Existence and stability of spike-type solutions to one dimensional Gierer-Meinhardt model with sub-diffusion

Yana Nec and Michael Ward

Mathematics Department, UBC

Introduction

- ‡ Mathematical model:
  activator, inhibitor, sub-diffusion – reaction equations
  ∗ loss of basic analysis tools

- ‡ Quasi-equilibrium patterns:
  slow evolution of an arbitrary set of spikes
  ∗ loss of inflexion invariance

- ‡ $\mathcal{O}(1)$ eigenvalues:
  instability onset thresholds
  ∗ no spectrum or related theorems
  ∗ no tracking of eigenvalues in the left half plane

sustained patterns for realistic diffusivities ratios

$$\mathcal{O}\left(\epsilon^2\right) \quad \rightsquigarrow \quad \mathcal{O}\left(\epsilon^{2\gamma}\right) \qquad 0 \leqslant \gamma < 1$$

Sub-diffusion and fractional derivatives

$$\langle r^2(t)\rangle \sim t^\gamma \qquad \text{stagnation} \longleftarrow 0 \leqslant \gamma \leqslant 1 \longrightarrow \text{normal}$$

$$\partial_t - \nabla^2 \quad \longmapsto \quad \partial_t^\gamma - \nabla^2$$

$$\frac{d^\gamma}{dt^\gamma} f(t) = -\frac{1}{\Gamma(-\gamma)} \int_0^t \frac{f(t) - f(t-\zeta)}{\zeta^{\gamma+1}}\, d\zeta \qquad 0 < \gamma < 1.$$

$$\partial_t^\gamma a = \epsilon^{2\gamma} a_{xx} - a + \frac{a^p}{h^q}, \qquad -1 < x < 1, \quad t > 0$$

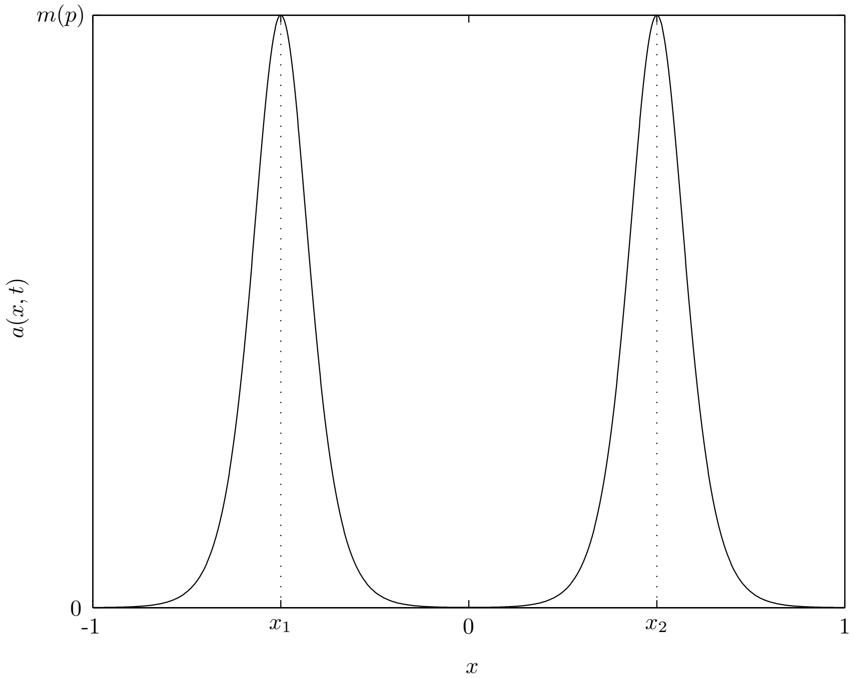
$$\tau_o\, \partial_t^\gamma h = D\, h_{xx} - h + \epsilon^{-\gamma} \frac{a^m}{h^s}, \qquad -1 < x < 1, \quad t > 0$$

$$a_x(\pm 1, t) = h_x(\pm 1, t) = 0$$

$$a(x,0) = a_0(x), \qquad h(x,0) = h_0(x)$$

$$\tau_o > 0 \qquad D = \vartheta_o^{-2} > 0 \qquad 0 < \epsilon \ll 1$$

$$p > 1, \quad q > 0, \quad m > 0, \quad s \geqslant 0, \quad 0 < \frac{p-1}{q} < \frac{m}{s+1}$$



Quasi-equilibrium

n-spike pattern: inner layer about spike foci

$$y_i(t) = \frac{x - x_i(\tau)}{\epsilon^\gamma} \qquad \tau = \epsilon^{\gamma+1} t \qquad i = \{0, \ldots, n-1\}$$

Inner solution

$$A(y_i, \tau) = a(x_i + \epsilon^\gamma y_i, \epsilon^{-(\gamma+1)}\tau) = A_i^{(0)} + \epsilon^\gamma A_i^{(1)} + \ldots$$

$$H(y_i, \tau) = h(x_i + \epsilon^\gamma y_i, \epsilon^{-(\gamma+1)}\tau) = H_i^{(0)} + \epsilon^\gamma H_i^{(1)} + \ldots$$

$*$ loss of inflexion invariance:

for $A(y_i(\tau)) \in C^\infty,\ y_i(\tau) \in C^\infty$ and $\epsilon \longrightarrow 0$

$$\partial_\tau^\gamma A\left(y_i(\tau)\right) \sim -\epsilon^{-\gamma^2} s \ \left|\frac{dx_i}{d\tau}\right|^\gamma \mathfrak{D}_{y_i}^\gamma A(y_i) \qquad s = \operatorname{sgn} \frac{dx_i}{d\tau}$$

$$\mathfrak{D}_{y_i}^\gamma A(y_i) = \frac{s}{\Gamma(-\gamma)} \int_0^\infty \frac{A(y_i) - A\left(y_i + s\xi\right)}{\xi^{\gamma+1}}\, d\xi$$

$$H_i^{(0)} = \bar{H}_i(\tau) \qquad A_i^{(0)} = \bar{H}_i^\beta(\tau) u(y_i) \qquad \beta = \frac{q}{p-1}$$

$$u'' - u + u^p = 0 \qquad \lim_{|y| \longrightarrow \infty} u(y) = 0$$

$$u(y) = \left\{ \frac{p+1}{2} \ \operatorname{sech}^2 \left(\frac{p-1}{2}\ y \right) \right\}^{\frac{1}{p-1}}$$

Approach to quasi-equilibrium

equation of motion for two symmetric spikes

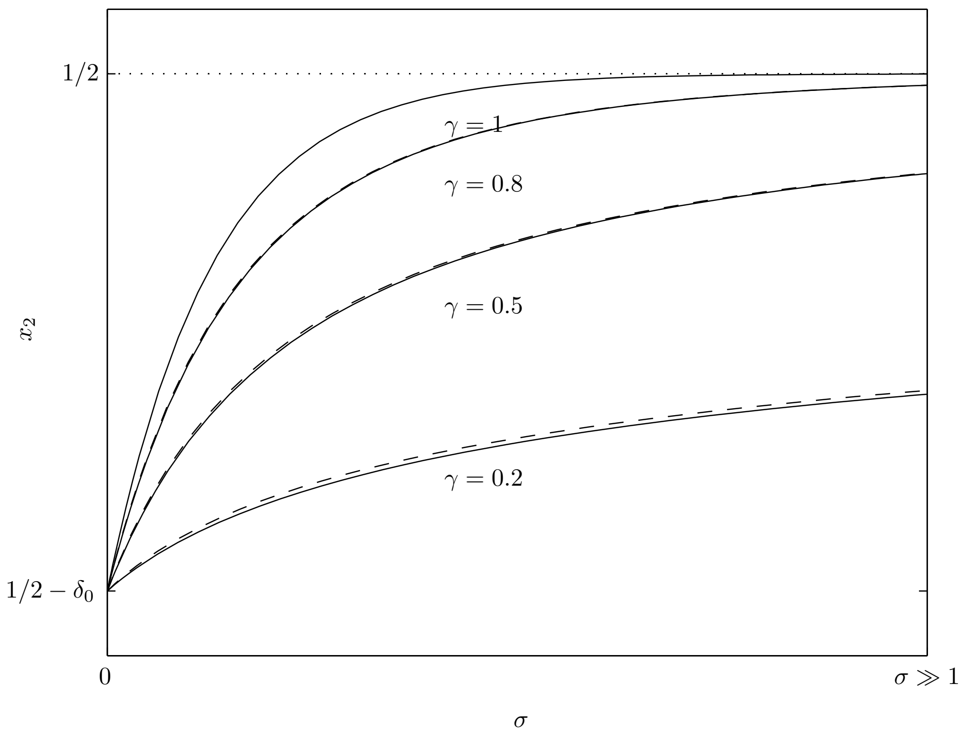
$$|x_1'|^\gamma = \frac{q\vartheta_o f(p;\gamma)}{2(p+1)} \Big\{ \mathrm{tgh}\Big(\vartheta_o(1-x_1)\Big) - \mathrm{tgh}\Big(\vartheta_o x_1\Big) \Big\}$$

linearised form

$$\delta' \sim -\mu\, \delta^{1/\gamma}\,, \qquad \mu = \left(\frac{q f(p;\gamma)}{(p+1)D}\, \mathrm{sech}^2\, \frac{\vartheta_o}{2} \right)^{1/\gamma}$$

solution with $\delta(0) = \delta_0$

$$\delta \sim \left(\delta_0^{(\gamma-1)/\gamma} + \frac{\mu(1-\gamma)\tau}{\gamma} \right)^{-\gamma/(1-\gamma)}$$



$O(1)$ **eigenvalues**

quasi-equilibrium

$$a_{qe} = \sum_{i=0}^{n-1} \bar{H}_i^{\beta}(\tau) u \left(\frac{x - x_i}{\epsilon^\gamma} \right)$$

$$h_{qe} = -b_m \sum_{i=0}^{n-1} \bar{H}_i^{\beta m - s}(\tau) G(x; x_i)$$

perturbations with exponential growth

$$a \sim a_{qe} + e^{\lambda t} \left(\tilde{a}^{(0)} + \epsilon^\gamma \tilde{a}^{(1)} + \dots \right)$$

$$h \sim h_{qe} + e^{\lambda t} \left(\tilde{h}^{(0)} + \epsilon^\gamma \tilde{h}^{(1)} + \dots \right)$$

$$\lambda(t) \sim \lambda^{(0)} + \epsilon^\gamma \lambda^{(1)}(t) + \dots \qquad \lambda^{(0)} = \text{const}$$

no tracking in the left-half plane

$$e^{-\lambda^{(0)} t} \frac{d^\gamma}{dt^\gamma} e^{\lambda^{(0)} t} = -\frac{1}{\Gamma(-\gamma)} \int_0^t \frac{1 - e^{-\lambda^{(0)} \zeta}}{\zeta^{\gamma+1}} \, d\zeta \sim$$

$$\lambda^{(0)\gamma} + \mathcal{O}\left(\epsilon^{\gamma(\gamma+1)} \right) \quad \text{if and only if} \quad \Re \lambda^{(0)} \geqslant 0$$

Non-local eigenvalue problem: n identical spikes

eigenfunction form

$$\tilde{a}^{(0)} \sim \sum_{i=0}^{n-1} \tilde{a}_i^{(0)} \left(\frac{x - x_i}{\epsilon^\gamma} \right) = \sum_{i=0}^{n-1} c_i \tilde{A}^{(0)}(y_i)$$

NLEP

$$\left(\frac{d^2}{dy^2} - 1 + pu^{p-1} \right) \tilde{A}^{(0)} = {\lambda^{(0)}}^\gamma \tilde{A}^{(0)} + \frac{\chi u^p}{b_m} \int_{-\infty}^{\infty} u^{m-1} \tilde{A}^{(0)} dy$$

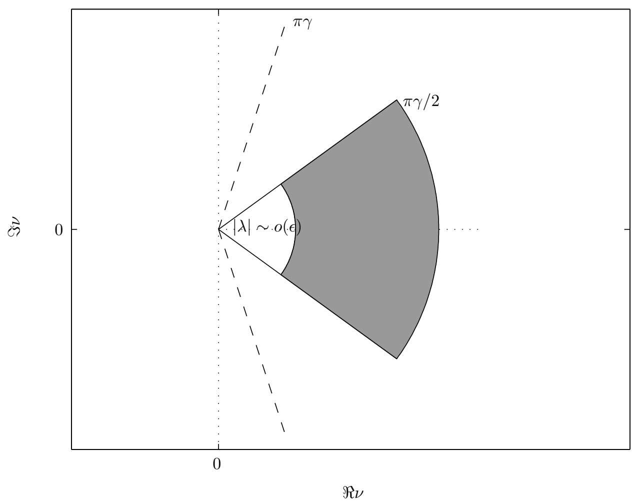
boundary condition $\quad \lim_{|y| \longrightarrow \infty} \tilde{A}^{(0)} = 0$

$$\chi = qm \left(s + \frac{\nu}{2} \frac{\vartheta}{\vartheta_o} \operatorname{ctgh} \frac{\vartheta_o}{n} \right)^{-1} \qquad \vartheta = \sqrt{\frac{1 + \tau_o {\lambda^{(0)}}^\gamma}{D}}$$

$$\nu = 2 \operatorname{csch} \frac{2\vartheta}{n} \left(\cosh \frac{2\vartheta}{n} - \cos \frac{\pi i}{n} \right), \qquad i = 0, \ldots, n-1$$

mapping to the problem for regular diffusion

$${\lambda^{(0)}}^\gamma \longmapsto \lambda^{(0)}$$



Stability theorems

Let $\chi > 0$ and either $m = 2,\ 1 < p \leqslant 5$ or $m = p + 1$. Then NLEP admits an eigenpair $\left\{\lambda^{(0)}, \tilde{A}^{(0)}\right\}$ such that $\Re{\lambda^{(0)}}^{\gamma} > 0$ if $\chi < p - 1$, and $\Re{\lambda^{(0)}}^{\gamma} < 0$ if $\chi > p - 1$.

Let $\tau_o \geqslant 0$ and $n \geqslant 2$. Then exists $\lambda^{(0)} > 0$ if

$$D > D_{\text{th}}\Big|_{\gamma=1} \stackrel{\text{def}}{=} \frac{4}{n^2 \ln^2\left(\alpha_n + \sqrt{\alpha_n^2 - 1}\right)}$$

$$\alpha_n = 1 + \frac{1}{\zeta}\left(1 + \cos\frac{\pi}{n}\right), \quad \zeta = \frac{qm}{p-1} - s - 1.$$

Let $\tau_o > 0$ and $n = 1$.

$$\frac{\chi}{b_m}\int_{-\infty}^{\infty} u^{m-1}\tilde{A}^{(0)}\,dy = 1$$

$$\left\{L_0 - {\lambda^{(0)}}^{\gamma}\right\}\widetilde{A}^{(0)} = u^p$$

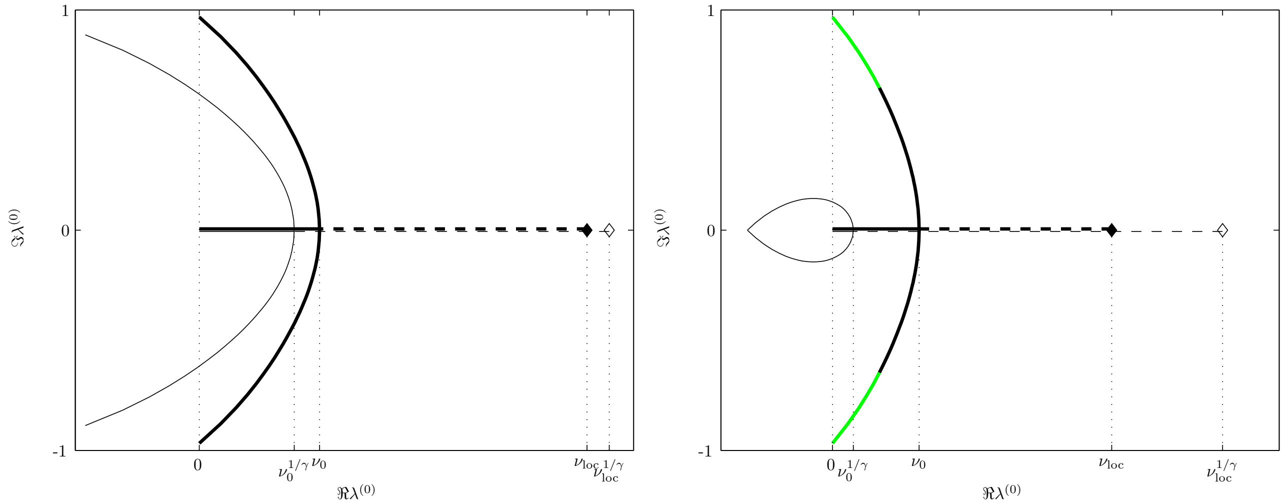
$\Longrightarrow$ seek roots of

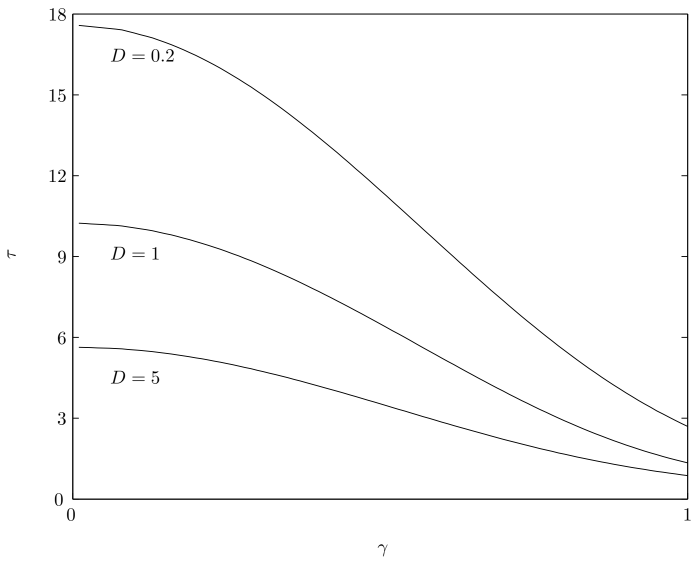
$$g\left(\lambda^{(0)}\right) = \frac{1}{\chi} - \frac{1}{b_m}\int_{-\infty}^{\infty} u^{m-1}\left\{L_0 - {\lambda^{(0)}}^{\gamma}\right\}^{-1} u^p dy$$

If the function $\Re g\Big|_{\lambda^{(0)} = \imath\lambda_\imath}$ possesses a unique root, then for any $D > 0$ the number of eigenvalues M in the open right half plane is either $M = 0$ or $M = 2$.

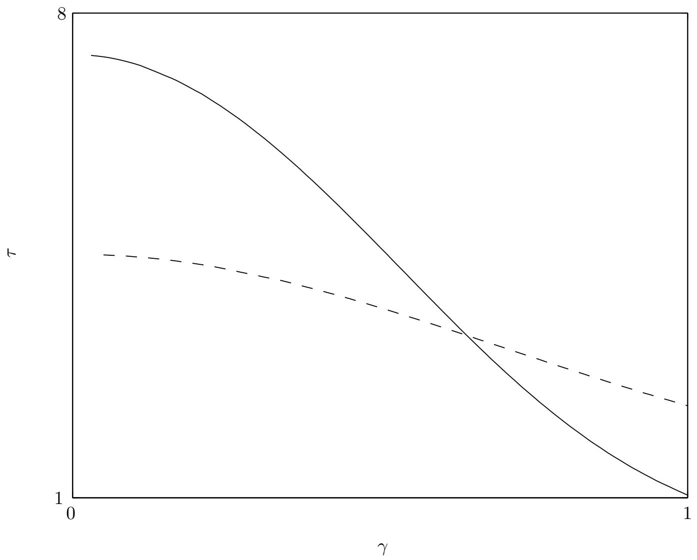
Eigenvalues in the right half plane

Hopf bifurcation threshold τ_o increases





Asynchronous mode dominates at $\gamma < \gamma_*$



Conclusion

- ↣ Fractional derivative generalises integer order derivatives. Many fundamental calculus and complex analysis tools become inapplicable.
- ↬ Sub-diffusion allows for pattern formation while keeping the ratio of diffusivities realistic.
- ⇝ Slow time and inner layer scales are γ-dependent. The homoclinic orbit is normal.
- ↠ Motion of spikes is slower than normal and the approach to equilibrium is algebraic rather than exponential.
- ↪ Non-local eigenvalue problem is fractional in $\lambda^{(0)}$. Stability theorems can be extended to the anomalous case and yield non-intuitive threshold changes.
- ⇀ Normally computed eigenvalues can be used to find the anomalous "eigenvalues". Instability due to real positive eigenvalues is essentially unaffected, except perhaps for $\gamma \ll 1$. The Hopf bifurcation threshold is higher and the asynchronous oscillation mode is dominant for γ below a certain threshold.

preprint at http://www.math.ubc.ca/~oulanka