Exponential asymptotic for fronts connecting an homogenous state and an hexagonal pattern

Gregory Kozyreff and Jon Chapman

Université libre de Bruxelles (ULB) [1]
Oxford University

July 25, 2011

[1] Fonds de la Recherche Scientifique -FNRS

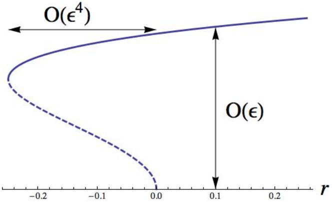
Outline

- Motivation: snaking 1D vs 2D
- Part 1: Sketch of the calculation technique in 1D
- Part 2: application to 2D localized hexagons

Required: subcritical Turing bifurcation. Simplest model: the Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} = ru + \nu u^2 - u^3 - \left(1 + \nabla^2\right)^2 u.$$

In 1D, subcritical Turing to roll pattern:

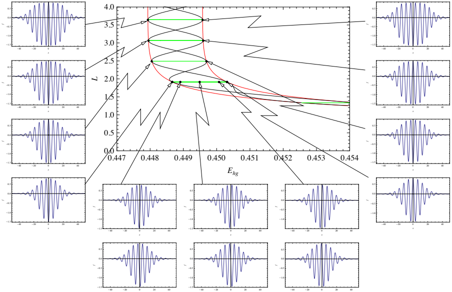


set $r = -\epsilon^4$, $u(x, y, t) = -\epsilon f(x)$, $\nu = 3E$:

$$\left(1 + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^2 f + \epsilon^4 f + 3E\epsilon f^2 + \epsilon^2 f^3 = 0.$$

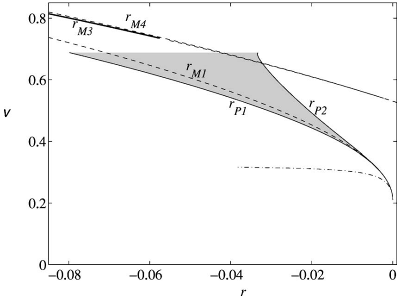
snaking 1D vs 2D

Inside the pinning range:



Woods & Champneys, Physica D 1999,
Hunt, Lord & Champneys (1999),
Beck et al 2009...

Pinning range:

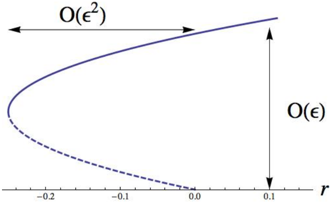


Burke and Knobloch, PRE 2006

For $\epsilon \ll 1$, the pinning range $\sim \epsilon^{-4}\exp(-\pi/\epsilon^2)$. (Kozyreff Chapman PRL 2006, Physica D 2009)

$$\frac{\partial u}{\partial t} = ru + \nu u^2 - u^3 - \left(1+\nabla^2\right)^2 u.$$

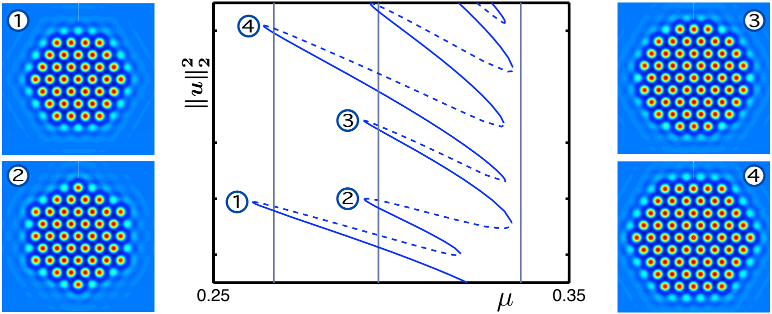
In 2D, subcritical bifurcation diagram for extended hexagon pattern:



set $r = -\epsilon^2$, $u(x, y, t) = -\epsilon f(x, y)$, $\nu = 3E$:

$$\left(1+\nabla^2\right)^2 f + \epsilon^2 f + 3E\epsilon f^2 + \epsilon^2 f^3 = 0.$$

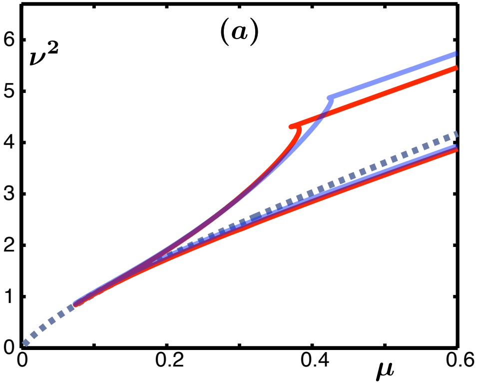
The snaking diagram for localized hexagons is more complicated.



Lloyd et al SIADS 2008

The width of the snaking oscillations depends on the growth direction.

How does the width of the pinning region scale for small ϵ?

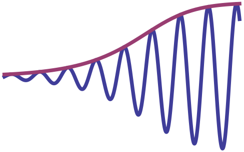


Lloyd et al.
SIADS 2008

Study

$$\left(1+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^2 f+\epsilon^4 f+3E\epsilon f^2+\epsilon^2 f^3=0, \qquad \epsilon \ll 1.$$

Look for front solutions $f \sim A(X)\left(e^{\mathrm{i}(x+\varphi)}+\text{c.c.}\right)+\epsilon f_1+\epsilon^2 f_2+\ldots,$



where $X=\epsilon^2 x$, φ arbitrary phase.
Define $\tilde{x}=x+\varphi$.

Standard multiple scale analysis yields a Ginzburg-Landau eq., which can be put in the form

$$\frac{\mathrm{d}^2 A}{\mathrm{d}X^2}+\frac{\partial V(A)}{\partial A}=0.$$

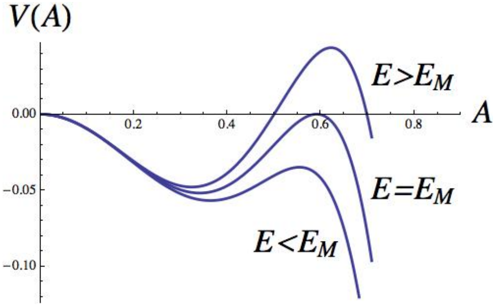
Calculation in 1D

This allows one to identify the Maxwell value $E_M(\epsilon)$:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}X^2} + \frac{\partial V(A)}{\partial A} = 0.$$

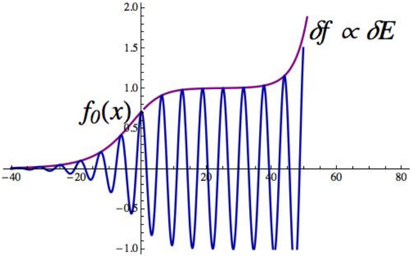
At $E = E_M$, front solution

$$|A| \propto \frac{1}{\sqrt{1+e^{-X}}}.$$



Calculation in 1D

Standard theory predicts that, away from E_M, say with $E = E_M + \delta E$, the front solution ceases to exist:



We have $f \sim f_0 + \epsilon f_1 + \ldots + \delta f$ with

$$\boxed{\delta f \propto \epsilon^{-2} \delta E \, e^{X} \left(e^{\mathrm{i}\tilde{x}} + \text{c.c.}\right)}$$

as $X \to \infty$.

The theory, however, misses some exponentially small terms, "$R_N(\varphi)$", which also appear in δf and which can counterbalance the divergence above. The balance between the two yields the finite pinning range

$$\delta E = \delta E(\varphi), \qquad 0 \le \varphi < 2\pi.$$

Calculation in 1D

In the multi-scale approach, one assumes $f \sim f_0(x,X) + \epsilon f_1(x,X) + \ldots \epsilon^n f_n(x,X)$, $X = \epsilon^2 x$. This makes the perturbation problem singular: a small parameter multiplies the highest derivative in X

$$\left(1+\frac{\partial^2}{\partial x^2}\right)^2 f \to \left(1+\left(\frac{\partial}{\partial x}+\epsilon^2\frac{\partial}{\partial X}\right)^2\right)^2 f$$

$$= \epsilon^8 \frac{\partial^4 f}{\partial X^4} + 4\epsilon^6 \frac{\partial^4 f}{\partial x \partial X^3} + 6\epsilon^4 \frac{\partial^4 f}{\partial x^2 \partial X^2} + 4\epsilon^2 \frac{\partial^4 f}{\partial x^3 X} + \left(1+\frac{\partial^2}{\partial x^2}\right)^2 f.$$

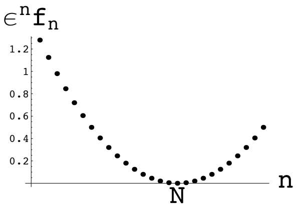
As a result, when solving at $O(\epsilon^n)$, one gets

$$f_n \propto \frac{\partial f_{n-2}}{\partial X}, \frac{\partial^2 f_{n-4}}{\partial X^2}, \frac{\partial^3 f_{n-6}}{\partial X^3}, \frac{\partial^4 f_{n-8}}{\partial X^4}.$$

Now notice that $f_0 \propto (e^{i\tilde{x}} + \text{c.c.})/\sqrt{1+e^{-X}}$ has singularities in the complex plane: $X_0 = \pm i\pi$. Hence,

$$f_0 \propto (X - i\pi)^{-1/2} \text{ and, at higher order, } f_n \propto (X - i\pi)^{-n/2-1/2}.$$

Besides, since the equation for f_n contains $\partial f_{n-2}/\partial X$, differentiation yields that $f_n \propto n f_{n-2}, n^2 f_{n-4}, \ldots$ and every second order, the terms in the asymptotic expansion get bigger by a factor n. Hence the series diverges.



One should therefore truncate the expansion and compute the remainder $R_N(\varphi)$.

Calculation in 1D

Let

$$f = \sum_{n=0}^{N-1} \epsilon^n f_n(x, X) + R_N.$$

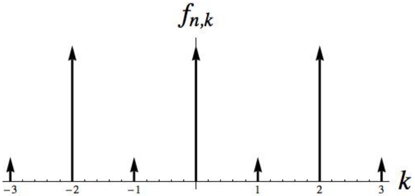
The equation for R_N is essentially the linearized S-H equation + inhomogeneous terms coming from the truncation.

$$\left(1+\partial_x^2\right)^2 R_N + 4\epsilon^2\left(1+\partial_x^2\right) R_{N_{xX}} + 2\epsilon^4\left(1+3\partial_x^2\right) R_{N_{XX}} + 4\epsilon^6 R_{N_{xXXX}}$$
$$+ \epsilon^8 R_{N_{XXXX}} + 3\epsilon^2(f_0^2 + \cdots)R_N + \epsilon^4 R_N + 6\epsilon E_M(f_0 + \epsilon f_1 + \cdots)R_N$$
$$\sim \epsilon^N\left(1+\partial_x^2\right)^2 f_N + \epsilon^{N+2}(-6f_{N-2_{xxXX}} - 4f_{N-4_{xXXX}} - f_{N-6_{XXXX}}$$
$$-2f_{N-2_{XX}}) + \epsilon^{N+4}(-4f_{N-2_{xXXX}} - f_{N-4_{XXXX}}) + \epsilon^{N+6}(-f_{N-2_{XXXX}}) + \cdots$$

For large n, one finds that

$$\epsilon^n f_n \sim \epsilon^n \frac{e^{in\pi/4}\Gamma(n/2+\alpha)}{(\mathrm{i}\pi - X)^{n/2+\alpha}} \left(F_0(X) + F_2(X)e^{2\mathrm{i}\tilde{x}}\right) + \text{c.c. (...plus other terms)}$$

when n is odd, where α is determined by an analysis in the vicinity of $X = \mathrm{i}\pi$.



Fourier components at n^{th} order.

The spectrum of the late-terms is thus essentially non resonant **but**... at some places, the denominator above oscillates violently.

1) Stirling formula:

$$\Gamma(n/2+\alpha) \sim \sqrt{2\pi n}(n/2)^{n/2+\alpha}e^{-n/2}$$

2) Just below the singularity, let $X = \mathrm{i}\pi - \mathrm{i}r + \xi$, with $\xi \ll r$.
Then

$$\mathrm{i}\pi - X = -\mathrm{i}r(1+\mathrm{i}\xi/r) \sim -\mathrm{i}re^{\mathrm{i}\xi/r+\frac{1}{2}(\xi/r)^2}.$$

Thus

$$\epsilon^n \frac{e^{\mathrm{i}n\pi/4}\Gamma(n/2+\alpha)}{(\mathrm{i}\pi - X)^{n/2+\alpha}} \sim \sqrt{2\pi n}\left(\frac{n}{2r}\right)^{\alpha}\left(\frac{\epsilon^2 n}{2r}\right)^{n/2} e^{-n\mathrm{i}\xi/(2r)-n(\xi/2r)^2}e^{-n}.$$

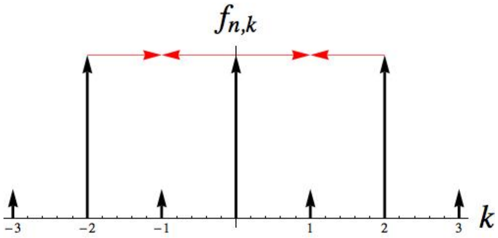
Optimal truncation: $n \sim 2r/\epsilon^2$.

$$\to \epsilon^n f_n \sim \sqrt{2\pi n}\left(\frac{n}{2r}\right)^{\alpha} e^{-\pi/\epsilon^2}\left(F_0(X)e^{-\mathrm{i}\tilde{x}} + F_2(X)e^{\mathrm{i}\tilde{x}}\right)e^{\mathrm{i}\varphi}e^{-\xi^2/2r\epsilon^2}.$$

We found that, in the region of the complex plane described by $X = \mathrm{i}\pi - \mathrm{i}r + \xi$, $\xi \ll r$,

$$\epsilon^n f_n \sim \sqrt{2\pi n}\left(\frac{n}{2r}\right)^{\alpha} e^{-\pi/\epsilon^2}\left(F_0(X)e^{-\mathrm{i}\tilde{x}} + F_2(X)e^{\mathrm{i}\tilde{x}}\right)e^{\mathrm{i}\varphi}e^{-\xi^2/2r\epsilon^2} + \text{c.c.}.$$

Hence, over a distance $\xi = O(\sqrt{2r}\epsilon)$, the late terms of the asymptotic series $\sum \epsilon^n f_n$ become resonant.



Calculation in 1D

This is what generate a remainder

$$R_N(\varphi) \propto e^{-\pi/\epsilon^2}\epsilon^{-6}\cos(\varphi+\ldots)e^{X}(e^{\mathrm{i}\tilde{x}}+\text{c.c.}), \quad \text{as } X\to\infty,$$

eventually yielding

$$\boxed{\delta E \propto \epsilon^{-4}e^{-\pi/\epsilon^2}\cos(\varphi+\ldots).}$$

(full story in Physica D 2009)

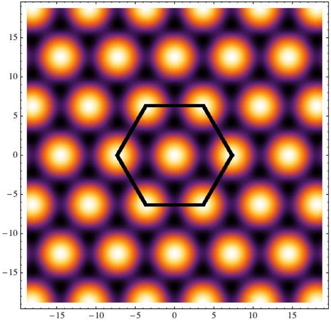
1D lessons

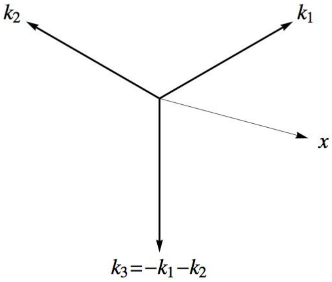
1. The physics of pinning is "beyond all orders".
2. One must look at singularities of the front.
3. The two nearest singularities, $\pm i\pi$, are joined by a Stokes line.
4. This is where the slow and fast scale interact and produce $R_N(\varphi)$.
5. $R_N(\varphi)$ compensates deviation δE from Maxwell to give finite pinning range.
6. $R_N(\varphi)$ also allows the pattern to accommodate distant boundary conditions in large but finite domains (not shown here, see PRL 2009.)
7. Not limited to SH. See numerous hydrodynamical studies and a recent study of the Lugiato-Lefever model in optics (to appear in PRA.)

Let us now consider hexagonal patterns. We thus study

$$\left(1+\nabla^2\right)^2 f+\epsilon^2 f+3E\epsilon f^2+\epsilon^2 f^3=0.$$

An extended pattern is given by $f \sim \sum_{i=1}^{3} A_i e^{\mathrm{i}\mathbf{k}_i\cdot\tilde{\mathbf{x}}} + \text{c.c.} + O(\epsilon)$, where $\tilde{\mathbf{x}} = \mathbf{x} - \varphi\hat{\mathbf{x}}$ and φ is an arbitrary phase.





Hexagons - coupled Ginzburg-Landau's

Consider a straight front perpendicular to the x-direction. With the slow scale $X = \epsilon x$, the multiple-scale analysis yields $E = \epsilon E_1$ and

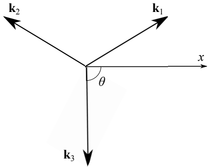
$$4k_{1,x}^2 \frac{\mathrm{d}^2 A_1}{\mathrm{d}X^2} = A_1 \left(1 + 3A_1^2 + 6A_2^2 + 6A_3^2\right) + 6E_1 A_2 A_3,$$
$$4k_{2,x}^2 \frac{\mathrm{d}^2 A_2}{\mathrm{d}X^2} = A_2 \left(1 + 3A_2^2 + 6A_3^2 + 6A_1^2\right) + 6E_1 A_3 A_1,$$
$$4k_{3,x}^2 \frac{\mathrm{d}^2 A_3}{\mathrm{d}X^2} = A_3 \left(1 + 3A_3^2 + 6A_1^2 + 6A_2^2\right) + 6E_1 A_1 A_2.$$

where we used the fact that A_i can be taken real and positive for hexagons.

- *No front solution between a hexagonal pattern and an homogenous solution is documented.*

Hexagons - a front solution

One case is doable: $k_{3x} = 0$, i.e. the front is aligned with one of the $\mathbf{k}_i$. This corresponds to one of the principal growth direction in Lloyd et al. 2008 paper. By symmetry, $A_1 = A_2$. We have

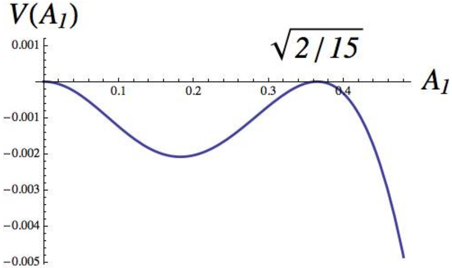


$$3\frac{\mathrm{d}^2 A_1}{\mathrm{d}X^2} = A_1\left(1 + 9A_1^2 + 6A_3^2\right) + 6E_1 A_1 A_3,$$
$$0 = A_3\left(1 + 3A_3^2 + 12A_1^2\right) + 6E_1 A_1^2.$$

By eliminating A_3 from the second equation, we get

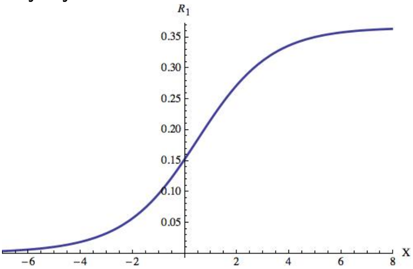
$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}X^2} + \frac{\partial V(A_1, E_1)}{\partial A_1} = 0,$$

where V is nasty but OK. The Maxwell point is $E_1 = -\sqrt{15/8}$.



A front solution is then obtained implicitly by

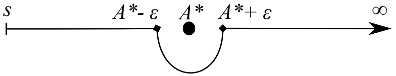
$$X(A_1) = X(S) + \int_S^{A_1} \frac{1}{\sqrt{-2V(A')}} \mathrm{d}A'.$$



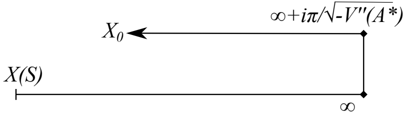
To locate the singularity, let A_1 tend to infinity through real values. A complex jump takes place at $A^* = \sqrt{2/15}$, where V vanishes:

$$X_0 = \lim_{\epsilon \to 0} X(S) + \int_S^{A^*-\epsilon} \frac{\mathrm{d}A_1}{\sqrt{-2V(A_1)}} + \frac{\mathrm{i}\pi}{\sqrt{-V''(A^*)}} - \int_{A^*+\epsilon}^{\infty} \frac{\mathrm{d}A_1}{\sqrt{-2V(A_1)}}.$$

A_1- plane:



X-plane:



Hexagons - a front solution

Hence

$$\text{Im}(X_0) = \frac{\pm\pi}{\sqrt{-V''(A^*)}} = \pm\sqrt{\frac{57}{26}}\pi \qquad (k_{3x} = 0)$$

only depends on the quadratic part of V near the constant amplitude $A^* = \sqrt{2/15}$. In other words,

$$\text{Im}(X_0) = \frac{\pm\pi}{\lambda},$$

where λ is an eigenvalue of the linearized dynamics around A^*: $A_1 = A^* + \delta A_1 e^{\pm\lambda X}$.

This observation can be applied to other front orientations, for which an effective 1D potential cannot be found. In particular, for a front normal to $\mathbf{k}_3$, we find in this way that

$$\text{Im}(X_0) = \frac{\pm\pi}{\lambda} = \frac{\pm\pi\sqrt{40}}{\sqrt{67 - \sqrt{2409}}}, \qquad (k_{3x} = 1).$$

Hexagons - near the singularity

As $X \to X_0$, we may assume that

$$A_1 \sim \frac{B_{0,1,0}}{X - X_0}, \quad A_2 \sim \frac{B_{0,0,-1}}{X - X_0}, \quad A_3 \sim \frac{B_{0,-1,-1}}{X - X_0},$$

where B_{n,m_1,m_2} refers to n^{th} order of the asymptotic expansion and to wave vector $\mathbf{q} = m_1\mathbf{k}_1 + m_2\mathbf{k}_2$. Substituting into the Ginzburg Landau equations, we find that

$$B_{0,1,0} = B_{0,0,1} = \sqrt{2/3}, \quad B_{0,-1,-1} = 0.$$

Hexagons - near the singularity

We may pursue the investigation to higher orders. With $B_{n,m_1,m_2} = B_{n,\mathbf{q}}$, $q = \sqrt{\mathbf{q}\cdot\mathbf{q}}$,

$$\left(1-q^2\right)^2 B_{n,\mathbf{q}} - 4\mathrm{i}nq_x\left(1-q^2\right)B_{n-1,\mathbf{q}} + 2n\left(n-1\right)\left(1-q^2-2q_x^2\right)B_{n-2,\mathbf{q}}$$
$$-4\mathrm{i}n\left(n-1\right)\left(n-2\right)q_x B_{n-3,\mathbf{q}} + n\left(n-1\right)\left(n-2\right)\left(n-3\right)B_{n-4,\mathbf{q}}$$
$$+\sum_{j=0}^{n-2}\sum_{m=0}^{n-2-j}\sum_{\mathbf{q}'}\sum_{\mathbf{q}''} B_{j,\mathbf{q}'}B_{m,\mathbf{q}''}B_{n-2-j-m,\mathbf{q}-\mathbf{q}'-\mathbf{q}''} = 0.$$

At every order, new wave vectors are excited by the nonlinearity. The recurrence relation invites us to look for solutions of the form

$$B_{n,\mathbf{q}} \sim \kappa^n \Gamma(n+\alpha_{\mathbf{q}}) b_{\mathbf{q}}$$

for large n. Through the offset $\alpha_{\mathbf{q}}$, some wave vectors dominates the others.

- κ is an eigenvalue of the recurrence relation.

The idea behind all this

From what precedes, we expect that, for large n,

$$\epsilon^n f_n \sim \text{const} \times \frac{\epsilon^n \kappa^n \Gamma(n+\alpha)}{(X-X_0)^{n+\alpha}} e^{\mathrm{i}\mathbf{q}\cdot\tilde{\mathbf{x}}},$$

for some dominating wave vector $\mathbf{q}$, with $\tilde{\mathbf{x}} = \mathbf{x} - \varphi\hat{\mathbf{x}}$. Following the same reasoning as in 1D, the factorial over power will turns this into

$$e^{\mathrm{i}\mathbf{q}\cdot\tilde{\mathbf{x}} - \mathrm{i}x/|\kappa| + \mathrm{i}X_0/(\epsilon|\kappa|) - \frac{1}{2}\xi^2/(r\epsilon|\kappa|)}.$$

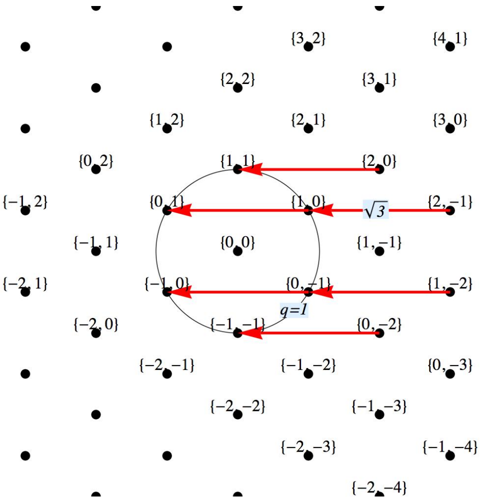
Hence, the dominating wave vector $\mathbf{q}$ can be brought onto some vectors of the basic triad $\pm\mathbf{k}_i$, $i = 1, 2, 3$ if the right eigenvalue κ is excited:

$$\mathbf{q} - \frac{1}{\kappa}\hat{\mathbf{x}} = \pm\mathbf{k}_i$$

The idea behind all this

When $k_{3x} = 0$, one expected eigenvalue is $-\mathrm{i}/\sqrt{3}$ and the corresponding set of dominating wave vectors are $2\mathbf{k}_1 - \mathbf{k}_2$, $\mathbf{k}_1 - 2\mathbf{k}_2$, $\mathbf{k}_1$, $-\mathbf{k}_2$, $2\mathbf{k}_1$, and $-2\mathbf{k}_2$.

The complementary singularity $\bar{X}_0$ will "activate" the wave vectors $2\mathbf{k}_2 - \mathbf{k}_1$, $\mathbf{k}_2 - 2\mathbf{k}_1$, $\mathbf{k}_2$, $-\mathbf{k}_1$, $2\mathbf{k}_2$, and $-2\mathbf{k}_1$



Outer analysis, with $k_{3x} = 0$

Indeed, we found that, among others, f_n contains the terms

$$\frac{\Gamma(n+4)}{(X-X_0)^{n+1}}\left(\frac{-\mathrm{i}}{\sqrt{3}}\right)^n \mathcal{B}\left(e^{\mathrm{i}(2\mathbf{k}_1-\mathbf{k}_2)\cdot\tilde{\mathbf{x}}}+e^{\mathrm{i}\mathbf{k}_1\cdot\tilde{\mathbf{x}}}+e^{-\mathrm{i}\mathbf{k}_2\cdot\tilde{\mathbf{x}}}+e^{\mathrm{i}(\mathbf{k}_1-2\mathbf{k}_2)\cdot\tilde{\mathbf{x}}}\right)$$

for $n \gg 1$ as $X \to X_0$.
Away from $X = X_0$, we are able to match this with the outer expansion

$$\frac{\Gamma(n+4)}{(X-X_0)^{n+4}}\left(\frac{-\mathrm{i}}{\sqrt{3}}\right)^n \left[F(X)\left(e^{\mathrm{i}(2\mathbf{k}_1-\mathbf{k}_2)\cdot\tilde{\mathbf{x}}}+e^{\mathrm{i}\mathbf{k}_1\cdot\tilde{\mathbf{x}}}+e^{-\mathrm{i}\mathbf{k}_2\cdot\tilde{\mathbf{x}}}+e^{\mathrm{i}(\mathbf{k}_1-2\mathbf{k}_2)\cdot\tilde{\mathbf{x}}}\right)\right.$$
$$\left.+\phi(X)\left(e^{2\mathrm{i}\mathbf{k}_1\cdot\tilde{\mathbf{x}}}+e^{-2\mathrm{i}\mathbf{k}_2\cdot\tilde{\mathbf{x}}}\right)\right],$$

where $F(X)$ and $\phi(X)$ satisfies the linearized amplitude equations:

$$\frac{\mathrm{d}^2 F}{\mathrm{d}X^2}+\frac{\partial^2 V(A_1,E_M)}{\partial A_1^2}F(X)=0, \qquad (F(X)\sim e^{\lambda X}, X\to\infty.)$$

Outer analysis, with $k_{3x} = 0$

As in the 1D problem, the late terms of the multiple-scale expansion will thus switch on an exponentially small hexagon amplitude in the remainder $R_N(\varphi)$.

$$R_N^{(S)} \sim -36\mathrm{i}\pi\epsilon^{-4}e^{\mathrm{i}(X_0/\epsilon-\varphi)/|\kappa|}\left[F(X)\left(e^{\mathrm{i}\mathbf{k}_1\cdot\tilde{\mathbf{x}}}+e^{-\mathrm{i}\mathbf{k}_1\cdot\tilde{\mathbf{x}}}+e^{\mathrm{i}\mathbf{k}_2\cdot\tilde{\mathbf{x}}}+e^{-\mathrm{i}\mathbf{k}_2\cdot\tilde{\mathbf{x}}}\right)\right.$$
$$\left.+\phi(X)\left(e^{\mathrm{i}\mathbf{k}_3\cdot\tilde{\mathbf{x}}}+e^{-\mathrm{i}\mathbf{k}_3\cdot\tilde{\mathbf{x}}}\right)\right]+\text{c.c.}$$

This amplitude grows with X and can compensate for a deviation δE from the Maxwell point $E = \epsilon E_M$. At the end of the day, we obtain

$$\boxed{\delta E = \frac{36\pi\Lambda\epsilon^{-3}e^{-\frac{\mathrm{Im}(X_0)}{\epsilon|\kappa|}}}{0.0164\ldots}\sin\left(\frac{\varphi-\mathrm{Re}(X_0)/\epsilon}{|\kappa|}-\chi\right).}$$

Discussion

In

$$\delta E \propto \epsilon^{-3} e^{-\frac{\mathrm{Im}(X_0)}{\epsilon|\kappa|}} \sin\left(\frac{\varphi - \mathrm{Re}(X_0)/\epsilon}{|\kappa|} - \chi\right),$$

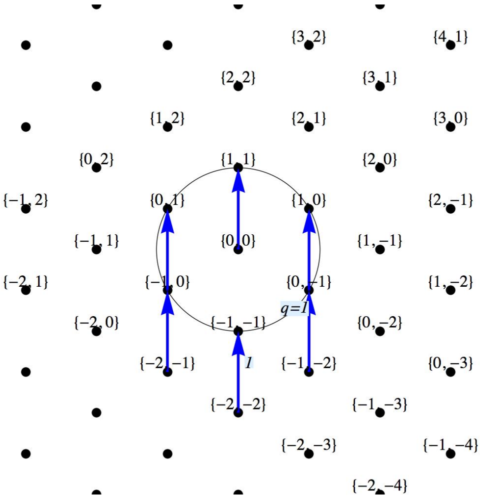
$\kappa = -\mathrm{i}/\sqrt{3}$ is the eigenvalue of the recurrence relation, which is associated to a shift in the lattice of wave vectors in the x-direction. More precisely, $|\kappa|^{-1} = \Delta k$. On the other hand, $\mathrm{Im}(X_0)$ was found to correspond to the rate at which the front tends to the constant amplitude $\sqrt{2/15}$. Hence, the exponential factor above can be written as

$$\exp\left(\frac{-\pi \Delta k}{\epsilon \lambda}\right).$$

It is thus controlled by the ratio of the actual periodic scale and relaxation scale in the direction normal to the front.

Discussion

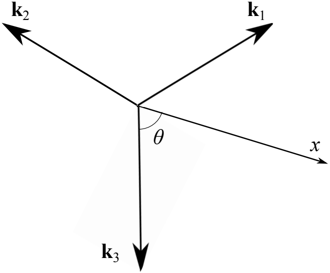
By the same token, if we consider a front oriented so that $k_{3x} = 1$, we expect $\kappa = -\mathrm{i}$



Discussion

The same formula is expected for the pinning range as with $k_{3x} = 0$ but this time with X_0 and κ corresponding to different direction. In general, we expect the pinning range to scale as

$$\epsilon^{-3} \exp\left(\frac{-\pi \Delta k(\theta)}{\epsilon \lambda(\theta)}\right).$$



In particular, $\frac{\Delta k(0)}{\lambda(0)} < \frac{\Delta k(\pi/2)}{\lambda(\pi/2)}$, and therefore the pinning range and snaking is wider when $\theta = 0$ (as in Lloyd 2008) . Due to $\Delta k(\theta)$, the pinning range is expected to be much smaller for directions different from $\theta = 0, \pi/2$ or equivalent.