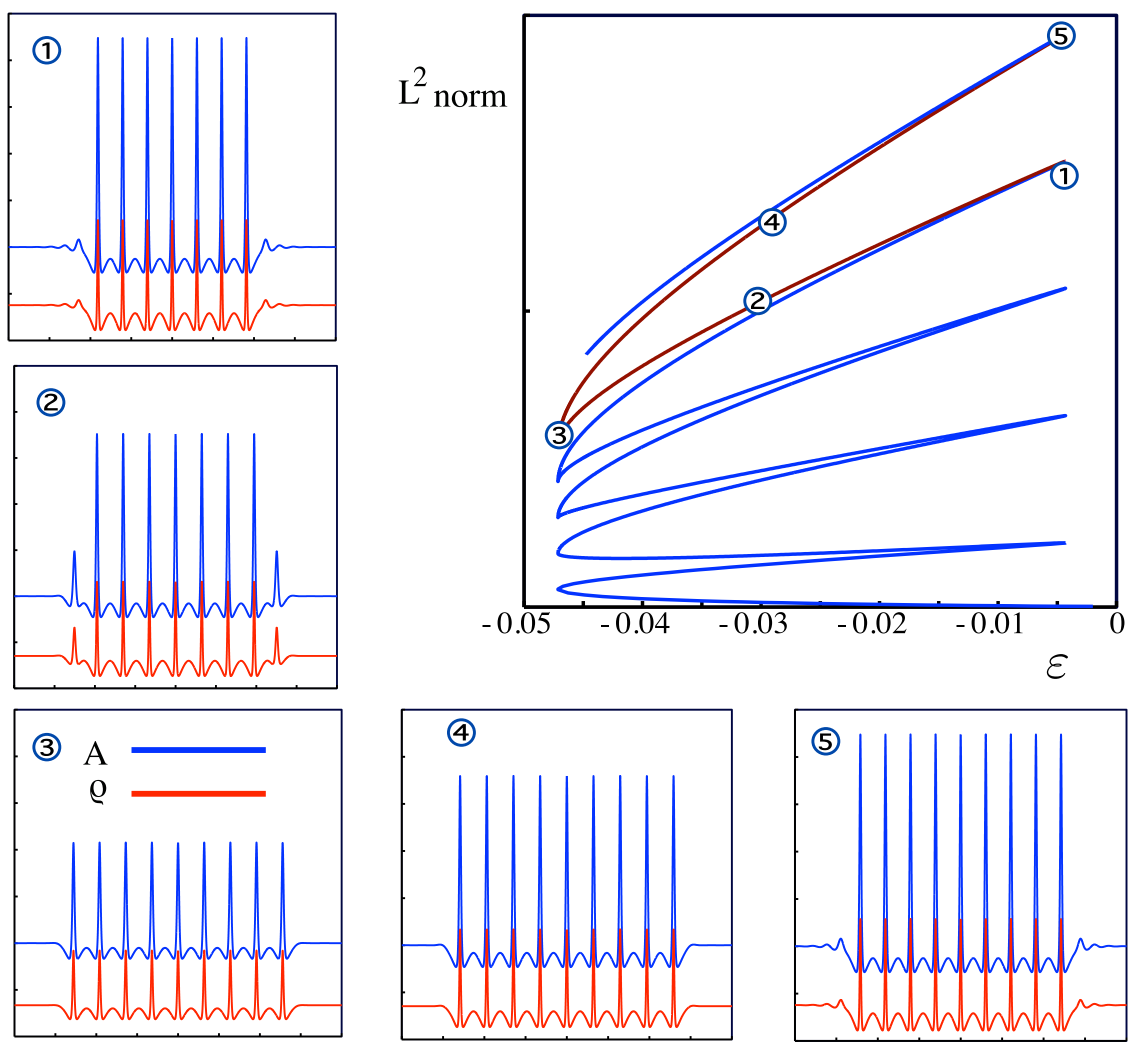
Homoclinic Snaking

- **Weakly nonlinear analysis:** $A(x,t) \approx \overline{A} + \epsilon^{\frac{1}{2}} P(X,T) e^{ix} + c.c.$

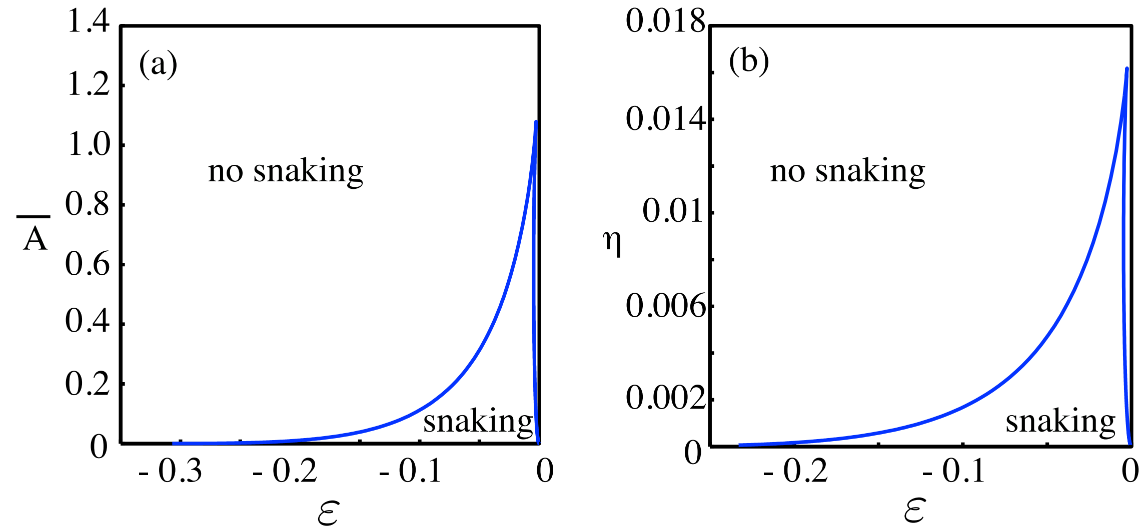
$$P_T = \eta P_{XX} + \epsilon P - \nu |P|^2 P$$

where: $A_0 = A_0^{\mathrm{crit}} - \epsilon \overline{A}$

- **Two pulses bifurcate off equilibrium...**



$\eta = 0.005$



Singular Limit Analysis

- **Change of coordinate:** $v = \dfrac{\rho}{A^2}$

- **Rescale:** $\eta \to \eta^2, \qquad A \to \dfrac{A}{D}, \qquad v \to D^2 v, \qquad \overline{A} \to \dfrac{\overline{A}}{D}, \qquad A_0 \to \dfrac{A_0}{D}$

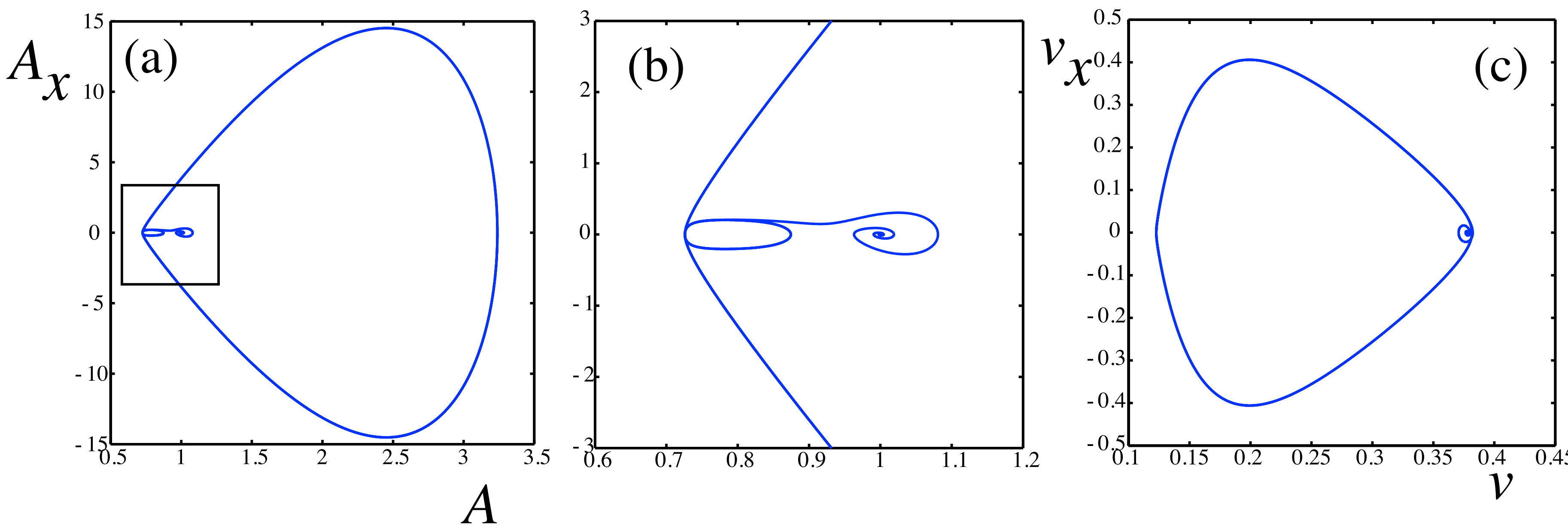
$$\eta^2 A_{xx} - A + vA^3 + A_0 = 0,$$

$$D(A^2 v_x)_x - vA^3 + \overline{A} - A_0 = 0$$

- **Kolokolnikov/Ward/Sun (2011) look at** $D \gg 1$

$$A(x) \sim \begin{cases} a\,\mathrm{sech}(x/\eta) & x = \mathcal{O}(\eta), \\ A_0 & x \gg \mathcal{O}(\eta) \end{cases} \qquad v(x) \sim v_0 \ll 1$$

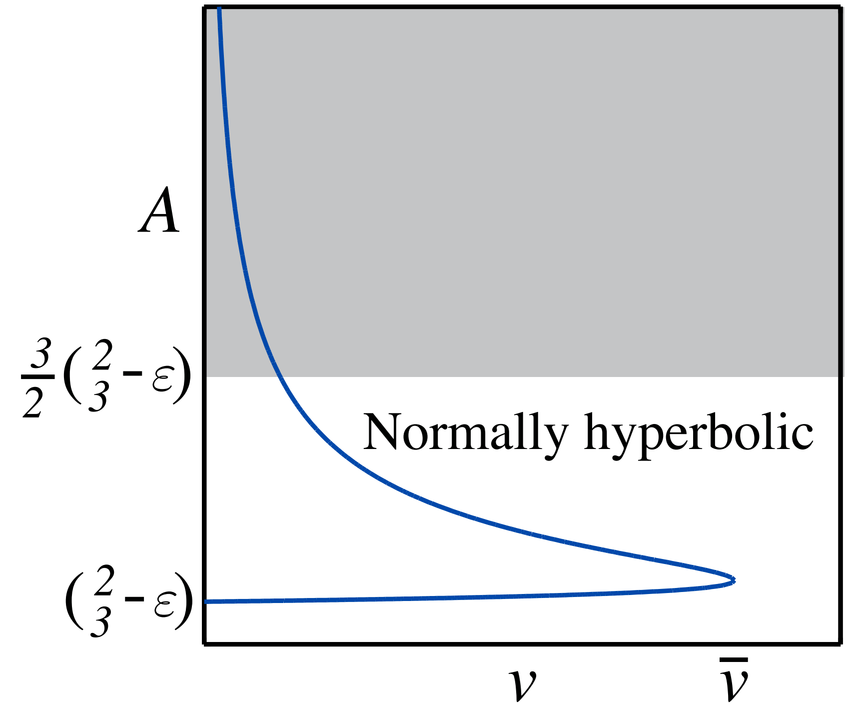
- **Interesting limit** $D \ll 1$ **1D Subcritical Turing:** $\sqrt{\dfrac{\eta^2 \overline{A}}{D}} \lesssim 0.157$



Slow limit

- **Slow system:**

$$\eta A_x = B,$$
$$\eta B_x = A - vA^3 - A_0,$$
$$v_x = \frac{w}{A^2},$$
$$w_x = vA^3 - \bar{A} + A_0,$$

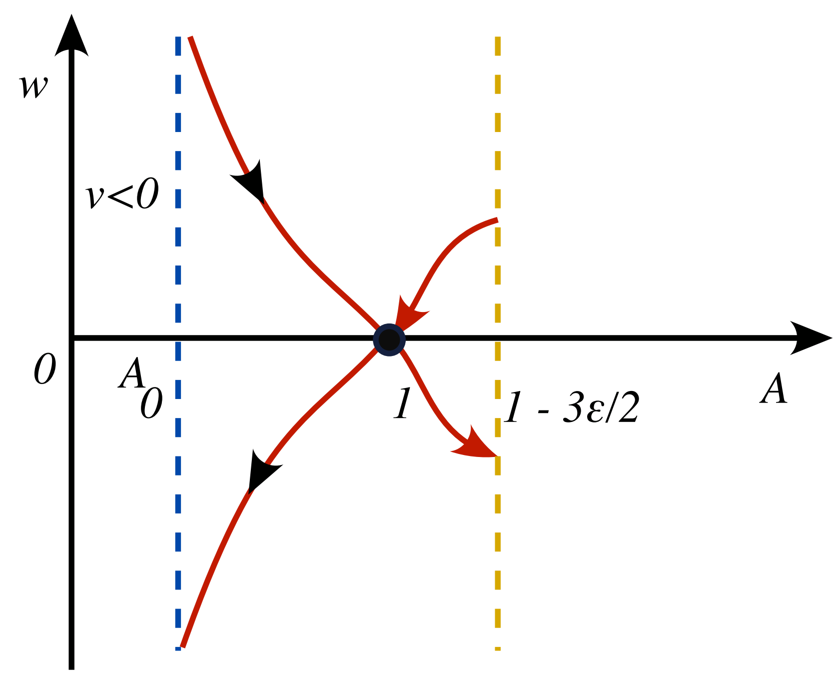


- **Setting** $\eta = 0$

$$B = 0, \qquad A - vA^3 - \left(\frac{2}{3} - \epsilon\right)\overline{A} = 0$$

- **Re-writing slow flow:**

$$A_x = \frac{A^2 w}{(2 - 3\epsilon - 2A)},$$
$$w_x = A - 1,$$



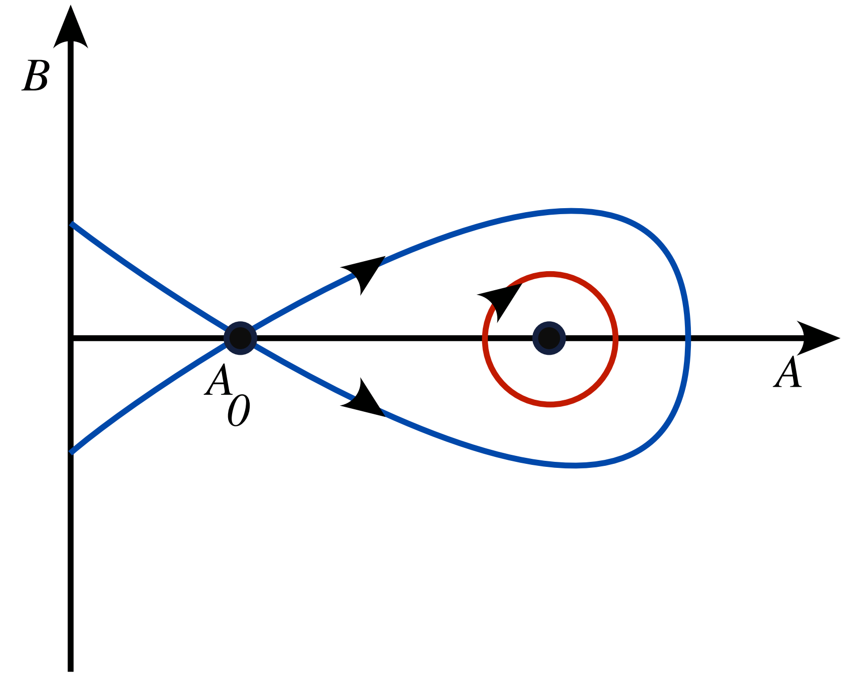
Fast limit

- **Fast system:**

$$A_x = B,$$
$$B_x = A - vA^3 - A_0,$$
$$v_x = \eta \frac{w}{A^2},$$
$$w_x = \eta v A^3 - \eta(\bar{A} - A_0),$$

- **Setting:** $\eta = 0$

$$v(y) = v_0 \qquad w(y) = w_0 \qquad A_{yy} = A - v_0 A^3 - \left(\frac{2}{3} - \epsilon\right)\overline{A}$$



- **Matching? cf. Gray Scott problem**

Ferrofluids

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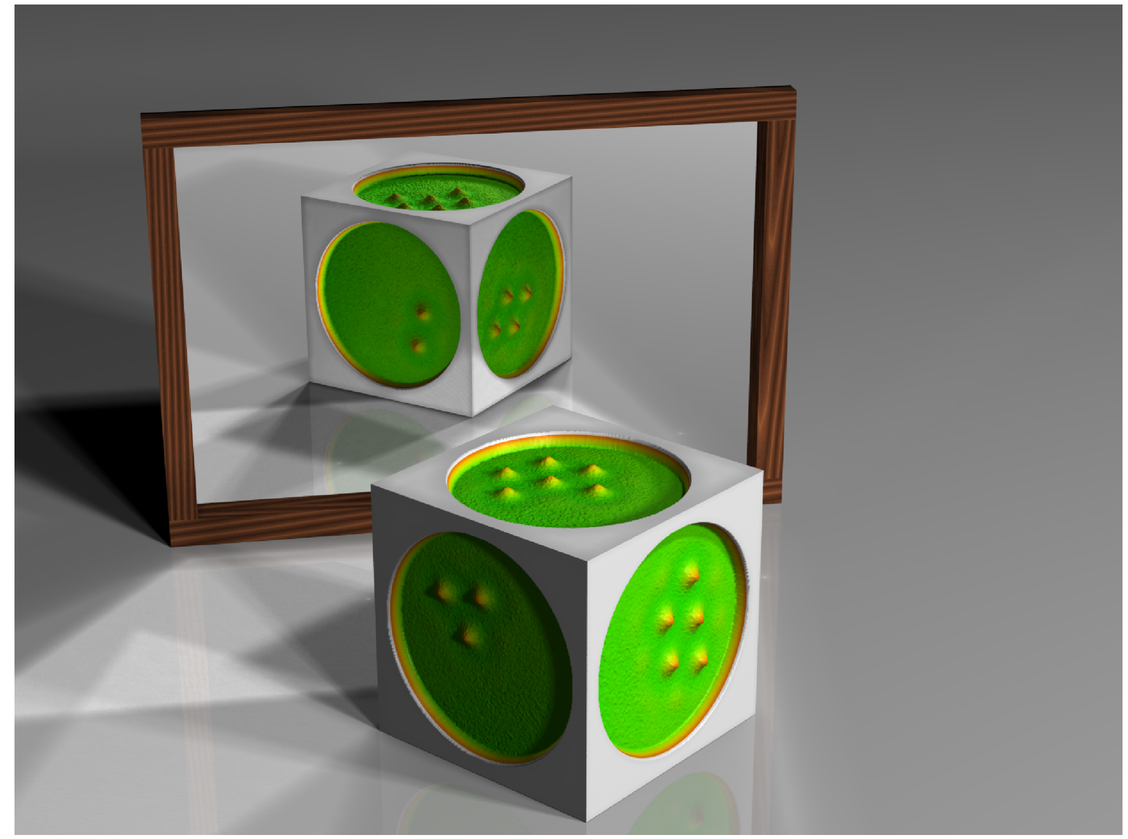


Ferrosolitons

[Richter & Barashenkov]

Ferro-hexagon patches

[Gollwitzer, Rehberg & Richter]

Q: Weakly interacting spots or hexagon patches?

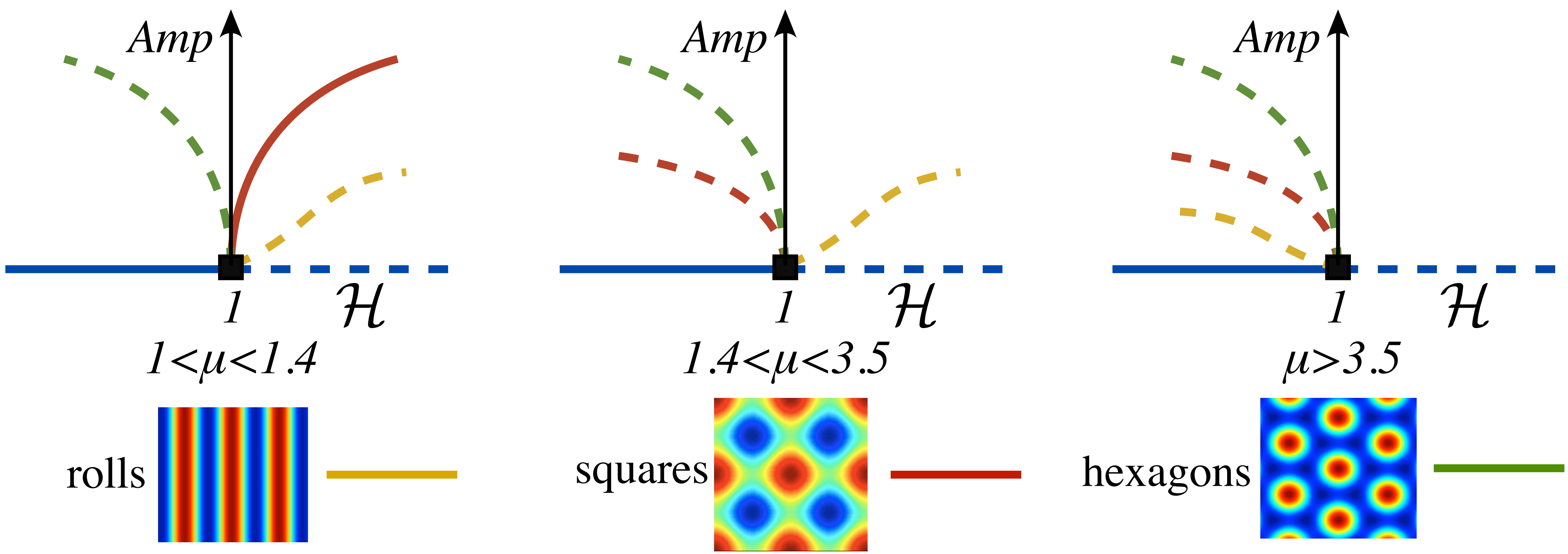
Periodic Ferro-patterns

Linear Magnetisation Law

2 non-dimensional params:

$\mathcal{H}$ **- rescaled uniform magnetic field** $\Rightarrow \mathcal{H} = 1$ **onset of instability**

μ **- permeability of ferrofluid**



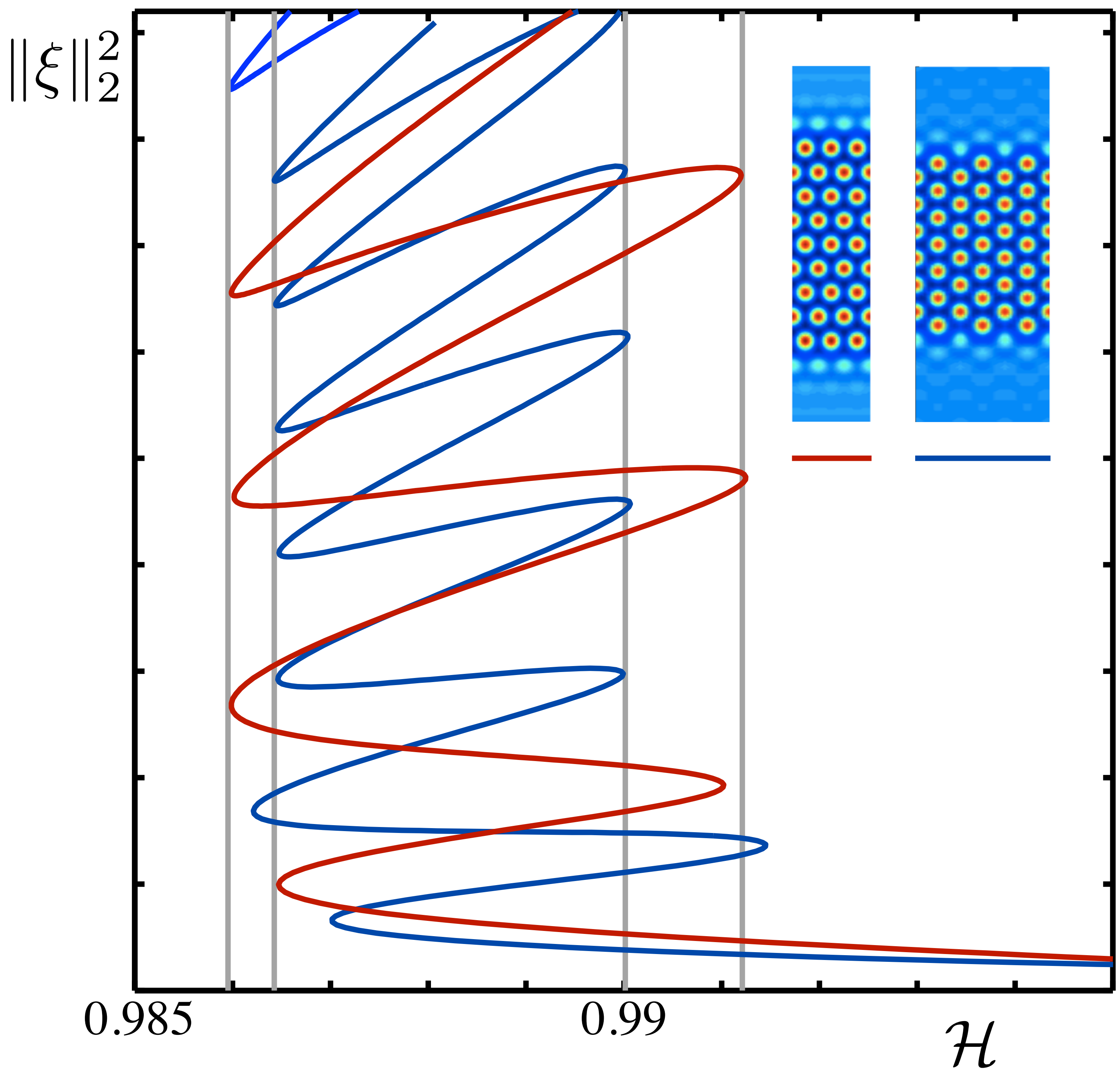
[Silber & Knobloch]

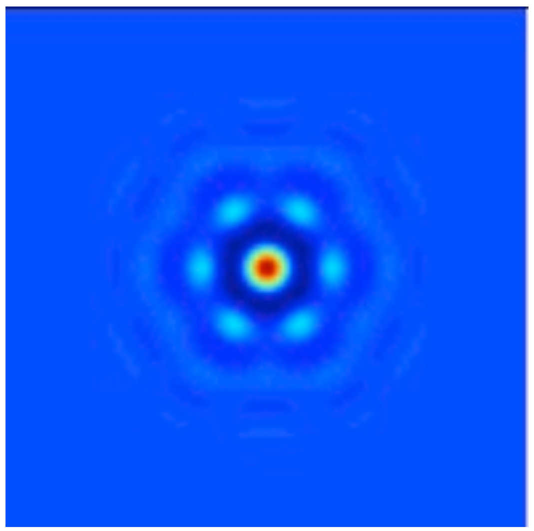
Richter & Barashenkov ferrofluid $\mu \sim 3.2$

Mystery: Spots should be unstable from SHeqn...

Hexagon Ferro-fronts

$\mu = 2.5$





Patch bifurcation off spot

$\mathcal{H} = 0.995$

Challenge:

Understand hexagon patches

- **Snaking breaks down** $\mu \geq 3$

Ferrofluid: Spatial Dynamics

- **Energy:** $\mathcal{E}(\phi,\chi,\xi) = \int\int \left[\frac{H_0^2}{2\mu_0}\left(\int_{\xi(x,y)}^{\infty} |\nabla\chi|^2 dz + \frac{1}{\mu}\int_{-\infty}^{\xi(x,y)} |\nabla\phi|^2 dz\right) + \frac{1}{2}\rho g \xi^2 + \tau\sqrt{1+|\nabla\xi|^2}\right] dxdy$

  Bcs: $\phi = \mu\chi, \qquad \phi_z - \chi_z = (\nabla\phi - \nabla\chi)\cdot\nabla\xi.$

- **Flatten out the free-surface:** $\tilde{z} = z - \xi(x,y)$
- **Legendre transformation -> Hamilton's equations**

$$
\begin{aligned}
\xi_x &= \frac{\tau W}{\sqrt{\tau^2 - W^2}}, \\
\eta_x &= \rho g \xi, \\
\phi_x &= \frac{\mu\mu_0}{H_0^2}\alpha + \frac{H_0^2}{\mu\mu_0}\phi_{\tilde{z}}\xi_x, \\
\chi_x &= \frac{\mu_0}{H_0^2}\beta + \frac{H_0^2}{\mu_0}\chi_{\tilde{z}}\xi_x, \\
\alpha_x &= -\frac{\mu\mu_0}{H_0^2}\phi_{zz} - \frac{\tau H_0^2}{\mu_0\mu}\alpha_z\xi_x, \\
\beta_x &= -\frac{\mu_0}{H_0^2}\chi_{zz} - \frac{\tau H_0^2}{\mu_0}\beta_z\xi_x
\end{aligned}
$$

Bcs:

$$\phi = \mu\chi$$

$$\phi_z - \chi_z = \xi_x\frac{\mu_0}{H_0^2}(\mu\alpha - \beta)$$

$$W = \eta + \frac{H_0^2}{\mu_0}\left[\int_0^{\infty}\beta\chi_{\tilde{z}}d\tilde{z} + \frac{1}{\mu}\int_{-\infty}^{0}\alpha\phi_{\tilde{z}}d\tilde{z}\right].$$

- **Equations similar to two-layer interface water waves...**

Conclusion

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- **Justification of Wulff construction for PDEs? (Peletier...)**
- **Snaking near the singular limit?**
- **Ferrofluids:**
  - **Matching Experiments and Numerics**
  - **1D/2D pulses interface near onset - normal form?**