Localized oscillations in a nonvariational Swift-Hohenberg equation

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BIRS Workshop,

Localized Multi-Dimensional Patterns in Dissipative Systems

Outline

- Swift-Hohenberg equation
- snakes-and-ladders structure of localized states
- localized states in a nonvariational SH equation (3 results)

Swift-Hohenberg equation

The generalized Swift-Hohenberg equation is a well known (and well studied) example of a pattern forming system with nonzero wavenumber at onset:

$$\partial_t u = r u - (\partial_x^2 + 1)^2 u + b u^2 - u^3$$

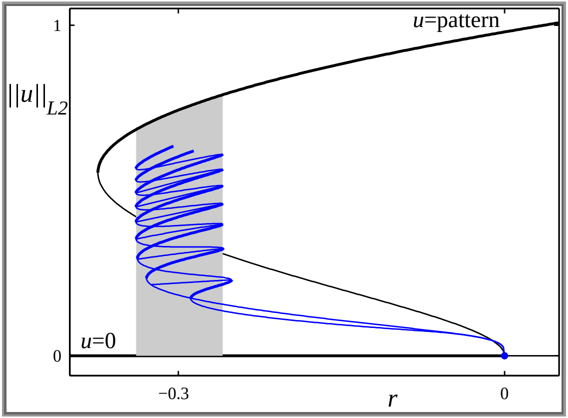
where $x \in R$, $u(x,t)$ is a real valued function, and (r, b) are real parameters.

The Swift-Hohenberg equation is variational so $\partial_t u = -\delta F / \delta u$ and $\partial_t F \leq 0$ along trajectories, for an appropriately defined 'energy' functional $F[u(x,t)]$.

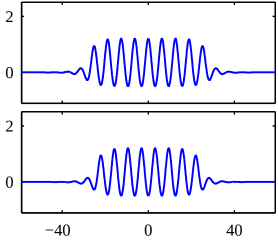
The Swift-Hohenberg equation is also conservative in x.

Snakes and ladders

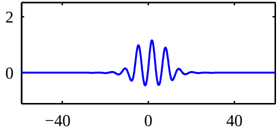
The Swift-Hohenberg equation contains stationary localized states organized in a snakes-and-ladders structure, which traces back to a subcritical bifurcation of the uniform state.



2 snaking branches ...



...and rungs



$b = 1.8$

A nonvariational Swift-Hohenberg equation

- The Swift-Hohenberg equation is unrealistic in most applications because it is variational in t and conservative in x.
- [Kozyreff and Tlidi, *Chaos* **17**, 037103 (2007)] The following equation is more applicable:

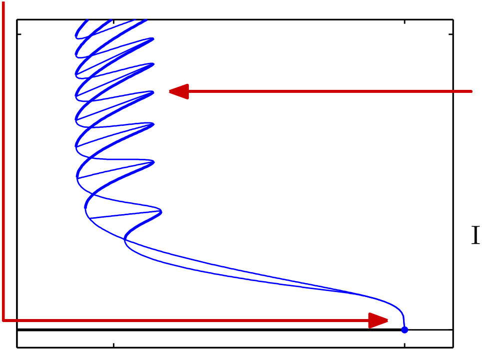
$$\partial_t u = ru - (\partial_x^2 + 1)^2 u + bu^2 - u^3 + \alpha_1 u_x^2 + \alpha_2 u u_{xx}$$

What is the effect of the nonvariational terms on the snakes-and-ladders structure of localized states?

NOTE: when $\alpha_2 = 2\alpha_1$ this equation remains variational / conservative.

A nonvariational Swift-Hohenberg equation

I: What happens to the bifurcation at $r = 0$, which creates the small amplitude localized states?



II: What aspects of the finite amplitude snakes-and-ladders structure persist?

III: What new behavior is introduced?

result I: the Hamiltonian-Hopf bifurcation

Snaking branches emerge from $r = 0$ in a Hamiltonian-Hopf bifurcation, with normal form:

$$A' = iA + B + iA\,P(\mu; |A|^2, \frac{i}{2}(A\bar{B} - \bar{A}B))$$
$$B' = iB + iB\,P(\mu; |A|^2, \frac{i}{2}(A\bar{B} - \bar{A}B)) + A\,Q(\mu; |A|^2, \frac{i}{2}(A\bar{B} - \bar{A}B)),$$

where A and B are complex variables, μ is the unfolding parameter, and P and Q are polynomials with real coefficients:

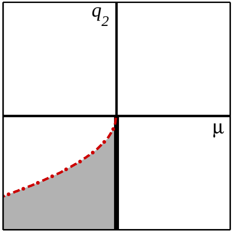
$$P(\mu; y, w) = p_1\mu + p_2 y + p_3 w + p_4 y^2 + p_5 wy + p_6 w^2$$
$$Q(\mu; y, w) = -q_1\mu + q_2 y + q_3 w + q_4 y^2 + q_5 wy + q_6 w^2 .$$

The phase-space structure depends crucially on q_2, q_4 and μ.

result I: the Hamiltonian-Hopf bifurcation

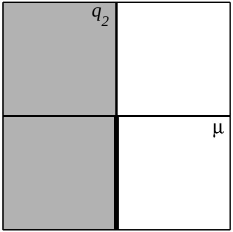
Homoclinic snaking is associated with a heteroclinic connection in the normal form, which is only present when: $\mu < 0$, $q_2 < 0$, $q_4 > 0$

- $\mu < 0 \Rightarrow$ the trivial state is hyperbolic
- $q_2 < 0 \Rightarrow$ the bifurcation is subcritical
- $q_4 > 0$, "good"



Woods and Champneys, *Physica D* **129**, 147 (1999)

$q_4 < 0$, "bad"



Dias and Iooss, *Eur. J. Mech. B* **15**, 367 (1996)

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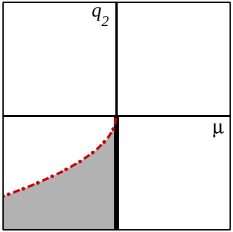
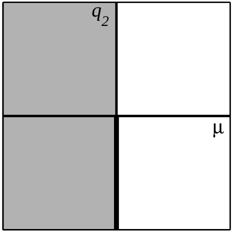


$q_4 < 0$, "bad"



For the variational Swift-Hohenberg equation:

- $(\mu, q_2) = (0, 0)$ is a point in parameter space, $(r, b) = (0, \sqrt{27/38})$
- the sign of q_4 is "good", $q_4|_{q_2=0} = 2202/361 > 0$

result I: the Hamiltonian-Hopf bifurcation

Normal form analysis gives q_2 in terms of the parameters in the nonvariational Swift-Hohenberg equation:

$$q_2(b, \alpha_1, \alpha_2) = (27 - \mathbf{p}^{\mathrm{T}} M \mathbf{p})/36$$

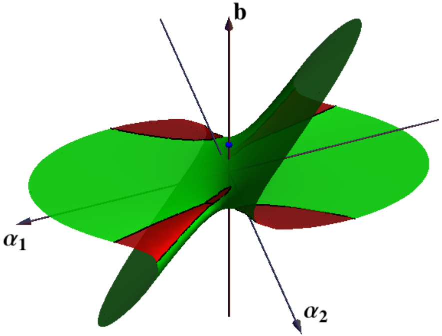
where $\mathbf{p} = [b \;\; \alpha_1 \;\; \alpha_2]^{\mathrm{T}}$

$$M = \begin{bmatrix} 38 & 17 & -61/2 \\ 17 & -4 & -17/2 \\ -61/2 & -17/2 & 23 \end{bmatrix}$$

The same calculation also gives a lengthy expression for $q_4(b, \alpha_1, \alpha_2)$.

result I: the Hamiltonian-Hopf bifurcation

In the nonvariational Swift-Hohenberg equation, the critical value $(\mu, q_2) = (0, 0)$ is a surface, and $q_4|_{q_2=0}$ changes sign on this surface.



$q_2 < 0$ outside the surface

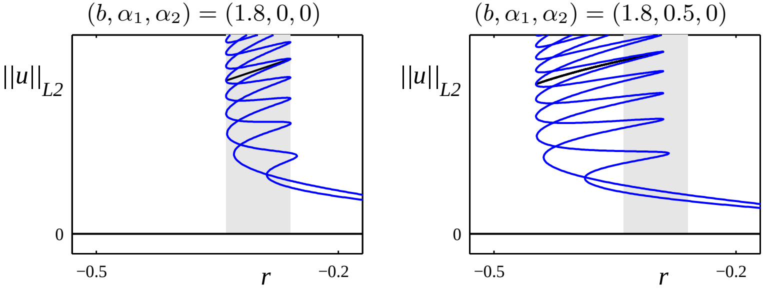
green: $q_4 > 0$ (good)
red: $q_4 < 0$ (bad)

$(b, \alpha_1, \alpha_2) = (1.8, 0, 0)$ is indicated

Which values of (b, α_1, α_2) lead to snaking?

result II: persistence

- the snakes-and-ladders structure persists to $\mathcal{O}(1)$ values of α_1, α_2.
- states on the snaking branches remain stationary, while states on the rungs travel.

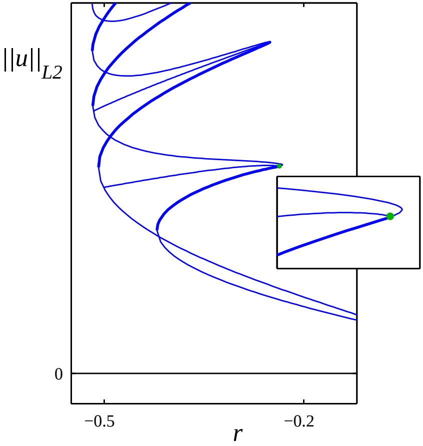


Noteworth because: many physical systems exhibit snaking of localized states, but are nonvariational (e.g., natural doubly diffusive convection: Bergeon and Knobloch, Phys. Fluids **20**, 034102, (2008))

result III: new behavior

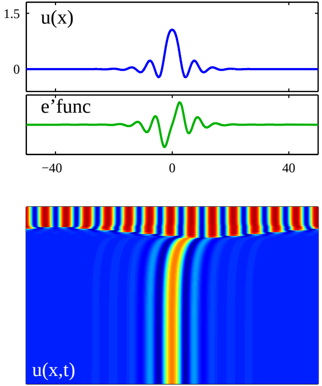
Secondary bifurcations reduce the range of existence of stable, stationary localized states.

$(b, \alpha_1, \alpha_2) = (1.8, 1.6, 0)$



unstable mode:

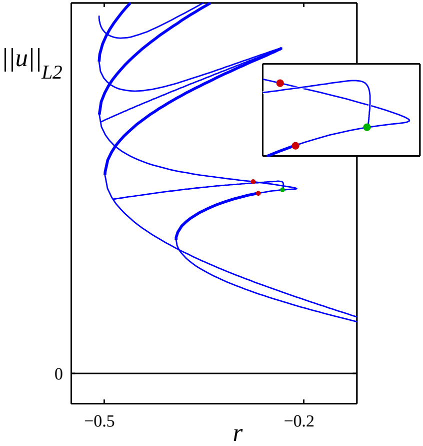
odd, with a real eigenvalue



result III: new behavior

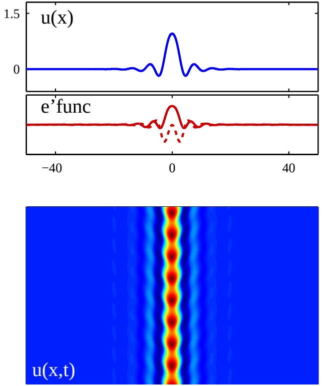
Secondary bifurcations reduce the range of existence of stable, stationary localized states.

$(b, \alpha_1, \alpha_2) = (1.8, 2, 0)$



unstable mode:

even, with a complex eigenvalue



Conclusions

Consider the effect of nonvariational terms on the snakes-and-ladders structure of localized states, in the context of the Swift-Hohenberg equation.

- result I: revisit the Hamiltonian-Hopf bifurcation
- result II: the snakes-and-ladders structure persists in the nonvariational system
- result III: new dynamics is also introduced, including localized Hopf bifurcations