Stable Localized Patterns in Cross Diffusion and Chemotaxis Systems

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Joint work with T. Kolokolnikov, M. Ward
Localized behavior in reaction-diffusion systems
July 25, 2011, BIRS

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Outline of My Talk

- Part I: Localized Solutions in Cross-Diffusion Systems
  (joint work with T. Kolokolnikov)
  (SIAM J. Appl. Math. 2011 online)

- Part II: Localized Solutions in Chemotaxis System Modeling LA Crimes
  (joint work with T. Kolokolnikov and M. Ward)

Part I: Cross-Diffusion Systems

We first discuss pattern formations in a cross-diffusion system

Standard Diffusion: $\nabla(J), J = \nabla u$

Self Diffusion: $J = a(x, u)\nabla u$

Cross-diffusion: $J = a(x, u, v)\nabla u$

Q: Can cross-diffusion create stable patterns?

A model of cross-diffusion

We consider cross-diffusion model of Shigesada, Kawasaki and Teramoto (1979)

$$\begin{cases} u_t = \Delta\left[(d_1 + \rho_{12} v)\, u\right] + u(a_1 - b_1 u - c_1 v) \\ v_t = \Delta\left[(d_2 + \rho_{21} u)\, v\right] + v(a_2 - b_1 u - c_1 v) \\ \text{Neumann B.C. on } [a, b] \end{cases} \qquad (1)$$

The kinetics are just the classic Lotka-Volterra competition model; d_1, d_2 represent self-diffusion
Cross-diffusion (ρ_{12}, ρ_{21}) represent inter-species avoidance: abundance of v will cause u to diffuse faster and vice-versa.

Without cross-diffusion, only constant solution is stable [Kishimoto, 1981].
A well-studied toy model [Lou, Ni, Yotsutani, Wu, Xu] is [after scaling]:

$$\begin{cases} u_t = (\rho v u)_{xx} + u(a_1 - b_1 u - c_1 v) \\ v_t = d v_{xx} + v(a_2 - b_1 u - c_1 v) \end{cases} \tag{2}$$

with the following assumptions:

$$d \ll 1; \quad \rho \gg 1; \text{ all other parameters are positive and of } O(1). \tag{3}$$

Biologically, when ρ is large, v acts as an inhibitor on u, so that u diffuses quickly in the regions of high concentration of v. This effect is believed to be responsible for the segregation of the two species.

Construction of steady state in 1D

- Lou, Ni, Yotsutani, 2004: Constructed a steady state in the form of a spike for u, and in the form of an inverted spike for v.
- More explicit computations [spike height] by Wu-Xu, 2010.
- Define

$$\tau = uv$$

so that

$$0 = dv_{xx} + a_2 v - b_2 \tau - c_2 v^2; \quad 0 = \rho \tau_{xx} + \tau \left(\frac{a_1}{v} - b_1 \frac{\tau}{v^2} - c_1 \right); \tag{4}$$

- In the limit $\rho \to \infty$ the shadow system is:

$$0 = dv_{xx} + a_2 v - b_2 \tau + c_2 v^2; \tag{5}$$

$$Lc_1 = \int_0^L \left(\frac{a_1}{v} - b_1 \frac{\tau}{v^2} \right). \tag{6}$$

(Keener, 1981, Nishiura)

- the solution to $dv_{xx} + a_2 v - b_2\tau + c_2 v^2 = 0$ can be written as $v = C_1 + C_2 \tanh^2(C_3 x)$. Matching the integral condition gives
- asymptotic behavior

$$v(x) \sim \frac{a_2}{2c_2}\left[\frac{3}{2}\tanh^2\left(\frac{x}{2\varepsilon}\right) + \delta\left(2 - 3\tanh^2\left(\frac{x}{2\varepsilon}\right)\right)\right];$$

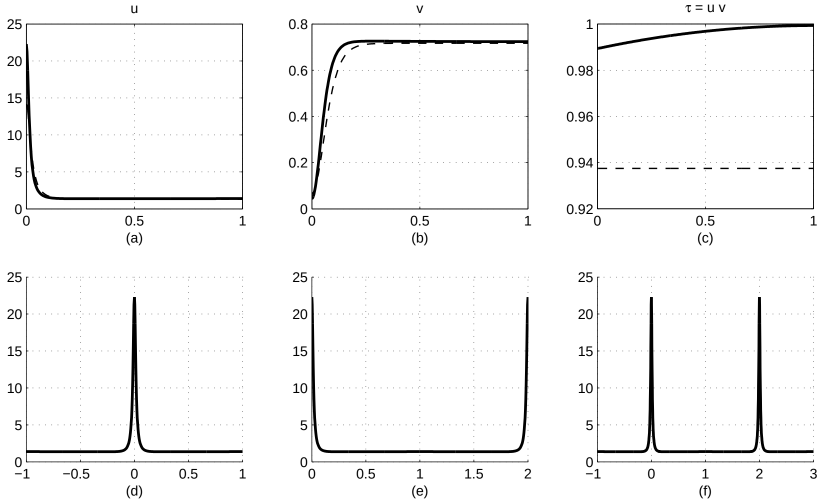
$$u \sim \frac{\tau_0}{v(x)}$$

where

$$\varepsilon := \sqrt{\frac{2d}{a_2}} \qquad \text{[spike width]}$$

$$\delta := (\varepsilon/L)^{2/3}\frac{3}{4}\left(\frac{b_1}{b_2}\frac{\pi}{2}\right)^{2/3}\left(4\frac{a_1}{a_2} - \frac{b_1}{b_2} - 3\frac{c_1}{c_2}\right)^{-2/3} \qquad \text{[spike height]}$$

$$\tau_0 := \frac{3}{16}\frac{a_2^2}{b_2 c_2};$$


u
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τ = u v
25
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0.92
(a)
(b)
(c)
−1
−0.5
1.5
2
3
(d)
(e)
(f)

- v has an inverted spike

$$v(x) \sim \frac{a_2}{2c_2}\left[w(0) - w\left(\frac{x}{2\varepsilon}\right) + \delta\left(2 - 3\tanh^2\left(\frac{x}{2\varepsilon}\right)\right)\right]$$

$$\boxed{\mathsf{w}_{yy} - w + w^2 = 0; \quad w \to 0 \text{ as } |y| \to \infty, \quad w'(0) = 0.}$$

- Note that $v(0) \sim \frac{a_2}{c_2}\delta = O(\varepsilon^{2/3}); \quad u(0) \sim O(\varepsilon^{-2/3})$.
- This construction works as long as

$$\left(4\frac{a_1}{a_2} - \frac{b_1}{b_2} - 3\frac{c_1}{c_2}\right) > 0.$$

- Question: is the solution stable ?

Linearized problem

Linearized equations are

$$\lambda\phi = d\phi_{xx} + a_2\phi - b_2\psi - c_2 2v\phi;$$

$$\lambda\left(\frac{1}{v}\psi - \frac{\tau}{v^2}\phi\right) = \rho\psi_{xx} + \left(\frac{a_1}{v} - b_1 2\frac{\tau}{v^2} - c_1\right)\psi + \left(-\frac{a_1\tau}{v^2} + 2b_1\frac{\tau^2}{v^3}\right)\phi.$$

Two kinds of eigenvalues

- large eigenvalues: $\lambda = O(1)$
- small eigenvalues:$\lambda = o(1)$

Principal stability result

Define

$$\rho_{K,\mathsf{small}} := d^{-1/3} L^{8/3} \frac{c_2}{2} \left(\frac{b_1}{b_2} \frac{\pi}{2} \right)^{-2/3} \frac{a_2^{1/3}}{2^{1/3}} \left(4 \frac{a_1}{a_2} - \frac{b_1}{b_2} - 3 \frac{c_1}{c_2} \right)^{5/3}; \tag{7}$$

$$\rho_b := 0.747 \rho_{K,\mathsf{small}}; \tag{8}$$

$$\rho_{K,\mathsf{large}} := \rho_{K,\mathsf{small}} \frac{2 \times 0.747}{1 - \cos\left[\pi \left(1 - 1/K\right)\right]}. \tag{9}$$

Then:

- A single boundary spike is stable for all ρ (not exponentially large in ε).
- A double-boundary steady state is stable if $\rho < \rho_b$ and is unstable otherwise. The instability is due to a large eigenvalue.
- A K-interior spike steady state with $K \geq 2$ is stable if $\rho < \min\left(\rho_{K,\text{small}}, \rho_{K,\text{large}}\right)$ and is unstable otherwise. When $K = 1$, it is stable provided that ρ is not exponentially large in ε.
- The critical scaling is

$$\rho = O(d^{-1/3}) = O(\varepsilon^{-2/3}) \gg 1.$$

Stability: small vs. large eigenvalues

- K spikes are always stable whenever $1 \ll \rho \ll d^{-1/3}$ and unstable when $K \geq 2$ and $\rho \gg d^{-1/3}$.
- Recall that $\rho_{K,\text{large}} := \rho_{K,\text{small}} \frac{2\times 0.747}{1-\cos[\pi(1-1/K)]}$ and

$$\frac{2 \times 0.747}{1 - \cos\left[\pi\left(1 - 1/K\right)\right]} = \begin{cases} 1.494 > 1, & K = 2 \\ 0.996 < 1, & K = 3 \\ 0.875 < 1, & K = 4 \end{cases}$$

- $\rho_{K,\text{large}} > \rho_{K,\text{small}}$ if $K = 2$ but $\rho_{K,\text{large}} < \rho_{K,\text{small}}$ if $K \geq 3$. It follows that the primary instability is due to small eigenvalues if $K = 2$but is due to large eigenvalues if $K \geq 3$. This is in agreement with numerical simulations.

Boundary Conditions

Possible boundary conditions (as in van der Ploeg-Doelman, Indiana Univ.Math. J. 2005):

Config type	Boundary conditions for ϕ
1 interior spike on $[-L, L]$ even eigenvalue	$\phi'(0) = 0 = \phi'(L)$
1 interior spike on $[-L, L]$ odd eigenvalue	$\phi(0) = 0 = \phi'(L)$
2 1/2-spikes at $[0, L]$	$\phi'(0) = 0 = \phi(L)$
K spikes on $[-L, (2K-1)L]$, Periodic BC	$\phi(L) = z\phi(-L), \quad \phi'(L) = z\phi'(-$ $z = \exp(2\pi ik/K), \quad k = 0 \ldots K -$
K spikes on $[-L, (2K-1)L]$, Neumann BC	$\phi(L) = z\phi(-L), \quad \phi'(L) = z\phi'(-$ $z = \exp(\pi ik/K), \quad k = 0 \ldots K -$

(same BC for ψ)

Reduced problem, large eigenvalues

- Using asymptotic matching, eventually we get a new point-weight eigenvalue problem (PWEP):

$$\begin{cases} \lambda\Phi = \Phi_{yy} - \Phi + 2w\Phi - \chi\Phi(0) \\ \Phi \text{ is even and is bounded as } |y| \to \infty \end{cases} \qquad \text{(PWEP)}$$

where $w(y) = \frac{3}{2}\operatorname{sech}^2\left(\frac{y}{2}\right)$ satisfies

$$w_{yy} - w + w^2 = 0; \quad w \to 0 \text{ as } |y| \to \infty, \quad w'(0) = 0.$$

  - For double-boundary spike,

$$\chi = \chi_b := \frac{\varepsilon^{-2/3}}{4\rho}\left(4\frac{a_1}{a_2} - \frac{b_1}{b_2} - 3\frac{c_1}{c_2}\right)^{5/3} c_2 \left(\frac{b_1}{b_2}\frac{\pi}{2}\right)^{-2/3} L^{8/3}.$$

  - For K spikes, Neumann BC, there are K choices for χ, namely

$$\chi = \frac{2}{1-\cos\frac{\pi k}{K}}\chi_b, \quad k = 0\ldots K-1 \text{ and } \chi = \text{very large positive.}$$

Analysis of $PWEP$ $\lambda\Phi = \Phi_{yy} - \Phi + 2w\Phi - \chi\Phi(0)$

- $\lambda = 0,\ \Phi = w_y$ is a solution [corresponds to translation invariance]
- If $\chi = 0$ then there is an unstable eigenvalue $\lambda_1 > 0$ and another eigenvalue $\lambda_3 < 0$.
- Decompose:

$$\Phi(y) = \Phi^\star + \Phi_0(y); \quad \text{where} \quad \Phi^\star = \lim_{y\to\pm\infty} \Phi(y).$$

Then

$$\lambda\Phi^\star = -\Phi^\star - \chi\left(\Phi_0(0) + \Phi^\star\right)$$

and Φ_0 satisfies

$$\lambda\Phi_0 = \Phi_{0yy} - \Phi_0 + 2w\Phi_0 + 2w\Phi^\star$$

so the PWEP becomes

$$\lambda\Phi_0 = \Phi_{0yy} - \Phi_0 + 2w\Phi_0 - \frac{2\chi}{\chi+\lambda+1}\Phi_0(0)w \qquad (10)$$

$$\lambda\Phi_0 = \Phi_{0yy} - \Phi_0 + 2w\Phi_0 - \frac{2\chi}{\chi+\lambda+1}\Phi_0(0)w$$

- Anzatz: if $\Phi_0 = w, \lambda = 0$ then $\chi = \frac{1}{2}$.
- Rigorous result: there is an unstable eigenvalue $\lambda > 0$ for all $\chi < \frac{1}{2}$
- The above two facts seem to suggest: stability when $\chi > \frac{1}{2}$???)
- In the limit $\chi \to \infty$, the limiting problem is

$$\lambda\Phi_0 = \Phi_{0,yy} - \Phi_0 + 2w\Phi_0 - 2\Phi_0(0)w \qquad (11)$$

Hypergeometric reduction

Theorem: the eigenvalues of $\lambda\Phi = \Phi_{yy} - \Phi + 2w\Phi - \chi\Phi(0)$
are given implicitly by:

$$\lambda = -1 - \chi + 2\chi\Phi_0(0)$$

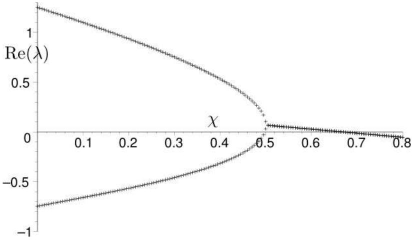
where

$$\Phi_0(0) = \frac{6\pi\lambda\left(\lambda+1\right)}{\sin\left(\pi\alpha\right)\left(4\lambda-5\right)\left(4\lambda+3\right)} - \frac{3}{2}\frac{1}{\lambda}{}_3F_2\left(\begin{array}{c} 1, 3, -1/2 \\ 2+\alpha,\ 2-\alpha \end{array};1\right)$$

$$\alpha = \sqrt{1+\lambda}$$

Similar idea has been used in
Doelman-Gardner-Kaper, Mem. AMS 2002, Indiana Univ. Math. J. 2001
Wei-Winter, MAA 2002, SIAM J.Math.Anal. 2003

- Numerical result+winding argument of Ward-Wei EJAM 2003: all $\lambda < 0$ whenever $\chi > 0.669$; stabilization is via a hopf bifurcation.



- desperate need for an analytical study of the limiting eigenvalue problem

$$\lambda\Phi_0 = \Phi_{0,yy} - \Phi_0 + 2w\Phi_0 - 2\Phi_0(0)w$$

Small eigenvalues

- Construct asymmetric spike steady states
- These bifurcate from the symmetric branch
- The instability thresholds for the small eigenvalues correspond precisely to this bifurcation point!
  Iron-Ward-Wei Phys D 2001
  van der Ploeg-Doelman, Indiana Univ.Math. J. 2005
  Proof is Needed !!!
- Main result: For 2 spikes, small eigenvalues is the dominant instability. For 3 or more, large eigenvalues dominate.

Radial equilibrium in two dimensions

Consider $\Omega \in \mathbb{R}^2$. Let w be the ground state in 2D:

$$\Delta w - w + w^2 = 0; \quad w \to 0 \text{ as } |y| \to \infty, \quad \max w = w(0)$$

and define

$$m := \max w(y) = w(0) \approx 2.39195.$$

Suppose that

$$\frac{a_1}{a_2}(2m-1) - (m-1)\frac{b_1}{b_2} - m\frac{c_1}{c_2} > 0 \tag{12}$$

and consider the asymptotic limit

$$d \ll 1; \quad \rho \gg 1. \tag{13}$$

If Ω is radially symmetric, there is a steady state at $x = 0$, in the form of an inverted spike for v. More precisely, we have

$$v(x) \sim \frac{1}{2m-1}\frac{a_2}{c_2}(1-2\delta)\left(w(0) - w\left(\frac{1-\delta}{\varepsilon}x\right) + (2m-1)\,\delta\right);$$

$$u \sim \frac{\tau_0}{v(x)}$$

$$\Delta w - w + w^2 = 0; \quad w \to 0 \text{ as } |y| \to \infty, \quad \max w = w(0)$$

where

$$\varepsilon := \sqrt{\frac{(2m-1)d}{a_2}}; \quad \tau_0 := \frac{(m-1)m}{(2m-1)^2}\frac{a_2^2}{b_2 c_2}.$$

$$\delta \sim \frac{\varepsilon^2}{|\Omega|}\frac{4\pi b_1 m}{b_2\left(2m-1\right)}\frac{1}{\left(\frac{a_1}{a_2}\left(2m-1\right)-\left(m-1\right)\frac{b_1}{b_2}-m\frac{c_1}{c_2}\right)};$$

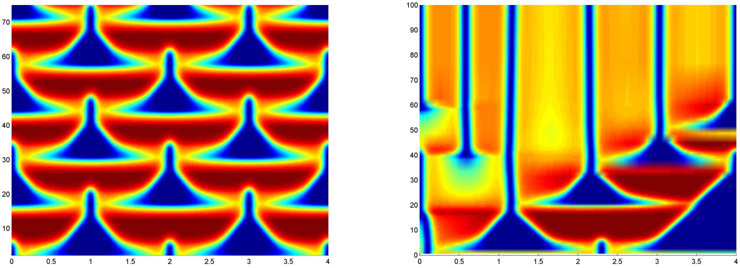
In particular,

$$v(0) \sim \frac{a_2}{c_2}\delta = O(d); \qquad u(0) \sim \frac{(m-1)m}{(2m-1)^2}\frac{a_2}{b_2}\frac{1}{\delta} = O\left(\frac{1}{d}\right). \quad (14)$$

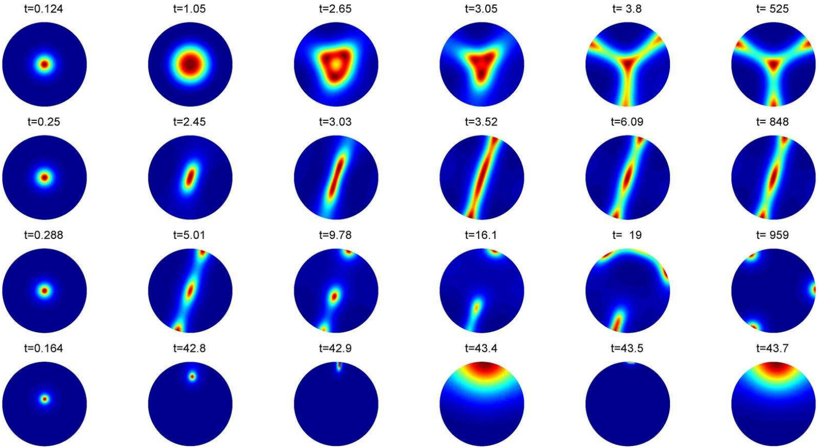
Stability:????

Interesting patterns: $\rho = O(1)$

- Spike insertion, spatio-temporal chaos



Sensitivity to initial conditions. The left and right figure differ only in the initial conditions. On the left, symmetric initial conditions result in an intricate a time-periodic solution. On the right, the initial condition is the same as on the left, except for a shift of 0.1 units to the right. dynamics eventually settle to a 5-spike stable pattern.



$$\rho = 50, \quad (a_1, b_1, c_1) = (5, 1, 1), \quad (a_2, b_2, c_2) = (5, 1, 5)$$

Row 1: $\rho = 2$. Spot splits into three spots. Row 2: $\rho = 4$. Initially, spot splits into two, final steady state consists of two boundary and one center spot. Row 3: $\rho = 6$. Row 4: $\rho = 500$. The interior

Part II: Localized Solutions in Crime Hotspot Model

UCLA Model of hot-spots in crime

- Recently proposed by Short Brantingham, Bertozzi et.al [PNAS, 2008].
- Very hot math: e.g. The New York Times, Dec 2010
- Crime is ubiquious but not uniformly distributed
  - some neigbourhoods are worse than others, leading to crime "hot spots"
  - Crime hotspots can persist for long time.

Fig. 1. Dynamic changes in residential burglary hotspots for two consecutive three-month periods beginning June 2001 in Long Beach, CA. These density maps were created using ArcGIS.

Figure taken from Short et al. *A statistical model of criminal behavi*

- Crime is temporally correlated:
  - Criminals often return to the spot of previous crime
  - If a home was broken into in the past, the likelyhood of subsequent breaking increases
  - Example: graffitti "tagging"
  - the motion of criminals towards higher attractiveness areas can be modeled by chemotaxis

- Two-component model

$$A_t = \varepsilon^2 A_{xx} - A + \rho A + A_0$$
$$\tau \rho_t = D\left(\rho_x - 2\frac{\rho}{A}A_x\right)_x - \rho A + \bar{A} - A_0.$$

  - $\rho(x,t) \equiv$ density of criminals;
  - $A(x,t) \equiv$ "attractiveness" of area to crime
  - $A_0 = O(1) \equiv$ "baseline attractiveness "
  - $D(-2\frac{\rho}{A}A_x)_x$ models the motion of criminals towards higher attractiveness areas
  - $\bar{A} - A_0 > 0$ is the baseline criminal feed rate
  - We assume here:

$$\varepsilon^2 \ll 1, \quad D \gg 1.$$

Hot-spot steady state

$$0 = \varepsilon^2 A_{xx} - A + \rho A + A_0; \quad 0 = D\left(\rho_x - 2\frac{\rho}{A}A_x\right)_x - \rho A + \bar{A} - A_0$$

- Key trick: $\rho_x - 2\frac{\rho}{A}A_x = A^2\left(\rho A^{-2}\right)_x$. This suggests the change of variables:

  $$v = \frac{\rho}{A^2};$$

  so that

  $$0 = \varepsilon^2 A_{xx} - A + vA^3 + A_0; \qquad 0 = D\left(A^2 v_x\right)_x - vA^3 + \bar{A} - A_0.$$

- "Shadow limit" Large D : $\quad v(x) \sim v_0;$

$$\varepsilon^2 A_{xx} - A + vA^3 + A_0 = 0; \qquad v_0 \int_0^L A^3 dx = \left(\bar{A} - A_0\right) L.$$

- Anzatz: $v_0 \ll 1, \quad A \sim v_0^{-1/2} w(y), \quad y = x/\varepsilon$ where w is the ground state,

$$w_{yy} - w + w^3 = 0, \quad w'(0) = 0, \quad w \to 0 \text{ as } |y| \to \infty;$$

then

$$v_0 \sim \frac{\left(\int_{-\infty}^{\infty} w^3 dy\right)^2}{4L^2 \left(\bar{A} - A_0\right)^2} \varepsilon^2;$$

$$A(x) \sim \begin{cases} \dfrac{2L(\bar{A} - A_0)}{\varepsilon \int w^3} w(x/\varepsilon), & x = O(\varepsilon) \\ A_0, & x \gg O(\varepsilon). \end{cases}$$

Critical Scaling

Based on previous computations, we now set

$$\Omega = (-1, 1),$$

$$A = A_0 + \frac{1}{\epsilon}\hat{A}, \quad v = \epsilon^2 \hat{v}$$

$$D = \frac{\hat{D}}{\epsilon}$$

Then the steady-state problem becomes

$$0 = \varepsilon^2 \hat{A}_{xx} - \hat{A} + \hat{v}(\epsilon A_0 + \hat{A})^3 \tag{15}$$

$$0 = \hat{D}\left((A_0 + \frac{1}{\epsilon}\hat{A})^2 \hat{v}_x\right)_x - \frac{1}{\epsilon}\hat{v}(\epsilon A_0 + \hat{A})^3 + \bar{A} - A_0. \tag{16}$$

Relation with A Schnakenberg Model

- The steady state problem in 1D is very close to the so-called Schnakenberg model

$$0 = \varepsilon^2 u_{xx} - u + vu^p$$
$$0 = Dv_{xx} + 1 - \frac{1}{\epsilon} vu^p$$

with $p = 3$

$$0 = \varepsilon^2 \hat{A}_{xx} - \hat{A} + \hat{v}\hat{A}^3 \tag{17}$$
$$0 = \hat{D}\left(A_0^2 \hat{v}_x\right)_x - \frac{1}{\epsilon}\hat{v}\hat{A}^3 + \bar{A} - A_0. \tag{18}$$

- To see this, we consider the following problem

$$\hat{D}(a(x)v_x)_x = f(x), v^{'}(0) = 0 \tag{19}$$

we have

$$v(x) - v(0) = \frac{1}{\hat{D}} \int_0^x K_a(x,s) f(s) ds \tag{20}$$

where

$$K_a(x,s) = \int_s^x \frac{1}{a(x)} dx$$

- Let us now consider $a(x) = (A_0 + \frac{\gamma}{\epsilon} w(\frac{x}{\epsilon}))^2$, where $w > 0$ and $w \sim e^{-|y|}$. Then

$$K_a(x,s) = K_{A_0^2}(x,s) + O(\epsilon|s-x|) + O(|[s,x] \cap (0, 2\epsilon \ln \frac{1}{\epsilon})|) \tag{21}$$

Main stability result (1D)

- Main result: Consider K spikes on the domain of size $2KL$. Then small eigenvalues become unstable if $D > D_{c,\text{small}}$; large eigenvalues become unstable if $D > D_{c,\text{small}}$ where

$$D_{c,\text{small}} \sim \frac{L^4}{\varepsilon^2} \frac{\left(\bar{A} - A_0\right)^3}{A_0^2 \pi^2}$$

$$D_{c,\text{large}} \sim D_{c,\text{small}} \left(\frac{2}{1 - \cos \frac{\pi}{K}} \right) > D_{c,\text{small}}$$

- Small eigenvalues become unstable before the large eigenvalues.

- Example: Take $L = 1, \bar{A} = 2, A_0 = 1,\ K = 2,\ \varepsilon = 0.07$. Then $D_{c,\text{small}} = 20.67, \quad D_{c,\text{large}} = 41.33.$
  - if $D = 15 \implies$ two spikes are stable
  - if $D = 30 \implies$ two spikes have very slow developing instability
  - if $D = 50 \implies$ two spikes have very fast developing instability
- very similar behavior to Schnakenberg model

$$0 = \varepsilon^2 u_{xx} - u + vu^3$$
$$0 = Dv_{xx} + 1 - \frac{1}{\epsilon} vu^3.$$

Iron-Wei-Winter, J.Math.Biol. 2003

Stability: large eigenvalues

- Step 1: Reduces to the nonlocal eigenvalue problem (NLEP):

$$\lambda\phi = \phi'' - \phi + 3w^2\phi - \chi\left(\int w^2\phi\right)w^3 \qquad \text{where } w'' - w + w^3 = 0 \tag{22}$$

with

$$\chi \sim \frac{3}{\int_{-\infty}^{\infty} w^3 dy}\left(1 + \varepsilon^2 D(1 - \cos\frac{\pi k}{K})\frac{A_0^2\pi^2}{4L^4\left(\bar{A} - A_0\right)^3}\right)^{-1}$$

- This is an oversimplified problem but captures the main characteristics

- Step 2: *Key identity*: $L_0 w^2 = 3w^2$, where $L_0\phi := \phi'' - \phi + 3w^2\phi$. Multiply

$$\lambda\phi = \phi'' - \phi + 3w^2\phi - \chi\left(\int w^2\phi\right) w^3$$

by w^2 and integrate to get

$$\lambda = 3 - \chi\int w^5 = 3 - \chi\frac{3}{2}\int w^3$$

Conclusion: (22) is stable iff $\chi > \frac{2}{\int w^3} \iff D > D_{c,\text{large}}$.

- This NLEP in 1D can be fully solved!!

Stability: small eigenvalues

- Compute asymmetric spikes
- They bifurcate from symmetric branch
- The bifurcation point is precisely when $D = D_{c,\text{small}}$.
- This is "cheating"... but it gets the correct threshold!!
- similar computations to
  Iron-Ward-Wei 2001 (for Gierer-Meinhardt system)
  Iron-Wei-Winter 2003 (for Schnakenberg model)

Two dimensions

$$\begin{cases} A_t = \varepsilon^2 \Delta A - A + \hat{v} A^3 + A_0 \\ \tau (A\hat{v})_t = D\nabla \cdot \left(A^2 \nabla \hat{v} \right) - \hat{v} A^3 + \bar{A} - A_0 \\ Neumann\ BC \end{cases}, \quad x \in \Omega$$

- Steady-state: construction is similar to 1D, but no reduction to Schnakenberg model
- Stability: of K hot-spots:
-
  - If $K = 1$, then a single hot-spot is stable with respect to large eigenvalues, as long as D is not exponentially large in $1/\varepsilon$.
  - If $K \geq 2$, then the steady state is stable with respect to large eigenvalues if

$$D < \frac{1}{\varepsilon^4} \ln \frac{1}{\varepsilon} \frac{\left(\bar{A} - A_0 \right)^3 |\Omega|^3 A_0^{-2}}{4\pi K^3 \left(\int_{\mathbb{R}^2} w^3 dy \right)^2}; \tag{23}$$

    and it is unstable otherwise.
- Instability thresholds occur when $D = O\left(\frac{\ln \varepsilon^{-1}}{\varepsilon^4 K^3} \right) \gg 1.$

Concluding Summary

- In both models, the instability thresholds occur close to the "shadow limit", i.e. the cross-diffusion term is very large.
- Steady-state computation is essentially a shadow system, but stability computations require more.
- Cross-diffusion (directed movement) can create stable multi-spike solutions even in the absence of spatial heterogenuity.
- Chemotaxis system (crime models) can also produce multiple stable patterns
- Stability analysis leads to novel, interesting and new eigenvalue problems